ESSENTIAL SPECTRA OF \( w \)-HYPONORMAL OPERATORS

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Abstract. Let \( K \) be the extension Hilbert space of a Hilbert space \( H \) and let \( \phi \) be the faithful *-representation of \( B(H) \) on \( K \). In this paper, we show that if \( T \) is an irreducible \( w \)-hyponormal operators such that \( \ker(T) \subset \ker(T^*) \) and \( T^*T - TT^* \) is compact, then \( \sigma_e(T) = \sigma_e(\phi(T)) \).

1. Introduction

Let \( H \) be a complex Hilbert space. The *-algebra of all bounded linear operators on \( H \) is denoted by \( B(H) \). For an operator \( T \) in \( B(H) \), we denote the spectrum, the point spectrum, the approximate point spectrum and the essential spectrum by \( \sigma(T) \), \( \sigma_p(T) \), \( \sigma_{ap}(T) \), and \( \sigma_e(T) \), respectively. A complex number \( z \) is a normal approximate propervalue of \( T \) if there exists a sequence \( \{x_n\} \) of unit vectors such that \( (T - z)x_n \to 0 \) and \( (T - z)^*x_n \to 0 \). The set of all normal approximate propervalue is called the normal approximate spectrum of \( T \) and it denote by \( \sigma_{na}(T) \).

Aluthge [1] first introduced \( p \)-hyponormality for operators; An operator \( T \) is said to be \( p \)-hyponormal for \( p \in (0, 1] \) if \( (T^*T)^p \geq (TT^*)^p \). If \( p = 1 \), \( T \) is called hyponormal and if \( p = \frac{1}{2} \), \( T \) is called semi-hyponormal. It is well known that a \( p \)-hyponormal operator is a \( q \)-hyponormal operator for \( 0 < q \leq p \) by the Löwner-Heinz theorem.

Let \( T = U|T| \) be the polar decomposition of \( T \), where \( U \) is a partial isometry, \( |T| \) is a positive square root of \( T^*T \) and \( \ker T = \ker |T| = \ker U \). Aluthge [1] introduced the operator \( \overline{T} = |T|^{1/2}U|T|^{1/2} \), which is called the Aluthge transformation of \( T \).

Received by the editors June 27, 2003 and, in revised form, November 12, 2003.
2000 Mathematics Subject Classification. 47B20, 47A10.
Key words and phrases. \( w \)-hyponormal, approximate point spectrum, essential spectrum, irreducible operator.
This paper was supported by research fund of Hanyang University, Seoul, Korea, 2002.

Aluthge & Wang [2] first introduced a new operator that an operator $T$ is said to be $w$-hyponormal if $|\bar{T}| \geq |T| \geq |T^*|$. Evidently, if $T$ is $w$-hyponormal, then $\bar{T}$ is semi-hyponormal.

They proved that if an operator $T$ is $p$-hyponormal, then it is $w$-hyponormal and also show the following results:

**Theorem 1.1** (Aluthge & Wang [4]).

1. An operator $T$ is $w$-hyponormal if and only if
   \[
   |T| \geq (|T^*|^p |T|^{q/2})^{1/a} \quad \text{and} \quad |T^*| \leq (|T^*|^p |T|^{q/2})^{1/b}.
   \]

2. If $T$ is a $w$-hyponormal operator, then $\sigma_{ap}(T) - \{0\} = \sigma_{na}(T) - \{0\}$.

Fujii, Jung, S. H. Lee, M. Y. Lee & Nakamoto [11] introduced a new class $A(p, q)$ of operators that for $p, q > 0$, an operator $T$ belongs to $A(p, q)$ if it satisfies an operator inequality

\[
(|T^*|^q |T|^{2p} |T^*|^q)^{\frac{1}{p+q}} \geq |T^*|^{2q}.
\]

Recently, Ito & Yamazaki [12] introduced a new class $wA(p, q)$ of operators that for $p, q > 0$, an operator $T$ belongs to $wA(p, q)$ if it satisfies an operator inequalities

\[
(|T^*|^q |T|^{2p} |T^*|^q)^{\frac{1}{p+q}} \geq |T^*|^{2q} \quad \text{and} \quad |T|^{2p} \geq (|T|^p |T^*|^{2q} |T|^p)^{\frac{p}{p+q}}.
\]

In Ito & Yamazaki [12], they obtained the following results:

**Theorem 1.2** (Ito & Yamazaki [12]). For each $p > 0$ and $q > 0$, the following assertions hold:

1. Class $A(p, q)$ coincides with class $wA(p, q)$.
2. Class $A(\frac{1}{2}, \frac{1}{2})$ coincides with the class of $w$-hyponormal operators (i.e., class $wA(\frac{1}{2}, \frac{1}{2})$).

An operator $T$ is said to be reducible if it has a nontrivial reducing subspace. If an operator is not reducible, then it is called irreducible.

Cha [6] constructed an extension $K$ of $\mathcal{H}$ by means of all weakly convergent sequences in $\mathcal{H}$ and the Banach Limit, and obtained the faithful $*$-representation $\phi$ of $B(\mathcal{H})$ on $K$.

In this paper, using the faithful $*$-representation $\phi$, for an irreducible $w$-hyponormal operator $T$ with $\ker(T) \subset \ker(T^*)$, we investigate the relation between the essential spectrum of $T$ and the essential spectrum of $\phi(T)$.
2. The main Theorem

Let $C^*(T)$ be the $C^*$-subalgebra of $B(\mathcal{H})$ generated by a single operator $T$ and identity. By a character on a $C^*$-algebra we mean a multiplicative linear functional. If $\mathcal{A}$ is a $C^*$-algebra with identity, then its commutator ideal is the closed ideal generated by the commutator $ab - ba$ for $a, b \in \mathcal{A}$.

Bunce [5] established a kind of the reciprocity among the character of single generated $C^*$-algebra and the approximate spectra of the generators and he proved the following theorem:

**Theorem 2.1** (Bunce [5]). If $T$ is a hyponormal operator, then for all $\lambda \in \sigma_{ap}(T)$ there is a character $\psi$ on the $C^*$-algebra $C^*(T)$ such that $\psi(T) = \lambda$.

Enomoto, Fujii & Tamaki [10] was generalized Bunce's result as following:

**Theorem 2.2** (Enomoto, Fujii & Tamaki [10]). A complex number $\lambda \in \sigma_{na}(T)$ if and only if there is a character $\psi$ of $C^*(T)$ such that $\psi(T) = \lambda$.

Let $C^*(T_i : i \in \Gamma)$ be the $C^*$-algebra generated by $\{T_i : i \in \Gamma\}$ and the identity operator, and let $\mathcal{I}$ be the commutator ideal of $C^*(T_i : i \in \Gamma)$.

S. G. Lee [13] obtained that the quotient algebra $C^*(T_i : i \in \Gamma)/\mathcal{I}$ is isometrically *-isomorphic to $C(\sigma_n(T_i : i \in \Gamma))$, where $\sigma_n(T_i : i \in \Gamma)$ is the joint normal spectrum of $\{T_i : i \in \Gamma\}$.

By Theorem 1.1 and Theorem 2.2, we have the following result:

**Corollary 2.3.** Let $T$ be a w-hyponormal operator with $\ker(T) \subset \ker(T^*)$. Then $\lambda \in \sigma_{ap}(T)$ if and only if there is a character $\psi$ of $C^*(T)$ such that $\psi(T) = \lambda$.

If $\Phi_\mathcal{A}$ is the set of all character on $\mathcal{A}$ and $M$ is the commutator ideal of $\mathcal{A}$, then $M = \bigcap\{h^{-1}(0) : h \in \Phi_\mathcal{A}\}$ and $\Phi_\mathcal{A}$ is the maximal ideal space of $\mathcal{A}/M$. With this statement, we have $\mathcal{A}/M \cong C(\Phi_\mathcal{A})$ under the Gel'fand transform, $a + M \rightarrow \hat{a}$, where $\hat{a}(h) = h(a)$ for $a$ in $\mathcal{A}$ and $h$ in $\Phi_\mathcal{A}$ (Conway [8, 9]).

**Lemma 2.4.** If an operator $T$ is w-hyponormal with $\ker(T) \subset \ker(T^*)$, there is an isometric *-isomorphism of $C^*(T)/M$ onto $C(\sigma_{ap}(T))$, where $A + M$ is mapped to the function $z$.

**Proof.** Let $\tau : \Phi_{C^*(T)} \rightarrow \sigma_{ap}(T)$ be defined by $\tau(\psi) = \psi(T)$. By Corollary 2.3 this map is surjective. If $\psi(T) = \psi'(T)$ for $\psi, \psi' \in \Phi_{C^*(T)}$, then $\psi = \psi'$, since $T$ is
generator of $C^*(T)$, and $\psi, \psi'$ are continuous on $C^*(T)$. So $\tau$ is injective. It is also easy to see that $\tau$ is continuous. Since $\Phi_{C^*(T)}$ is compact, $\tau$ is a homeomorphism. Thus the map $\tau^\#: C(\sigma_{ap}(T)) \to C(\Phi_{C^*(T)})$ defined by $\tau^\#(f) = f \circ \tau$ is an isometric $*$-isomorphism. We define a map $\rho : C(\sigma_{ap}(T)) \to C^*(T)/M$ so that the following diagram commutes:

$$
\begin{array}{ccc}
C^*(T)/M & \xrightarrow{\gamma} & C(\Phi_{C^*(T)}) \\
\downarrow{\rho} & & \downarrow{\tau^\#} \\
C(\sigma_{ap}(T))
\end{array}
$$

where the Gel'fand transform $\gamma : C^*(T)/M \to C(\Phi_{C^*(T)})$ is an isometric $*$-isomorphism. Then $\rho$ is clearly an isometric $*$-isomorphism. □

Cha [6] introduced an extension $K$ of $H$ by means of all weakly convergent sequences in $H$ and the Banach Limit, and obtained the faithful $*$-representation $\phi$ of $B(H)$ on $K$.

In order to show our results, we use the following propositions.

**Proposition 2.5** (Cha [6, 7]). There exists a faithful $*$-representation $\phi$ of $B(H)$ on $K$ with the following properties:

1. $\|\phi(T)\| = \|T\|$.  
2. $\sigma(T) = \sigma(\phi(T))$.  
3. $\sigma_{ap}(T) = \sigma_p(\phi(T))$.
4. If $T$ is a compact operator on $H$, then so is $\phi(T)$ on $K$.
5. If $T$ is an irreducible operator on $H$, then so is $\phi(T)$ on $K$.

**Remark.** The Proposition 2.5 (5) does not mean a representation of a $C^*$-algebra is irreducible. It implies the concept of a simple irreducible operators.

**Proposition 2.6** (Cha [7]). We have the following properties.

1. The $C^*$-algebra $C^*(T)$ is isometrically $*$-isomorphic to the $C^*$-algebra $C^*(\phi(T))$.
2. If $M$ is the maximal ideal of $C^*(T)$, then $\phi(M)$ is the maximal ideal of $C^*(\phi(T))$.
3. Let $\Phi_{C^*(T)}$ and $\Phi_{C^*(\phi(T))}$ be the maximal ideal space of $C^*(T)$ and $C^*(\phi(T))$, respectively. Then $\Phi_{C^*(T)}$ and $\Phi_{C^*(\phi(T))}$ are isometrically isomorphic.

**Proposition 2.7** (Cha [7]). We have the following properties.

1. $M = \bigcap\{f^{-1}(0) : f \in \Phi_{C^*(T)}\} \cong N = \bigcap\{h^{-1}(0) : h \in \Phi_{C^*(\phi(T))}\}$.
2. $C^*(T)/M \cong C^*(\phi(T))/N$. 


Related to above propositions, we obtained the following results for \( w \)-hyponormal operators.

To show this property, we need the following proposition:

**Proposition 2.8.** If an operator \( T \) is \( w \)-hyponormal, then so is \( \phi(T) \).

**Proof.** Since operators in \( A(\frac{1}{2}, \frac{3}{2}) \) is \( w \)-hyponormal, we need only to show that

\[
(|\phi(T)^*|^{1/2}|\phi(T)||\phi(T)^*|^{1/2})^{1/2} \geq |\phi(T)^*|.
\]

It is easily check that \( |\phi(T)| = \phi(|T|) \) and \( \phi \) preserves the positive property. Thus we have

\[
|\phi(T)^*| = \phi(|T^*|) \\
\leq \phi((|T|^*|^{1/2}|T||T^*|^{1/2})^{1/2}) \\
= (|\phi(T)^*|^{1/2}|\phi(T)||\phi(T)^*|^{1/2})^{1/2}.
\]

Therefore,

\[
(|\phi(T)^*|^{1/2}|\phi(T)||\phi(T)^*|^{1/2})^{1/2} \geq |\phi(T)^*|.
\]

Thus, \( \phi(T) \) is \( w \)-hyponormal. \( \square \)

With the notation of Proposition 2.5, Proposition 2.7 and Lemma 2.4, we have the following:

**Theorem 2.9.** If \( T \) is a \( w \)-hyponormal operator with \( \ker(T) \subset \ker(T^*) \), then

\[
C^*(T)/M \cong C^*(\phi(T))/N \cong C(\sigma_p(\phi(T))).
\]

We need the following proposition in order to give proofs of Theorem 2.11 and Corollary 2.12.

**Proposition 2.10** (Conway [9]). If \( T \) is an irreducible operator such that \( T^*T - TT^* \) is compact, then the commutator ideal \( M \) of \( C^*(T) \) is \( K(\mathcal{H}) \), where \( K(\mathcal{H}) \) is the ideal of all compact operators on \( \mathcal{H} \).

We have the results for irreducible \( w \)-hyponormal operators.

**Theorem 2.11.** If \( T \) is an irreducible \( w \)-hyponormal operators such that \( \ker(T) \subset \ker(T^*) \) and \( T^*T - TT^* \) is compact, then

\[
\sigma_{ap}(T) = \sigma_e(T) \text{ and } \sigma_p(\phi(T)) = \sigma_e(\phi(T)).
\]
Proof. The fact that $\sigma_{ap}(T) = \sigma_e(T)$ follows immediately from Proposition 2.10 and Lemma 2.4. The second assertion is clear from Proposition 2.10 and Proposition 2.5. \hfill \Box

It is easy to see that if $T$ is a Fredholm operator on $\mathcal{H}$, then so is $\phi(T)$ on $\mathcal{K}$, and so $\sigma_e(\phi(T)) \subset \sigma_e(T)$ for any operator $T$.

Corollary 2.12. If $T$ is an irreducible $w$-hyponormal operators such that $\ker(T) \subset \ker(T^*)$ and $T^*T - TT^*$ is compact, then $\sigma_e(T) = \sigma_e(\phi(T))$.

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