A NOTE ON STRONG REDUCEDNESS IN NEAR-RINGS

YONG UK CHO

ABSTRACT. Let $N$ be a right near-ring. $N$ is said to be strongly reduced if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$, or equivalently, for $a \in N$ and any positive integer $n$, $a^n \in N_c$ implies $a \in N_c$, where $N_c$ denotes the constant part of $N$.

We will show that strong reducedness is equivalent to condition (ii) of Reddy and Murty's property (*) (cf. [Reddy & Murty: On strongly regular near-rings. Proc. Edinburgh Math. Soc. (2) 27 (1984), no. 1, 61–64]), and that condition (i) of Reddy and Murty's property (*) follows from strong reducedness. Also, we will investigate some characterizations of strongly reduced near-rings and their properties. Using strong reducedness, we characterize left strongly regular near-rings and $(P_0)$-near-rings.

1. INTRODUCTION

Throughout this paper we will work with right near-rings. For notations and basic concepts, we shall refer to Pilz [7].

Let $N$ be a right near-ring. $N$ is said to be left strongly regular if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$. Right strong regularity is defined in a symmetric way. Mason [4] introduced these notions and characterized left strongly regular zero-symmetric unital near-rings. Several authors (cf. Hongan [2], Mason [5], Murty [6] and Reddy & Murty [8]) have studied them. In particular, Reddy & Murty [8] extended some results in Mason [4] to the non-zero symmetric case. They observed that every left strongly regular near-ring has an interesting property. In this paper, we consider the property (it is called Reddy and Murty's property (*)) in Reddy & Murty [8]:

(i) For any $a, b \in N$, $ab = 0$ implies $ba = b0$.
(ii) For $a \in N$, $a^3 = a^2$ implies $a^2 = a$.

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Let $N_c$ denote the constant part of $N$, that is, $N_c = \{ a | a = a0, a \in N \}$.

Now we define a new concept for near-rings, that is, a near-ring $N$ is said to be **strongly reduced** if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$.

Recall that a near-ring $N$ is reduced if, for $a \in N$, $a^2 = 0$ implies $a = 0$. As we shall show later, a strongly reduced near-ring $N$ is reduced. We will show that strong reducedness is equivalent to condition (ii) of Reddy and Murty's property (*) and condition (i) of Reddy and Murty's property (*) follows from strong reducedness. Consequently, we see that condition (i) of Reddy and Murty's property (*) is not needed.

Left or right strongly regular near-rings form one of the important classes of strongly reduced near-rings. We will investigate some properties of strongly reduced near-rings. Using strong reducedness, we characterize left strongly regular near-rings and $(P_0)$-near-rings.

2. Results

A subnear-ring $H$ of a near-ring $N$ is said to be **left invariant** if $NH \subseteq H$, **right invariant** if $HN \subseteq H$ and **invariant** if it is both left and right invariant. For a subset $S$ of $N$, \( \langle S \rangle \), $|S|$ and $\langle S \rangle$ (resp.) stand for the left invariant, right invariant and invariant (resp.) subnear-rings of $N$ generated by $S$. For any element $a \in N$, \( \langle a \rangle \), $|a|$ and $\langle a \rangle$ (resp.) are called the **principal left invariant**, **principal right invariant** and **principal invariant** (resp.) subnear-rings of $N$ generated by $a$.

There are slightly generalized new concepts of left strong regularity and right strong regularity. A near-ring $N$ is said to be **quasi left strongly regular** if $a \in \langle a^2 \rangle$ for each $a \in N$, **quasi right strongly regular** if $a \in \langle a^2 \rangle$ for each $a \in N$.

There are lots of quasi left (resp. right) strongly regular near-rings which are not left (resp. right) strongly regular.

First, we introduce the following lemma.

**Lemma 1.** We have the following properties.

1. The direct product of strongly reduced near-rings is strongly reduced.
(2) Every subnear-ring of a strongly reduced near-ring is strongly reduced.
(3) Every homomorphic image of a strongly reduced constant near-ring is strongly reduced.

Proof. (3) A constant near-ring is strongly reduced, and the homomorphic image of a constant near-ring is constant.

Now we give some sufficient conditions for quasi left strongly regular near-rings or quasi right strongly regular near-rings to be strongly reduced.

**Proposition 1.** We have the following properties.

(1) All quasi left strongly regular near-rings and quasi right strongly regular near-rings are strongly reduced. In particular, right or left strongly regular near-rings are strongly reduced.

(2) Every integral near-ring $N$ is strongly reduced. Hence a subdirect product of integral near-rings is strongly reduced.

Proof. (1) Note that the constant part $N_c$ is an invariant subnear-ring of $N$. Suppose $N$ is a quasi left strongly regular near-ring. Then $a \in \langle a^2 \rangle$ for each $a \in N$. If $a^2 \in N_c$ then $a \in \langle a^2 \rangle \subseteq N_c$. Hence $N$ is strongly reduced. Similarly, all quasi right strongly regular near-rings are strongly reduced.

(2) Let $a \in N$ with $a^2 \in N_c$. Then $(a - a^2)a = 0$, and hence $a = a^2 \in N_c$.

**Proposition 2.** If $N$ is a unital quasi left strongly regular near-ring, then every completely prime ideal is maximal.

Proof. Let $P$ be a completely prime ideal which is not maximal, so suppose that $P \subseteq M$ for some maximal $M$. Let $a \in M \setminus P$. Since $N$ is quasi left strongly regular, we see that $a = a^2$ or $a = xa^2$ for some $x \in N$. Then $0 = (1-a)a$ or $0 = (1- xa)a$. Since $P$ is completely prime, $1-a \in P \subseteq M$ or $1- xa \in P \subseteq M$. In any case, $1 \in M$, this is a contradiction.

From now on, we consider on strongly reduced near-rings and left strongly regular near-rings. Now, we state some basic and useful properties of a strongly reduced near-ring.

**Proposition 3.** Let $N$ be a strongly reduced near-ring and let $a, b \in N$. Then we have the following properties.

(1) $N$ is reduced.
(2) If \(ab^n \in N_c\) for any positive integer \(n\), then \(\{ab, ba\} \cup aN_b \cup bN_a \subseteq N_c\). In particular, \(ab \in N_c\) implies \(ba \in N_c\), \(aN_b \subseteq N_c\) and \(bN_a \subseteq N_c\).

(3) If \(ab^n = 0\) for any positive integer \(n\), then \(ab = 0\) and \(ba = b0\). In particular, \(ab = 0\) implies \(ba = b0\), that is, \(N\) has condition (i) of Reddy and Murty's property (*).

Proof. (1) Assume that \(a^2 = 0\). Then \(a^2 \in N_c\), hence \(a \in N_c\). Then we see \(a = a0 = a0a = aa = 0\).

(2) First, suppose \(ab \in N_c\). Then \((ba)^2 = babab = bab0a = bab0 \in N_c\). Since \(N\) is strongly reduced, we have \(ba \in N_c\). Then we obtain \(xba \in N_c\) for each \(x \in N\), whence \((axb)^2 \in N_c\). By the strong reducibility of \(N\), we obtain \(axb \in N_c\) for each \(a \in N\). Since \(ba \in N_c\), we also obtain \(bNa \subseteq N_c\). Now suppose \(ab^n \in N_c\). Then \((ba)^n \in N_c\) by the above argument. Since \(N\) is strongly reduced, this implies \(ab \in N_c\). Hence by the first paragraph, the claim is proved.

(3) If \(ab^n = 0\) for some \(n \geq 1\), then \(ab \in N_c\) by (2). Hence \(ab = ab^{n-1} = ab^n = 0\). Then \((ba)^2 = babab = b0 \in N_c\). Hence \(ba \in N_c\). Therefore \((ba)^2 - ba \in N_c\). Then \((ba)^2 - ba = (ba)^2 - ba\) is \(babab - bab = b0 - b0 = 0\). Hence we obtain \(ba = (ba)^2 = b0\). \(\square\)

Clearly, if \(N\) is a zero-symmetric near-ring, then \(N\) is strongly reduced if and only if \(N\) is reduced. The following example shows that, in general, a reduced near-ring is not necessarily strongly reduced.

**Example 1.** Let \(N = \{0, 1, 2, 3, 4, 5\}\) be an additive group of integers modulo 6 and multiplication as follows (see Pilz [7] for near-rings of low order; \(\mathbb{Z}_6\) No. 32):

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Clearly, this near-ring \(N\) is reduced. The constant part of \(N\) is \(\{0, 2, 4\}\). We see that this near-ring \(N\) is not strongly reduced, because \(1^2 = 4\) is a constant element but 1 is not a constant element. On the other hand, this near-ring \(N\) is an example of \(\pi\)-regular but not a regular near-ring.

**Example 2.** Let \(V = \{0, a, b, c\}\) be Klein's four group under addition.
(1) We define multiplication as follows (see Pilz [7] near-rings of low order; V No. 20):

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & 0 & a & b & c \\
c & a & 0 & c & b \\
\end{array}
\]

The constant part of this near-ring is \( \{0, a\} \). Clearly, this near-ring is reduced and strongly reduced.

(2) We have multiplication table as follows (see Pilz [7] near-rings of low order; V No. 19):

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & 0 & 0 & 0 & b \\
c & a & a & a & c \\
\end{array}
\]

The constant part of this near-ring is \( \{0, a\} \). Obviously, this near-ring is not reduced, for \( b^2 = 0 \); and it is also not strongly reduced.

Now we consider polynomial near-rings over commutative unital rings and polynomial near-rings on groups (cf. Lausch & Nöbauer [3, §8.11 and §9.11], Pilz [7, §7.61]). Let \( R \) be a commutative ring with unity 1, \( G \) an additive group, \( x \) an indeterminate variable, \( R[x] \) the set of all polynomials over \( R \) and

\[
G[x] = \{ a_0 + n_1 x + a_1 + n_2 x + a_2 + \cdots + a_{t-1} + n_t x + a_t \ |
\]

\[ t \in \mathbb{N}_0, a_i \in G, n_i \in \mathbb{Z}^* \text{ and } a_1 \neq 0, a_2 \neq 0, \ldots, a_{t-1} \neq 0 \}.
\]

Then \( (R[x], +, \circ) \) and \( (G[x], +, \circ) \) are near-rings with unity \( x \) respectively, where \( \circ \) is substitution. In this case, we say that \( R[x] \) is a polynomial near-ring over \( R \) and \( G[x] \) is a polynomial near-ring on \( G \). We see that

\[
(R[x])_c = R \text{ and } (R[x])_0 = \left\{ \sum_{i=1}^{n} a_i x^i \mid i \in \mathbb{Z}^+ \right\},
\]

so that \( R[x] = (R[x])_c + (R[x])_0 \).

Next, for any \( f(x) \in R[x] \), the map \( f : R \rightarrow R \) given by \( a \mapsto f(x) \circ a = f(a) \) is called the polynomial function induced by \( f(x) \). We let \( P(R) = \{ f \mid f(x) \in R[x] \} \) be the set of all polynomial functions on \( R \). Similarly, one can define \( f \) for \( f(x) \in G[x] \) and let \( P(G) \) be the set of all polynomial functions on \( G \). It is well known that
$P(R)$ and $P(G)$ are subnear-rings of $M(R)$ (resp. $M(G)$), and they are called the near-rings of polynomial functions on $R$ (resp. on $G$) (cf. Pilz [7, §7.65 and §7.66]).

**Example 3.** Consider the group $(\mathbb{Z}_2, +)$ and the commutative ring $(\mathbb{Z}_2, +, \cdot)$. The two kinds of near-rings (see Pilz [7] for near-rings of low order; $\mathbb{Z}_2$ No. 2 and $\mathbb{Z}_2$ No. 3) on a group $(\mathbb{Z}_2, +)$ are strongly reduced, and $\mathbb{Z}_2[x]$ and $P(\mathbb{Z}_2) = \{0, 1, x, x+1\}$ are strongly reduced.

**Example 4.** The four kinds of near-rings (see Pilz [7] for near-rings of low order; $\mathbb{Z}_4$ No. 8, $\mathbb{Z}_4$ No. 9, $\mathbb{Z}_4$ No. 10 and $\mathbb{Z}_4$ No. 11) on a group $(\mathbb{Z}_4, +)$ are strongly reduced. However, $\mathbb{Z}_4[x]$ and $P(\mathbb{Z}_4) = \{0, 1, x, 2x, \cdots\}$ are not strongly reduced.

We give equivalent conditions for a near-ring $N$ to be strongly reduced.

**Theorem 1.** The following statements are equivalent for a near-ring $N$:

1. $N$ is strongly reduced.
2. For $a \in N$, $a^2 = a^2$ implies $a^2 = a$, that is, $N$ has condition (ii) of Reddy and Murty's property (*).
3. If $a^{n+1} = xa^{n+1}$ for $a, x \in N$ and some nonnegative integer $n$, then $a = xa = ax$.

**Proof.** (1) $\Rightarrow$ (3). Suppose $a^{n+1} = xa^{n+1}$ for some $n \geq 0$. We will show $a = xa = ax$. If $n = 0$, then immediately $a = xa$. Now $(a - ax)a = a^2 - axa = a^2 - a^2 = 0 \in N_c$. Hence $(a - ax)^2 = a(a - ax) - ax(a - ax) \in N_c$ by property (2) of Proposition 3, and so $a - ax \in N_c$. Therefore $a - ax = (a - ax)a = 0$. If $n \geq 1$, then $(a - xa)a^n = 0$. Hence $(a - xa)a = 0$ by property (3) of Proposition 3, and so $(a - xa)^2 \in N_c$ by property (2) of Proposition 3. Since $N$ is strongly reduced, we have $a - xa \in N_c$. Then $a - xa = (a - xa)a = 0$, that is $a = xa$. Obviously as above $a = ax$.

(3) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (1). Assume $a^2 \in N_c$. Then $a^3 = a^2a = a^2$. By condition (2), this implies $a = a^2 \in N_c$.

Left strongly regular near-rings has been studied by several authors (cf. Lausch & Nöbauer [3], Mason [4, 5], Murty [6], Reddy & Murty [8], etc.) Since all left strongly regular near-rings are strongly reduced, the following is a generalization of Reddy & Murty [8, Theorem 3].

**Lemma 2.** Let $N$ be a strongly reduced near-ring and let $a, x \in N$. If $a^n = xa^{n+1}$ for some positive integer $n$, then $a = xa^2 = axa$ and $ax = xa$. 
Proof. Assume that $a^n = xa^{n+1}$ for some $n \geq 1$. By condition (3) of Theorem 1, $a = xa^2 = axa$. Then $(ax - xa)a = 0$. Hence, by property (2) of Proposition 3, $(ax - xa)^2 = ax(ax - xa) - xa(ax - xa) \in N_e$. Since $N$ is strongly reduced, $ax - xa \in N_e$. Hence $ax - xa = (ax - xa)a = 0$. 

A near-ring $N$ is said to be left strongly $\pi$-regular if, for each $a \in N$, there exists a positive integer $n$ and an element $x \in N$ such that $a^n = xa^{n+1}$. This equation is equivalent to $a^n = ya^{2n}$, for some $y \in N$. Here we give some characterizations of left strongly regular near-rings.

Theorem 2. Let $N$ be a near-ring. Then the following statements are equivalent:

1. $N$ is left strongly regular.
2. $N$ is strongly reduced and left strongly $\pi$-regular.
3. For each $a \in N$, there exists $x, y \in N$ such that $a = xa^2ya$.
4. For each $a \in N$, $a \in \langle a^2 \rangle \cap aNa$.

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) follow easily from property (1) of Proposition 1 and Lemma 2.

(3) $\Rightarrow$ (1). The hypothesis implies $N$ is strongly reduced. If $a = xa^2ya$, then

$ya = yxa^2(ya)$. By Theorem 1, $ya = yayxa^2$. Thus $a = xa^2yayxa^2$. This implies that $N$ is left strongly regular.

(4) $\Rightarrow$ (3). Since $a \in \langle a^2 \rangle$ for each $a \in N$, $N$ is strongly reduced by an argument similar to that in the proof for property (1) of Proposition 1. Hence $N$ satisfies (3) in Theorem 1. Since $a \in aNa$, there exists $x \in N$ such that $a = axa$. Hence $a = (ax)a = a(ax) = a^2x$. Then we have $a = axa = (a^2x)xa = a^2x^2a = a^2x^2a^2x^2a$. (3) holds.

A near-ring is said to be periodic if, for each $a \in N$, there exist distinct positive integers $m, n$ such that $a^m = a^n$. A near-ring $N$ is called a $(P_0)$-near-ring if, for each $a \in N$, there exists an integer $n \geq 1$ such that $a = a^n$ (see [7, §9.4, p. 289]). Obviously a $(P_0)$-near-ring is strongly reduced. Hence the proof of the following corollary follows directly from Lemma 2.

Corollary 1. Let $N$ be a near-ring. Then the following statements are equivalent:

1. $N$ is periodic and strongly reduced.
2. $N$ is a $(P_0)$-near-ring.

As a special case of this corollary, we have
Corollary 2. Let $N$ be a finite near-ring. Then the following statements are equivalent:

1. $N$ is strongly reduced.
2. $N$ is left strongly regular.
3. $N$ is a $(P_0)$-near-ring.

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References


Department of Mathematics, Silla University, San 1-1, Gwaebop-dong, Sasang-gu, Busan 617-736, Korea
Email address: yucho@silla.ac.kr