ANNIHILATOR CONDITIONS ON RINGS AND NEAR-RINGS

YONG UK CHO

ABSTRACT. In this paper, we initiate the study of some annihilator conditions on polynomials which were used by Kaplansky [Rings of operators. W. A. Benjamin, Inc., New York, 1968] to abstract the algebra of bounded linear operators on a Hilbert spaces with Baer condition. On the other hand, p.p.-rings were introduced by Hattori [A foundation of torsion theory for modules over general rings. Nagoya Math. J. 17 (1960) 147–158] to study the torsion theory. The purpose of this paper is to introduce the near-rings with Baer condition and near-rings with p.p. condition which are somewhat different from ring case, and to extend a results of Armendariz [A note on extensions of Baer and P.P.-rings. J. Austral. Math. Soc. 18 (1974), 470–473] and Jøndrup [p.p. rings and finitely generated flat ideals. Proc. Amer. Math. Soc. 28 (1971) 431–433].

1. INTRODUCTION

Kaplansky [5] introduced the Baer rings as rings in which every left (right) annihilator ideal is generated by an idempotent. On the other hand, Hattori [3] introduced the left p.p.-rings as rings in which any principal left ideal is projective. In this paper we introduce Baer near-rings and p.p.-near-rings and study some of their properties and give some examples. Let $G$ be an additively written (but not necessarily abelian) group with zero 0 and $M_0(G) = \{ f : G \to G \mid f(0) = 0 \}$ the near-ring of all zero fixing mappings on $G$. We show that $M_0(G)$ is a Baer near-ring. As a corollary, we show that every zero-symmetric near-ring can be embedded into a Baer near-ring. Let $R$ be a commutative ring with identity. It is well known that $R$ is a Baer (resp. p.p.-) ring if and only if the polynomial ring $R[x]$ is a Baer (resp. p.p.-) ring (see e.g., Armendariz [1] and Jøndrup [4]). Corresponding to this result, we will prove that the zero-symmetric part of $R[x]$ is a Baer (resp. p.p.-) near-ring if and only if

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$R$ is a Baer (resp. p.p.-) ring. Finally we study the structure of a zero-symmetric reduced p. p.-near-ring with identity.

2. BAER NEAR-RINGS AND P.P.-NEAR-RINGS

A (right) near-ring is a set $N$ with two binary operations $+$ and $\cdot$ such that $(N, +)$ is a not necessarily abelian group with identity $0$, $(N, \cdot)$ is a semigroup and $(x + y)z = xz + yz$ for all $x, y, z \in N$. Some basic definitions and concepts in near-ring theory can be found in Meldrum [6] and Pilz [7].

For a subset $S$ of a near-ring $N$, the set $\{n \in N | NnS = 0\}$ is called the annihilator of $S$ in $N$ which is denoted by $\text{Ann}_N(S) = \text{Ann}(S)$.

A near-ring $N$ is called a Baer near-ring if, for any subset $S$ of $N$, $\text{Ann}(S) = \text{Ann}(e)$ for some idempotent $e \in N$. The following proposition is obvious.

**Proposition 1.** Let $N_i$ ($i \in I$) be a family of near-rings. Then the direct product $\prod_{i \in I} N_i$ is a Bear near-ring if and only if $N_i$ is a Bear near-ring for each $i \in I$.

A near-ring $N$ is said to be integral if $N$ has no nonzero divisors of zero (cf. Pilz [7, 1.14, p. 11]).

**Example 1.**

1. Every integral near-ring with identity is a Baer near-ring.
2. Every constant near-ring is a Baer near-ring.
3. A direct product of integral near-rings with identity is a Baer near-ring.

Let $G$ be an additively written (but not necessarily abelian) group with zero $0$ and $M_0(G) = \{f : G \to G \mid f(0) = 0\}$ the near-ring of all zero fixing mappings on $G$ (see Pilz [7, 1.4, p. 8]). Beidleman [2, Theorem 1] proved that $M_0(G)$ is a regular near-ring. We shall prove that $M_0(G)$ is Baer.

**Theorem 1.** The near-ring $M_0(G)$ is a Baer near-ring.

**Proof.** Let $S$ be a subset of $M_0(G)$ and let $H = \{s(g) \mid s \in S, g \in G\}$. Let $e$ be a mapping on $G$ such that if $x \in H$, then $e(x) = x$ and $e(y) = 0$ for any $y \in G - H$. Then $e$ is an idempotent of $M_0(G)$ and $\text{Ann}(S) = \text{Ann}(e)$. This implies that $M_0(G)$ is a Baer near-ring. □
Corollary 1. Every zero-symmetric near-ring can be embedded into a Baer near-ring.

Proof. By Pilz [7, 1.102, p. 11], every zero-symmetric near-ring can be embedded into a zero-symmetric near-ring with identity. Let $N$ be a zero-symmetric near-ring with identity. By Theorem 1, $M_0(N)$ is a Baer near-ring. For any $r \in N$, the mapping $f_r : t \in N \rightarrow rt \in N$ is an element of $M_0(N)$. Since $N$ contains an identity, the mapping $f : N \rightarrow M_0(N) ; r \mapsto f_r$ is a near-ring monomorphism. □

An associative ring $R$ called a left p.p.-ring if every principal left ideal of $R$ is projective. This is equivalent to the condition that, for any $a \in R$, $\text{Ann}(a) = \text{Ann}(e)$ for some idempotent $e \in R$. A right p.p.-ring is defined in a symmetric way.

Now we call a near-ring $N$ a p.p.-near-ring if, for any $a \in N$, $\text{Ann}(a) = \text{Ann}(e)$ for some idempotent $e \in N$. Clearly a Baer near-ring is a p.p.-near-ring.

Following Beidleman [2], we call a near-ring $N$ regular if, for any $x \in N$, there exists $y \in N$ such that $xyx = x$.

Example 2. Every regular near-ring is a p.p.-near-ring. In fact, for any $x \in N$, there exists $y \in N$ such that $xyx = x$. Then $xy$ is an idempotent and $\text{Ann}(x) = \text{Ann}(xy)$.

Let $R$ be a commutative ring with identity and let $R[x]$ denote the set of all polynomials in one indeterminate over $R$. Under usual addition $+$ and substitution $\circ$ of polynomials, $(R[x], +, \circ)$ becomes a near-ring. Following Pilz [7, 7.78, p. 221], $R_0[x]$ denotes the zero symmetric part of $R[x]$, that is

$$R_0[x] = \left\{ \sum_{i=1}^{n} a_i x^i \mid a_i \in R, n \geq 1 \right\}.$$

The following is a near-ring theoretic modification of Jøndrup [4, Theorem 2.1].

Theorem 2. Let $R$ be a commutative ring with identity. Then the following conditions are equivalent:
1) $R_0[x]$ is a p.p.-near-ring.
2) $R$ is a p.p.-ring.

Proof.

1) $\Rightarrow$ 2). First we claim that $R$ is reduced. Suppose that $a \in R$ with $a^2 = 0$. By hypothesis, there exists an idempotent $f \in R_0[x]$ such that $\text{Ann}(ax) = \text{Ann}(f)$. Let $f = a_1 x + a_2 x^2 + \cdots + a_n x^n$ with $a_i \in R$. Since $f$ is an idempotent, we have
$a_1^2 = a_1$. Since $ax \in \text{Ann}(af)$, $ax \circ f = af = 0$. In particular, $aa_1 = 0$. Since $x - f \in \text{Ann}(f)$, $0 = (x - f) \circ ax = ax^2 - f(ax)$. Hence $ax^2 = a_1ax = 0$, that is $a = 0$. This proves that $R$ is reduced. Since $R$ is reduced, the set of idempotents of $R_0[x]$ is just \{ex | e^2 = e \in R\}. Now let $r$ be an arbitrary element of $R$. By hypothesis, there exists an idempotent $e \in R$ such that $\text{Ann}(rx) = \text{Ann}(ex)$. Clearly this implies that \{s \in R | sr = 0\} = R(1 - e). Hence $R$ is a p.p.-ring.

2) $\Rightarrow$ 1). Let $f = a_1x + \cdots + a_nx^n \in R_0[x]$ and $g = b_1x + \cdots + b_mx^m \in R_0[x]$. First we claim that $f \circ g = 0$ if and only if $a_ib_j = 0$ for all $i, j$. It suffices to prove the 'only if' part. Let $P$ be an arbitrary prime ideal of $R$ and let $\bar{f}$ and $\bar{g}$ denote the image of $f$ and $g$ in $(R/P)[x]$ respectively. Since $R/P$ is an integral domain and since $\bar{f} \circ \bar{g} = 0$, we can easily see that either $\bar{f} = 0$ or $\bar{g} = 0$ holds. Hence $a_ib_j \in P$ for all $i, j$. Since $P$ is an arbitrary prime ideal, this implies that $a_ib_j \in \text{Rad}(R)$, where $\text{Rad}(R)$ denote the prime radical of $R$. Since $R$ is a commutative p.p.-ring, $R$ is reduced and hence $\text{Rad}(R) = 0$. This proves our claim. Therefore $a_1, \ldots, a_n \in \text{Ann}_R(b_1, \ldots, b_m)$. Since $R$ is a p.p.-ring, for each $i$, there exists an idempotent $e_i \in R$ such that $\text{Ann}(b_i) = \text{Ann}(e_i)$. If $n = 2$, then $f = e_1 + e_2 - e_1e_2$ is an idempotent and $\text{Ann}_R(b_1, b_2) = \text{Ann}(f)$. Using induction on $n$, we can find an idempotent $e$ of $R$ such that $\text{Ann}_R(b_1, \ldots, b_m) = \text{Ann}(e)$. Then $ex$ is an idempotent of $R_0[x]$ and $\text{Ann}(g) = \text{Ann}(ex)$. Therefore $R_0[x]$ is a p.p.-near-ring.

The next theorem gives more examples of Baer near-rings.

**Theorem 3.** Let $R$ be a commutative ring with identity. Then the following conditions are equivalent:

1) $R_0[x]$ is a Baer near-ring.
2) $R$ is a Baer ring.

**Proof.**

1) $\Rightarrow$ 2). Let $T$ be a subset of $R$ and consider the subset $S = \{tx | t \in T\}$ of $R_0[x]$. As saw in the proof of 1) $\Rightarrow$ 2) of Theorem 2, the set of idempotents of $R_0[x]$ is just \{ex | e^2 = e \in R\}. Since $R_0[x]$ is Baer, $\text{Ann}(S) = \text{Ann}(ex)$ for some idempotent $e \in R$. Then we can easily see that $\text{Ann}_R(T) = \text{Ann}_R(e)$. Hence $R$ is a Baer ring.

2) $\Rightarrow$ 1). Let $S$ be a subset of $R_0[x]$ and consider the set $T$ of all coefficients of $g(x) \in S$. Let $f = a_1x + \cdots + a_nx^n \in \text{Ann}(S)$. As saw in the proof of 2) $\Rightarrow$ 1) of Theorem 2, $a_i \in \text{Ann}_R(T)$ for all $i$. Since $R$ is a Baer ring, there exists an idempotent
e such that $\text{Ann}_R(T) = \text{Ann}_R(e)$. Now we can easily see that $\text{Ann}(S) = \text{Ann}(ex)$. This proves that $R_0[x]$ is a Baer near-ring.

\[\square\]

**Corollary 2.** Let $R$ be a commutative ring with identity. Then the following conditions are equivalent:

1) $R$ is a von Neumann regular ring.
2) $(R/I)_0[x]$ is a p.p.-near-ring for all ideals $I$ of $R$.

**Proof.**

1) $\Rightarrow$ 2). If $R$ is regular, then $R/I$ is regular for every ideal $I$ of $R$, so that $R/I$ is a p.p.-ring. Hence this follows from Theorem 1.

2) $\Rightarrow$ 1). As saw in the proof of 1) $\Rightarrow$ 2) of Theorem 2, $R/I$ is reduced for every ideal $I$ of $R$. Let $a \in R$ and consider the ideal $Ra^2$ of $R$. Since $R/Ra^2$ is reduced and since $a + Ra^2 \in R/Ra^2$ is nilpotent, we have $a \in Ra^2$. This implies that $R$ is von Neumann regular. \[\square\]

Let $R$ be an associative ring with identity and let $M$ be a unital left $R$-module. If we define a multiplication on the additive group $R \oplus M$ by $(a, b) \circ (c, d) = (ac, ad + b)$ for $(a, b), (c, d) \in R \oplus M$, $R \oplus M$ becomes a near-ring with identity $(1, 0)$.

**Theorem 4.** Let $R$ be an associative ring with identity and let $M$ be a unital left $R$-module. Then the following conditions are equivalent:

1) $R \oplus M$ is a p.p.-near-ring.
2) $R$ is a left p.p.-ring.

**Proof.**

2) $\Rightarrow$ 1). We can easily see that, for $(c, d) \in R \oplus M$,

$$\text{Ann}(c, d) = \{(a, -ad) \mid a \in \text{Ann}(s)\}.$$  

Since $R$ is a left p.p.-ring, there is an idempotent $e \in R$ such that $\text{Ann}_R(c) = \text{Ann}(e)$. Then $(e, (1 - e)d)$ is an idempotent of $R \oplus M$ and $\text{Ann}(c, d) = \text{Ann}(e, (1 - e)d)$.

1) $\Rightarrow$ 2). We first note that the set of all idempotents of $R \oplus M$ is equal to $\{(e, (1 - e)x) \mid e = e^2 \in R, x \in M\}$. Hence, for any $c \in R$, there exists an idempotent $e \in R$ and an $x \in M$ such that $\text{Ann}(c, 0) = \text{Ann}(e, (1 - e)x)$. By the way, $\text{Ann}(c, 0) = \{(a, 0) \mid a \in \text{Ann}(c)\}$. On the other hand, $(1 - c, -(1 - e)x) \in \text{Ann}(e, (1 - e)x)$. Hence $(1 - c)x = 0$, and so $\text{Ann}(c, 0) = \text{Ann}(e, 0)$. This implies $\text{Ann}(c) = \text{Ann}(e)$. Therefore $R$ is a left p.p.-ring. \[\square\]
A near-ring with no non-zero nilpotent elements is said to be reduced. For the rest of this paper, we shall study the structure of zero-symmetric reduced p.p.-near-rings with identity.

**Proposition 2.** Let $N$ be a zero-symmetric reduced p.p.-near-ring with identity. Then, for any finitely many elements $a_1, \ldots, a_n \in N$, there exists an idempotent $e \in N$ such that $\text{Ann}(a_1, \ldots, a_n) = \text{Ann}(e)$.

**Proof.** Since $N$ is a p.p.-near-ring, there exist idempotents $e_1, \ldots, e_n \in N$ such that $\text{Ann}(a_i) = \text{Ann}(e_i)$ for each $i$. By Ramakotaiah & Sambasiva Rao [8, Lemma 0.2] (or Pilz [7, Proposition 9.43(b), p. 304]), all idempotents of $N$ is central. Then, by the same method as in the proof of 2) \(\Rightarrow 1)\) of Theorem 2, we can find an idempotent $e \in N$ such that $\text{Ann}(a_1, \ldots, a_n) = \text{Ann}(e)$. \(\Box\)

**Proposition 3.** Let $N$ be a zero-symmetric reduced p.p.-near-ring with identity. If $N$ has no infinitely many nonzero orthogonal idempotents, then $N$ is a direct sum of finitely many integral near-rings.

**Proof.** Let $\text{Ann}(a)$ be a minimal element in $\{\text{Ann}(t) \neq 0 \mid t \in N\}$. By hypothesis, there exists an idempotent $e_1 \in N$ such that $\text{Ann}(a) = \text{Ann}(e_1)$. We claim that $N(1 - e_1)$ is an integral near-ring. Let $b, c \in N(1 - e_1)$ such that $bc = 0$ and $c \neq 0$. Then $\text{Ann}(c + e_1) \subseteq \text{Ann}(e_1)$. By minimality of $\text{Ann}(e_1)$, we conclude that $\text{Ann}(c + e_1) = 0$. Clearly $b \in \text{Ann}(c + e_1)$, whence $b = 0$. This proves our claim.

Next we choose a minimal element $\text{Ann}(e_2)$ with $e_2 = e_2^2$ in $\{\text{Ann}(t) \neq 0 \mid t \in Ne_1\}$. Then we can also show that $N(e_1 - e_2)$ is an integral near-ring. Continuing this process, we obtain orthogonal idempotents $e_0 = 1, e_1, e_2, \ldots$ of $N$ such that $N(e_i - e_{i+1})$ is integral near-ring for each $i = 0, 1, \ldots$. Since

$$1 - e_1, e_1 - e_2, \ldots, e_{n-1} - e_n, \ldots$$

are orthogonal idempotents, by hypothesis there exists a natural number $n$ such that $e_n = 0$. Then $N = N(1 - e_1) \oplus \cdots \oplus N(e_{n-2} - e_{n-1}) \oplus Ne_{n-1}$ and $N(1 - e_1), \ldots, N(e_{n-2} - e_{n-1}), Ne_{n-1}$ are all integral near-rings. \(\Box\)

**Proposition 4.** Let $N$ be a zero-symmetric reduced p.p.-near-ring with identity. Then, for any $a \in N$, there exists a non zero-divisor $d \in N$ and an idempotent $e \in N$ such that $a = ed$.

**Proof.** By hypothesis, there exists an idempotent $e \in N$ such that $\text{Ann}(a) = \text{Ann}(e)$. Since every idempotent of $N$ is central, we have $a = ea = e(a + (1 - e))$. By
Ramakotaiah & Sambasiva Rao [8, Lemma 0.1] \( xy = 0 \), whence \( x, y \in N \) implies \( yx = 0 \). Using this property we can easily see that \( a+(1-e) \) is a non zero-divisor. □

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REFERENCES


Department of Mathematics, Silla University, 1-1 San, Gwaebeg-dong, Sasang-gu, Busan 617-736, Korea
Email address: yucho@silla.ac.kr