ON THE FEKETE-SZEGÖ PROBLEM FOR CERTAIN ANALYTIC FUNCTIONS

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Abstract. Let $\mathcal{CS}_\alpha(\beta)$ denote the class of normalized strongly $\alpha$-close-to-convex functions of order $\beta$, defined in the open unit disk $\mathcal{U}$ of $\mathbb{C}$ by

\[ \left| \arg \left\{ \frac{(1 - \alpha) f(z)}{g(z)} + \alpha \frac{zf'(z)}{g(z)} \right\} \right| \leq \frac{\pi}{2} \beta \quad (\alpha, \beta \geq 0), \]

such that $g \in \mathcal{S}^*$, the class of normalized starlike functions. In this paper, we obtain the sharp Fekete-Szegö inequalities for functions belonging to $\mathcal{CS}_\alpha(\beta)$.

1. Introduction

Let $S$ denote the class of analytic functions $f$ of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]  \hspace{1cm} (1.1)

which are univalent in the open unit disk $\mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \}$ & let $S^*$ be the subclass of $S$ consisting of all starlike functions. A classical theorem of Fekete and Szegö [4] states that, for $f \in S$ given by (1.1),

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq 0, \\ 1 + 2e^{-2\mu/(1-\mu)} & \text{if } 0 \leq \mu < 1, \\ 4\mu - 3 & \text{if } \mu \geq 1, \end{cases} \]

The inequality is sharp in the sense that for each $\mu$, there exists a function in $S$ such that equality holds. There are also several results of this type in the literature. Various interesting developments involving the Fekete-Szegö problem can be found in Abdel-Gawad & Thomas [1], Keogh & Merkes [7] and London [8].

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We denote by $\mathcal{K}(\beta)$ the class of strongly close-to-convex functions of order $\beta$. Thus $f \in \mathcal{K}(\beta)$ if and only if there exists $g \in S^*$ such that

$$\left| \arg \frac{zf'(z)}{g(z)} \right| \leq \frac{\pi}{2} \beta \quad (\beta \geq 0; \ z \in \mathcal{U}).$$

For $0 \leq \beta \leq 1$, the class $\mathcal{K}(\beta)$ is a subclass of close-to-convex functions introduced by Kaplan [6] and hence contains only univalent functions. However, Goodman [5] showed that $\mathcal{K}(\beta)$ can contain functions with infinite valence for $\beta > 1$. The Fekete-Szegő problems for $\mathcal{K}(1)$ and $\mathcal{K}(\beta)$, respectively, have been solved by Keogh & Merkes [7] and London [8], respectively.

We now introduce a new class which covers the class $\mathcal{K}(\beta)$ as follows:

**Definition.** A function $f \in S$, given by (1.1) is said to be strongly $\alpha$-close-to-convex of order $\beta$ if there exists a function $g \in S^*$ such that

$$\left| \arg \left\{ (1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{zf'(z)}{g(z)} \right\} \right| \leq \frac{\pi}{2} \beta \quad (\alpha, \beta \geq 0; \ z \in \mathcal{U}). \quad (1.2)$$

We denote by $\mathcal{CS}_\alpha(\beta)$ the class of strongly $\alpha$-close-to-convex functions of order $\beta$. We note that $\mathcal{CS}_0(1) = \mathcal{CS}$, the class of close-to-star functions introduced by Reade [10] and $\mathcal{CS}_1(\beta) = \mathcal{K}(\beta)$.

The purpose of the present paper is to prove the sharp Fekete-Szegő inequalities for the functions belonging to the class $\mathcal{CS}_\alpha(\beta)$.

2. **Main Results**

**Theorem.** Let $f \in \mathcal{CS}_\alpha(\beta)$ and be given by (1.1). Then for $\alpha, \beta \geq 0$, we have

$$2(1 + 2\alpha)|a_3 - \mu a_2^2| \leq \begin{cases} 1 + \frac{2(1 + \beta)(1 + \alpha)^2 - 2(1 + 2\alpha)\mu}{(1 + \alpha)^2} & \text{if } \mu \leq \frac{\beta(1 + \alpha)^2}{2(1 + \beta)(1 + 2\alpha)}, \\ 1 + 2\beta + \frac{2((1 + \alpha)^2 - 2(1 + 2\alpha)\mu)}{(1 + \alpha)^2 - \beta((1 + \alpha)^2 - 2(1 + 2\alpha)\mu)} & \text{if } \frac{\beta(1 + \alpha)^2}{2(1 + \beta)(1 + 2\alpha)} \leq \mu \leq \frac{(1 + \alpha)^2}{2(1 + 2\alpha)}, \\ 1 + 2\beta & \text{if } \frac{(1 + \alpha)^2}{2(1 + 2\alpha)} \leq \mu \leq \frac{(\beta + 2)(1 + \alpha)^2}{2(\beta + 1)(1 + 2\alpha)}, \\ -1 + \frac{2(1 + \beta)^2(2(1 + 2\alpha)\mu - (1 + \alpha)^2)}{(1 + \alpha)^2} & \text{if } \mu \geq \frac{(\beta + 2)(1 + \alpha)^2}{2(\beta + 1)(1 + 2\alpha)}. \end{cases}$$

For each $\mu$, there is a function in $\mathcal{CS}_\alpha(\beta)$ such that equality holds in all cases.
To prove above Theorem, we need the following.

**Lemma.** Let \( p \) be analytic in \( \mathcal{U} \) and satisfying \( \Re p(z) > 0 \) for \( z \in \mathcal{U} \), with \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \). Then
\[
| p_n | \leq 2 \tag{2.1}
\]
and
\[
| p_2 - \frac{p_1^2}{2} | \leq 2 - \frac{|p_1|^2}{2}. \tag{2.2}
\]

The inequality (2.1) can be first proved by Carathéodory [2] (also, see Duren [3], p. 41) and the inequality (2.2) can be found in [Pommerenke [9], p. 166].

**Proof of Theorem.** Let \( f \in CS_\alpha(\beta) \). Then it follows from (1.2) that we may write
\[
(1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{zf'(z)}{g(z)} = p^\beta(z), \tag{2.3}
\]
where \( g \) is starlike and \( p \) has positive real part. Let \( g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \), and let \( p(z) \) be given as in Lemma. Then by equating coefficients of both side of (2.3), we obtain
\[
(1 + \alpha)a_2 = b_2 + \beta p_1
\]
and
\[
(1 + 2\alpha)a_3 = b_3 + \beta p_1 b_2 + \frac{\beta(\beta - 1)}{2} p_1^2 + \beta p_2.
\]
So, with
\[
x = \frac{(1 + \alpha)^2 - 2(1 + 2\alpha)\mu}{(1 + \alpha)^2},
\]
we have
\[
(1 + 2\alpha)(a_3 - \mu a_3^2) = b_3 + \frac{1}{2} (x - 1)b_2^2 + \beta(p_2 + \frac{1}{2}(\beta x - 1)p_1^2) + \beta x p_1 b_2. \tag{2.4}
\]
Since rotations of \( f \) also belong to \( CS_\alpha(\beta) \), without loss of generality, we may assume that \( a_3 - \mu a_3^2 \) is positive. Thus we now estimate \( \Re (a_3 - \mu a_3^2) \).

Since \( g \in \mathcal{S}^* \), there exists \( h(z) = 1 + k_1 z + k_2 z^2 + \cdots \) \((|z| < 1)\) with positive real part such that \( zg'(z) = g(z)h(z) \), and so equating coefficients, we have \( b_2 = k_1 \) and \( b_3 = (k_2 + k_1^2)/2 \). Hence, by Lemma,
\[
\Re \left( b_3 + \frac{1}{2}(x - 1)b_2^2 \right) = \frac{1}{2} \Re \left( k_2 - \frac{1}{2}k_1^2 \right) + \frac{1 + 2x}{4} \Re k_1^2 \leq 1 - \rho^2 + (1 + 2x)\rho^2 \cos 2\phi, \tag{2.5}
\]
where \( b_2 = k_1 = 2\rho e^{i\phi} \) for some \( \rho \) in \([0,1]\). We also have
\[
\text{Re} \left( p_2 + \frac{1}{2}(\beta x - 1)p_1^2 \right) = \text{Re} \left( p_2 - \frac{1}{2}p_1^2 \right) + \frac{1}{2}\beta x \text{Re} \ p_1^2 \\
\leq 2(1 - r^2) + 2\beta x r^2 \cos 2\theta, \tag{2.6}
\]
where \( p_1 = 2r e^{i\theta} \) for some \( r \) in \([0,1]\). From (2.4), (2.5) and (2.6), we obtain
\[
\text{Re} (1 + 2\alpha)(a_3 - \mu a_2) \\
\leq 1 - \rho^2 + (1 + 2x)\rho^2 \cos 2\phi + 2\beta(1 - r^2 + r^2 \beta x \cos 2\theta) + 4\beta x r \rho \cos (\theta + \phi), \tag{2.7}
\]
and we now proceed to maximize the right-hand side of (2.7). This function will be denoted \( \psi(x) \) whenever all parameters except \( x \) are held constant.

At first, we assume that
\[
\frac{\beta(1 + \alpha)^2}{2(1 + \beta)(1 + 2\alpha)} \leq \mu \leq \frac{(1 + \alpha)^2}{2(1 + 2\alpha)},
\]
so that \( 0 \leq x \leq 1/(1 + \beta) \). Since the expression \(-t^2 + t^2 \beta x \cos 2\theta + 2xt\) is the largest when \( t = x/(1 - \beta x \cos 2\theta) \), we have
\[
-t^2 + t^2 \beta x \cos 2\theta + 2xt \leq \frac{x^2}{1 - \beta x \cos 2\theta} \leq \frac{x^2}{1 - \beta x}.
\]
Thus
\[
\psi(x) \leq 1 + 2x + 2\beta \left( 1 + \frac{x^2}{1 - \beta x} \right) \\
= 1 + 2\beta + \frac{2((1 + \alpha)^2 - 2(1 + 2\alpha)\mu)}{(1 + \alpha)^2 - \beta((1 + \alpha)^2 - 2(1 + 2\alpha)\mu)}
\]
and with (2.7) this establishes the second inequality in the theorem. Equality occurs only if
\[
p_1 = \frac{2((1 + \alpha)^2 - 2(1 + 2\alpha)\mu)}{(1 + \alpha)^2 - \beta((1 + \alpha)^2 - 2(1 + 2\alpha)\mu)}, \quad p_2 = 2, \quad b_2 = 2, \quad b_3 = 3,
\]
and the corresponding function \( f \) is defined by
\[
(1 - \alpha)f(z) + \alpha zf'(z) = \frac{z}{(1 - z)^2} \left( \frac{1 + z}{1 - z} \right)^{\lambda} \left( \frac{1 + \lambda}{1 - z} + (1 - \lambda) \frac{1 - z}{1 + z} \right)^{\beta},
\]
where
\[
\lambda = \frac{(1 + \alpha)^2 + (1 - \beta)((1 + \alpha)^2 - 2(1 + 2\alpha)\mu)}{2((1 + \alpha)^2 - \beta((1 + \alpha)^2 - 2(1 + 2\alpha)\mu))}.
\]
We now prove the first inequity. Let
\[
\mu \leq \frac{\beta(1 + \alpha)^2}{2(\beta + 1)(1 + 2\alpha)},
\]
so that \( x \geq 1/(1 + \beta) \). With \( x_0 = 1/(1 + \beta) \), we have

\[
\psi(x) = \psi(x_0) + 2(x - x_0)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\rho \beta r \cos(\theta + \phi)) \\
\leq \psi(x_0) + 2(x - x_0)(1 + \beta)^2 \\
\leq 1 + \frac{2(1 + \beta)^2((1 + \alpha)^2 - 2(1 + 2\alpha)\mu)}{(1 + \alpha)^2},
\]

as required. Equality occurs only if \( p_1 = p_2 = 2, b_2 = 2, b_3 = 3 \), and the corresponding function \( f \) is defined by

\[
(1 - \alpha)f(z) + \alpha zf'(z) = \frac{z}{(1 - z)^2} \left( \frac{1 + z}{1 - z} \right)^\beta.
\]

Let \( x_1 = -1/(1 + \beta) \). We shall find that \( \psi(x_1) \leq 1 + 2\beta \), and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

\[
\psi(x) \leq \psi(x_1) + 2|x - x_1|(1 + \beta)^2 \\
\leq -1 + \frac{2(1 + \beta)^2(2(1 + 2\alpha)\mu - (1 + \alpha)^2)}{(1 + \alpha)^2},
\]

if \( x \leq x_1 \), that is,

\[
\mu \geq \frac{(\beta + 2)(1 + \alpha)^2}{2(\beta + 1)(1 + 2\alpha)}.
\]

Equality occurs only if \( p_1 = 2i, p_2 = -2, b_2 = 2i, b_3 = -3 \), and the corresponding function \( f \) is defined by

\[
(1 - \alpha)f(z) + \alpha zf'(z) = \frac{z}{(1 - iz)^2} \left( \frac{1 + iz}{1 - iz} \right)^\beta.
\]

Also, for \( 0 \leq \lambda \leq 1 \),

\[
\psi(\lambda x_1) = \lambda \psi(x_1) + (1 - \lambda)\psi(0) \\
\leq \lambda(1 + 2\beta) + (1 - \lambda)(1 + 2\beta) = 1 + 2\beta,
\]

so \( \psi(x) \leq 1 + 2\beta \) for \( x_1 \leq x \leq 0 \), i.e.,

\[
\frac{(1 + \alpha)^2}{2(1 + 2\alpha)} \leq \mu \leq \frac{(\beta + 2)(1 + \alpha)^2}{2(\beta + 1)(1 + 2\alpha)}.
\]

Equality occurs only if \( p_1 = b_2 = 0, p_2 = 2, b_3 = 1 \), and the corresponding function \( f \) is defined by

\[
(1 - \alpha)f(z) + \alpha zf'(z) = \frac{z(1 + z^2)^\beta}{(1 - z^2)^{1 + \beta}}.
\]
We now show that $\psi(x_1) \leq 1 + 2\beta$. Since

$$-(1 - \beta x_1 \cos 2\theta)t^2 + 2x_1 \rho \cos(\theta + \phi)t$$

$$= (1 - \beta x_1 \cos 2\theta) \left(t - \frac{x_1 \rho \cos(\theta + \phi)}{1 - \beta x_1 \cos 2\theta}\right)^2 + \frac{x_1^2 \rho^2 \cos^2(\theta + \phi)}{1 - \beta x_1 \cos 2\theta}$$

and

$$1 - \beta x_1 \cos 2\theta = 1 + \frac{\beta}{1 + \beta} \cos 2\theta \geq 1 - \frac{\beta}{1 + \beta} \geq 0,$$

we have

$$\psi(x_1) - (1 + 2\beta) \leq \rho^2 \left(-1 + (1 + 2x_1) \cos 2\phi + \frac{\beta x_1^2(1 + \cos 2(\theta + \phi))}{1 - \beta x_1 \cos 2\theta}\right).$$

Thus we consider the inequality

$$\beta x_1^2(1 + \cos 2(\theta + \phi)) + (1 - \beta x_1 \cos 2\theta)(-1 + (1 + 2x_1) \cos 2\phi) \leq 0.$$

After some simplifications, this becomes

$$\beta^2 (\cos 2\phi - 1)(\cos 2\theta + 1) - \beta(1 + \cos 2\theta + \sin 2\theta \sin 2\phi) - 1 - \cos 2\phi \leq 0,$$

which is true if

$$2\beta^2 \sin^2 \phi \cos^2 \phi + 2\beta \cos \theta \sin \theta \cos \phi \sin \phi + \cos^2 \phi \geq 0. \quad (2.8)$$

Now, for all real $t$,

$$2t^2 + 2t \sin \theta \cos \phi + \cos^2 \phi \geq 0,$$

so, by taking $t = \beta \sin \phi \cos \theta$, we obtain (2.8). This completes the proof of Theorem.

For the case $\alpha = 0$ in Theorem, we have the following. □

**Corollary.** Let $f \in C\mathcal{S}_0(\beta)$ and be given by (1.1). Then for $\beta \geq 0$, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
1 + 2(1 + \beta)^2(1 - 2\mu) & \text{if } \mu \leq \frac{\beta}{2(1 + \beta)}, \\
1 + 2\beta + \frac{2(1 - 2\mu)}{1 - \beta(1 - 2\mu)} & \text{if } \frac{\beta}{2(1 + \beta)} \leq \mu \leq \frac{1}{2}, \\
1 + 2\beta & \text{if } \frac{1}{2} \leq \mu \leq \frac{2 + \beta}{2(1 + \beta)}, \\
-1 + \frac{2(1 + \beta)^2(2(1 + 2\alpha)\mu - (1 + 3\alpha))}{(1 + \alpha)^2} & \text{if } \mu \geq \frac{2 + \beta}{2(1 + \beta)}.
\end{cases}$$

For each $\mu$, there is a function in $C\mathcal{S}_0(\beta)$ such that equality holds in all cases.
Remark. (i) Putting $\alpha = \beta = 1$ in Theorem, we have the result by Keogh & Merkes [7].

(ii) Taking $\alpha = 1$ in Theorem, we obtain the corresponding results by Abdel-Gawad & Thomas [1] and London [8].

REFERENCES


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