ON TWO-DIMENSIONAL LANDSBERG SPACE
WITH A SPECIAL \((\alpha, \beta)\)-METRIC

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ABSTRACT. In the present paper, we treat a Finsler space with a special \((\alpha, \beta)\)-
metric \(L(\alpha, \beta) = c_1\alpha + c_2\beta + \alpha^2/\beta\) satisfying some conditions. We find a condition
that a Finsler space with a special \((\alpha, \beta)\)-metric be a Berwald space. Then it is shown
that if a two-dimensional Finsler space with a special \((\alpha, \beta)\)-metric is a Landsberg
space, then it is a Berwald space.

1. INTRODUCTION

In the Cartan connection \(CT\), a Finsler space is called Landsberg space, if the
covariant derivative \(C_{hijk}\) of the \(C\)-torsion tensor \(C_{hi} = \partial_h\partial_i\partial_j(L^2/4)\) satisfies
\(C_{hijk}(x, y)y^k = 0\). A Berwald space is characterized by \(C_{hijk} = 0\). Berwald spaces
are specially interesting and important, because the connection is linear, and many
examples of a Berwald space have been known. But any concrete example of a
Landsberg space which is not a Berwald space is not known yet. If a Finsler space
is a Landsberg space and satisfies some additional conditions, then it is merely
a Berwald space (cf. Bácsó & Matsumoto [3]). On the other hand, in the two-
dimensional case, a general Finsler space is a Landsberg space, if and only if its
main scalar \(I(x, y)\) satisfies \(I_{ij}y^i = 0\) (cf. Matsumoto [6]).

The purpose of the present paper is to find a two-dimensional Landsberg space
with a special \((\alpha, \beta)\)-metric \(L(\alpha, \beta) = c_1\alpha + c_2\beta + \alpha^2/\beta\) satisfying some conditions,
where \(c_1, c_2\) are constants and \(c_1 \neq 0\). First we find the condition that a Finsler
space with a special \((\alpha, \beta)\)-metric be a Berwald space (see Theorem 3.1). Next we
determine the difference vector and the main scalar of \(F^2\) with the metric above.

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Finally we derive the condition that a two-dimensional Finsler space $F^2$ with a special $(\alpha, \beta)$-metric be a Landsberg space, and we show that if $F^2$ with the metric above is a Landsberg space, then it is a Berwald space (see Theorem 4.1).

2. Preliminaries

Let $F^n = (M^n, L(\alpha, \beta))$ be an $n$-dimensional Finsler space with an $(\alpha, \beta)$-metric and $R^n = (M^n, \alpha)$ the associated Riemannian space, where $\alpha^2 = a_{ij}(x)y^iy^j$, $\beta = b_i(x)y^i$. We put $(a_{ij}) = (a_{ij})^{-1}$.

The Riemannian metric $\alpha$ is not supposed to be positive-definite and we shall restrict our discussions to a domain of $(x, y)$ where $\beta$ does not vanish. The covariant differentiation in the Levi-Civita connection $(\gamma^i_jk(x))$ of $R^n$ is denoted by the semi-colon. Let us list the symbols here for the late use:

\[
\begin{align*}
2r_{ij} &= b_{ij} + b_{ji} & 2s_{ij} &= b_{ij} - b_{ji} & r^i_j &= a^i_i r_{ij} & s^i_j &= a^i_i s_{ij} \\
&= r^i_j, & & & = b^i_j, & & = b^i_j,
\end{align*}
\]

\[
L_\alpha = \partial L / \partial \alpha, \quad L_\beta = \partial L / \partial \beta, \quad L_{\alpha\alpha} = \partial^2 L / \partial \alpha \partial \beta \quad \text{and} \quad y_k = a_{kr}y^r.
\]

The Berwald connection $B\Gamma = (G^i_jk, G^i_j, 0)$ of $F^n$ plays one of the leading roles in the present paper. Denote by $B^i_jk$ the difference tensor Matsumoto [7] of $G^i_jk$ from $\gamma^i_jk$:

\[
G^i_jk(x, y) = \gamma^i_jk(x) + B^i_jk(x, y).
\]

With the subscript 0, the transvection by $y^i$, we have

\[
(2.1) \quad G^i_j0 = \gamma^i_j0 + B^i_j0,
\]

and then $B^i_j = \partial_j B^i_0$ and $B^i_jk = \partial_k B^i_j$. On account of Matsumoto [7], the Berwald connection $B\Gamma$ of a Finsler space with $(\alpha, \beta)$-metric $L(\alpha, \beta)$ is given by (2.1) and (2.2), where $B^i_jk$ are the components of a Finsler tensor of $(1, 2)$-type which is determined by

\[
(2.3) \quad L_\alpha B^i_jk y^i y_k = a L_\beta (b_{ij} - B^k_i b_k) y^i.
\]

According to Matsumoto [7], $B^i(x, y)$ is called the difference vector. If

\[
\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha} \neq 0,
\]

where $\gamma^2 = b^2 \alpha^2 - \beta^2$, then $B^i$ is written as follows:

\[
(2.4) \quad B^i = \frac{E}{\alpha} y^i + \frac{a L_\beta}{L_\alpha} s^i_0 - \frac{a L_{\alpha\alpha}}{L_\alpha} C_x \left( \frac{1}{\alpha} y^i - \frac{\alpha}{\beta} y^i \right),
\]
where

\[ E = \frac{\beta L_\beta}{L} C^* \quad \text{and} \quad C^* = \frac{\alpha \beta (r_0 L_\alpha - 2 \alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha^2 L_{\alpha\alpha})}. \]

Furthermore, by means of Hashiguchi, Hōjō & Matsumoto [4] we have

\[ \alpha_i = -\frac{L_\beta}{L_\alpha} \beta_\mu. \]

\[ \beta y^i = r_{00} - 2 b_r B^r. \]

\[ b_i^2 = 2(r_0 + s_0). \]

\[ \gamma^2 y^i = 2(r_0 + s_0) \alpha^2 - 2 \left( \frac{L_\beta}{L_\alpha} b_i^2 + \beta \right) (r_{00} - 2 b_r B^r). \]

The following Lemmas have been shown:

**Lemma 2.1** (Bácsó & Matsumoto [2]). If \( \alpha^2 \equiv 0 \pmod{\beta} \), that is, \( a_{ij}(x)y^i y^j \) contains \( b_i(x)y^i \) as a factor, then the dimension \( n \) is equal to two and \( b^2 \) vanishes. In this case we have \( \delta = d_i(x)y^i \) satisfying \( \alpha^2 = \beta \delta \) and \( d_i b^i = 2 \).

**Lemma 2.2** (Hashiguchi, Hōjō & Matsumoto [4]). We consider a two-dimensional case.

1. If \( b^2 \neq 0 \), then there exist a sign \( \epsilon = \pm 1 \) and \( \delta = d_i(x)y^i \) such that \( \alpha^2 = \beta^2/b_i^2 + \epsilon \delta^2 \) and \( d_i b^i = 0 \).

2. If \( b^2 = 0 \), then there exists \( \delta = d_i(x)y^i \) such that \( \alpha^2 = \beta \delta \) and \( d_i b^i = 2 \).

If there are two functions \( f(x) \) and \( g(x) \) satisfying \( f \alpha^2 + g \beta^2 = 0 \), then \( f = g = 0 \) is obvious, because \( f \neq 0 \) implies a contradiction \( \alpha^2 = (-g/f) \beta^2 \).

Throughout the paper, we shall say “homogeneous polynomial(s) in \((y^i)\) of degree \( r\)” as \( hp(r) \) for brevity. Thus \( \gamma_0 y^0 \) are \( hp(2) \).

### 3. Berwald Space

In the present section, we find the condition that a Finsler space \( F^n \) with a special \((\alpha, \beta)\)-metric be a Berwald space.

Let \( F^n = (M^n, L(\alpha, \beta)) \) be an \( n \)-dimensional Finsler space with a special \((\alpha, \beta)\)-metric given by

\[ L(\alpha, \beta) = c_1 \alpha + c_2 \beta + \alpha^2 / \beta, \]

where \( c_1, c_2 \) are constants, and \( c_1 \neq 0 \).
We shall assume $b^2 \neq 0$. If $b^2 = 0$, then from Lemma 2.2 we have $\alpha^2 = \beta \delta$, so $L = c_1 \alpha + (c_2 \beta + \delta)$, which is a Randers metric. So the assumption $b^2 \neq 0$ is reasonable.

Then from the above we have

$$(3.2) \quad L_\alpha = c_1 + 2\alpha/\beta, \quad L_\beta = c_2 - \alpha^2/\beta^2, \quad L_{\alpha\alpha} = 2/\beta.$$ 

Substituting (3.2) into (2.3), we obtain

$$(3.3) \quad c_1 \beta^2 B_j^k i y^i y^k + \alpha \{2\beta B_j^k i y^i y^k + (\alpha^2 - c_2 \beta^2) (b_{ji} - B_j^k b_k) y^i \} = 0.$$ 

Assume that the Finsler space with (3.1) be a Berwald space, that is, $G_j^i k = G_j^i k(x)$. Then we have $B_j^k i = B_j^k i(x)$, so the left-hand side of (3.3) has a form

$$(3.4) \quad P(x, y) + \alpha Q(x, s) = 0,$$ 

where $P, Q$ are polynomials in $(y^i)$ while $\alpha$ is irrational in $(y^i)$. Hence the above (3.3) shows $P = Q = 0$. By Lemma 2.1, the assumption $b^2 \neq 0$ implies $\alpha^2 - c_2 \beta^2 \neq 0$. Thus we have

$$(3.5) \quad B_j^k i a_{kh} y^i y^h = 0 \quad \text{and} \quad (b_{ji} - B_j^k i b_k) y^i = 0.$$ 

The former yields $B_j^k i a_{kh} + B_h^k i a_{kj} = 0$, so we have $B_j^k i = 0$. Then the latter leads to $b_{ji} = 0$ directly.

Conversely, if $b_{ij} = 0$, then $(\gamma_j^i k, \gamma_0^i j, 0)$ becomes the Berwald connection of $F^n$ due to the well-known Okada's axioms. Thus $F^n$ is a Berwald space. Therefore we have:

**Theorem 3.1.** The Finsler space $F^n$ with a special $(\alpha, \beta)$-metric (3.1) satisfying $b^2 \neq 0$ is a Berwald space if and only if $b_{ji} = 0$, and then the Berwald connection is essentially Riemannian $(\gamma_j^i k, \gamma_0^i j, 0)$.

### 4. Two-dimensional Landsberg space

In the present section, we find the necessary and sufficient conditions that a two-dimensional Finsler space with a special $(\alpha, \beta)$-metric (3.1) be a Landsberg space.

The difference vector $B^i$ of the Finsler space has been first given in Shibata, Shimada, Azuma & Yasuda [11]. Here, by means of (2.4) and (3.2), we have

$$(4.1) \quad 2B^i = \frac{AB}{\beta(c_1 \beta + 2\alpha) L \Omega} \left(y^i + \frac{2\alpha^3 L L^i}{B^i} \right) + \frac{2\alpha(c_2 \beta^2 - \alpha^2)}{\beta(c_1 \beta + 2\alpha) s^i}.$$
where

\[ A = \beta(2\alpha + c_1 \beta)r_{00} + 2\alpha(\alpha^2 - c_2 \beta^2)s_0, \]
\[ B = c_1c_2 \beta^3 - 3c_1\alpha^2 \beta - 4\alpha^3, \]
\[ \Omega = c_1 \beta^3 + 2b^2 \alpha^3. \]

It is trivial that \( \beta \neq 0, \ c_1 \beta + 2\alpha \neq 0 \) and \( \Omega \neq 0 \), because \( \alpha \) is irrational in (\( y^i \)). It follows from (4.1) that

(4.2) \[ r_{00} - 2b_r B^r = \frac{\alpha(c_1 \beta + 2\alpha)A}{L \Omega}. \]

Now we deal with the necessary and sufficient conditions that a two-dimensional Finsler space \( F^2 \) with (3.1) be a Landsberg space. It is well known that in the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar \( I_{ij}y^i = 0 \). Owing to Antonelli, Ingarden & Matsumoto [1], Kitayama, Azuma & Matsumoto [5], the main scalar \( I \) of a two-dimensional Finsler space \( F^2 \) with (3.1) is obtained as follows:

(4.3) \[ \varepsilon I^2 = \frac{9\gamma^2 M^2}{4\alpha^3 \Omega^3}, \text{ where } M = c_1c_2 \beta^5 - c_1\alpha^2 \beta^3 - 2c_1b^2 \alpha^4 \beta - 4b^2 \alpha^5. \]

The covariant differentiation of (4.3) leads to

(4.4) \[ 4\alpha^2 \beta^2 L \Omega^4 \varepsilon I^2 \]
\[ = 9M(\alpha \beta \Omega M \gamma^2_{ii} + 2\alpha \beta \Omega \gamma^2 M_{ii} - \beta \Omega \gamma^2 M \alpha_{ii} - \alpha \Omega \gamma^2 M \beta_{ii} - 3\alpha \beta \gamma^2 M \Omega_{ii}). \]

Trasvecting (4.4) by \( y^i \), we have

(4.5) \[ 4\alpha^2 \beta^2 L \Omega^4 \varepsilon I^2 y^i = 9M(U \gamma^2_{ij} y^i + Q M_{ij} y^i - R \alpha_{ij} y^i - S \beta_{ij} y^i - T \Omega_{ij} y^i), \]

where

\[ U = c_1^2c_2 \alpha^9 - c_1^2 \alpha^3 \beta^7 + 2c_1c_2b^2 \alpha^4 \beta^6 - 2c_1^2b^2 \alpha^5 \beta^5 - 6c_1b^2 \alpha^6 \beta^4 - 4c_1b^4 \alpha^8 \beta^2 - 8b^6 \alpha^9 \beta, \]
\[ Q = -2c_1 \beta \alpha^6 + 2c_1 b^2 \alpha^3 \beta^4 - 4b^2 \alpha^4 \beta^3 + 4b^4 \alpha^6 \beta, \]
\[ R = -c_1^2c_2 \beta^{11} + c_1^2(c_2 b^2 + 1) \alpha^2 \beta^9 - 2c_1c_2 b^2 \alpha^3 \beta^8 + c_1^2b^2 \alpha^4 \beta^7 + 2c_1 b^2(c_2 b^2 + 3) \alpha^6 \beta^6 - 2c_1b^4 \alpha^6 \beta^5 - 2c_1b^4 \alpha^7 \beta^4 + 8b^4 \alpha^8 \beta^3 - 4c_1b^6 \alpha^9 \beta^2 - 8b^6 \alpha^{10} \beta, \]
\[ S = -c_1^2c_2 \alpha \beta^{10} + c_1^2(c_2 b^2 + 1) \alpha^3 \beta^8 - 2c_1c_2 b^2 \alpha^4 \beta^7 + c_1^2b^2 \alpha^5 \beta^6 + 2c_1b^2(c_2 b^2 + 3) \alpha^6 \beta^5 - 2c_1b^4 \alpha^7 \beta^4 - 2c_1b^4 \alpha^8 \beta^3 + 8b^4 \alpha^9 \beta^2 - 4c_1b^6 \alpha^{10} \beta - 8b^6 \alpha^{11}, \]
\[ T = -3c_1c_2\alpha^8 + 3c_1(c_2b^2 + 1)\alpha^2\beta^6 + 3c_1b^2\alpha^5\beta^4 + 12b^2\alpha^6\beta^3 - 6c_1b^4\alpha^7\beta^2 - 12b^4\alpha^8\beta. \]

Thus the equation (4.5) is rewritten in the form
\[ 4\alpha^2\beta^2L^2\Omega^2\varepsilon I^2_y = 9M(U\gamma_i^2y^i + V\alpha_i^2y^i + W\beta_i^2y^i + X\delta_i^2y^i), \]

where
\[ V = c_1^2c_2\beta^{11} - c_1^2(c_2b^2 - 3)\alpha^2\beta^9 + 20c_1c_2b^2\alpha^3\beta^8 - 13c_1b^2\alpha^4\beta^7 - 4c_1b^2(5c_2b^2 - 18)\alpha^2\beta^9 + 14c_1^2b^6\alpha^6\beta^5 - 32c_1b^2\alpha^7\beta^4 - 72c_1b^6\alpha^9\beta^2, \]
\[ W = -4c_1^2\alpha^3\beta^8 - 18c_1c_2b^2\alpha^4\beta^7 - 12c_1b^2\alpha^5\beta^6 + 6c_1b^2(3c_1c_2b^2 - 5)\alpha^6\beta^5 - 2c_1b^4(1 - 9)\alpha^7\beta^4 + 34c_1b^4\alpha^8\beta^3 - 8b^4\alpha^9\beta^2 - 4c_1b^6\alpha^{10}\beta + 8b^6\alpha^{11}, \]
\[ X = 6c_1c_2\alpha^4\beta^8 + 4c_1^2b^2\alpha^5\beta^7 - 2c_1(3c_2b^2 - 1)\alpha^6\beta^6 - 4c_1^2b^2\alpha^7\beta^5 - 6c_1b^2\alpha^8\beta^4 - 8b^2\alpha^9\beta^3 + 4c_1b^6\alpha^{10}\beta^2 + 8b^6\alpha^{11}\beta. \]

Consequently, the two-dimensional Finsler space \( P^2 \) with (3.1) is a Landsberg space, if and only if
\[ U\gamma_i^2y^i + V\alpha_i^2y^i + W\beta_i^2y^i + X\delta_i^2y^i = 0, \]
where \( M \neq 0 \). If \( M = 0 \), then \( b^2 = 0 \), namely, it is a contradiction.

By means of (2.5), (2.6), (2.7) and (2.8), the equation above is written as
\[ 2\beta(c_1\beta + 2\alpha)(\alpha^2U + X)(r_0 + s_0) + [(\alpha^2 - c_2\beta^2)V + \beta(c_1\beta + 2\alpha)W - 2\{c_1\beta^3 + (c_2b^2 + 2)\alpha\beta^2 - b^2\alpha^3\}U](r_0 - 2b_rB^r) = 0. \]

Substituting (4.2), \( U, V, W \) and \( X \) into (4.8), we obtain
\[ [2c_1^2c_2\alpha^3\beta^{14} + 2c_1^2c_2(c_1^2 + 6c_2)\alpha^4\beta^{13} + 20c_1^3c_2\alpha^5\beta^{12} + 2c_1^2(3c_1^2 - 2c_2b^2 + 8c_2)\alpha^6\beta^{11} + 2c_1(12c_2b^2 - 8c_1^2c_2b^2 + 5c_1^3)\alpha^7\beta^{10} + 4c_1^2(2c_2b^2 - 3c_1^2b^2 + 1)\alpha^8\beta^9 + 8c_1b^2(2c_2 - 3c_1^2 - 2c_2b^2)\alpha^9\beta^8 - 20c_1^2b^2(2c_2b^2 + 1)\alpha^{10}\beta^7 - 8c_1b^2(3c_1^2b^2 + 8c_2b^2 + 1)\alpha^{11}\beta^6 - 8b^2(9c_1^2 + 4c_2)\alpha^{12}\beta^5 - 80c_1b^4\alpha^{13}\beta^4 - 32b^4\alpha^{14}\beta^3](r_0 + s_0) + [c_1^3c_2\alpha^4\beta^{15} - c_1^2c_2(5c_1^2 - 2c_2)\alpha^2\beta^{14} - c_1^2c_2(2c_2b^2 + c_2 - 6)\alpha^3\beta^{13} + 4c_1^2(4c_2b^2 - c_1^2 + 8c_2)\alpha^4\beta^{12} + c_1(-31c_1^2c_2b^2 + 40c_2b^2 + c_1^2c_2 - 3c_1^2)\alpha^5\beta^{11} - 2c_1^2(12c_2b^2 + 4c_1^2b^2 + 99c_2b^2 - c_1 - 1)\alpha^6\beta^{10} + c_1b^2(40c_1c_2b^2 - 48c_2b^2 - 16c_1^2 + 13c_1 - 272c_2 - 36)\alpha^7\beta^9 \]
\[ + 2c_1b^2\{c_1b^2(72c_2 - c_1^2 + 9c_1) + 67c_1 - 72)\alpha^8\beta^8 \\
+ 4b^2\{c_1b^2(3c_2^2 + 18c_1 - 4c_2) + 54c_1 - 36)\alpha^9\beta^7 \\
+ 8b^4(10c_1^2c_2b^2 + 14c_1^2 + 9c_1 - 40c_2)\alpha^{10}\beta^6 \\
+ 4c_1b^2\{(84c_2 + c_1^2)b^4 + 84b^2 - 2c_1)\alpha^{11}\beta^5 \\
- 8b^2\{(7c_1^2 - 44c_2)b^4 - 44b^2 + 2c_1^2 + 2c_1)\alpha^{12}\beta^4 \\
- 32c_1b^2(9b^4 + 1)\alpha^{13}\beta^3 - 320b^6\alpha^{14}\beta^2]\gamma_{00} \\
+ [- 2c_1^3c_2^2\alpha^{15} + 10c_1^3c_2\alpha^3\beta^{14} + 2c_1^2c_2^2(2c_2b^2 + c_2 - 15)\alpha^4\beta^{13} \\
- 2c_1c_2(32c_2^2b^2 + 3c_1^2)\alpha^5\beta^{12} + 2c_1^2c_2(29c_2b^2 - 3c_2 + 15)\alpha^6\beta^{11} \\
+ 4c_1(2c_2b^2(6c_2^2b^2 + 2c_1^2 + 34c_2 + 5) - c_1^2)\alpha^7\beta^{10} \\
- 2c_1\{c_2b^2(40c_1c_2b^2 + 13c_1 - 36) - c_1(-31c_2b^2 + c_2 + 1)\}\alpha^8\beta^9 \\
- 2b^2\{c_1c_2b^2(60c_2 - c_1^2 + 9c_1) + 8c_1^2 + 190c_2c_1 - 36c_2)\alpha^9\beta^8 \\
+ 2b^2\{-4c_2b^2(-6c_1^2 + 9c_1 - 40c_2) + c_1(13c_2 - 16)\}\alpha^{10}\beta^7 \\
- 2c_1\{c_1b^2(40c_2b^2 + 4c_2 + c_1^2 - 9c_1) + 54c_1 - 36)\alpha^{11}\beta^6 \\
- 8b^2\{(84c_2 - 4c_1^2 - 9c_1)b^2 - 8b^2 - 2c_1c_2)\alpha^{12}\beta^5 \\
+ 32c_1b^2(9c_2b^4 + 5b^2 + c_2)\alpha^{13}\beta^4 + 4b^2\{(84c_2 + c_1^2)b^4 + 44b^2 - 2c_1)\alpha^{14}\beta^3 \\
- 32c_1b^2(4b^4 + 1)\alpha^{15}\beta^2 - 320b^6\alpha^{16}\beta]\zeta_{00} = 0. \\

Separating (4.9) in the rational and irrational terms with respect to \(y^i\), we have

\[ (4.10) \quad \{\alpha^4\beta^2D_1(r_0 + s_0) + \alpha^2\beta E_1r_0 + 2\alpha^2F_1s_0 \}
\]

\[ + \alpha\{\alpha^2\beta^3D_2(r_0 + s_0) + \beta^2E_2r_0 + 2\alpha^2\beta F_2s_0 \} = 0, \]

where

\[ D_1 = 2c_1^2c_2(c_1^2 + 6c_2)\beta^{10} + 2c_1^2(3c_1^2 - 2c_2b^2 + 8c_2)\alpha^2\beta^8 \\
+ 4c_1^2(2c_2b^2 - 3c_1^2b^2 + 1)\alpha^4\beta^6 - 20c_1^2b^2(2c_2b^2 + 1)\alpha^6\beta^4 \\
- 8b^4(9c_1^2 + 4c_2)\alpha^8\beta^2 - 32b^4\alpha^{10}, \]

\[ D_2 = 2c_1^3c_2^2\beta^{10} + 20c_1^3c_2\alpha^2\beta^8 + 2c_1(12c_2b^2 - 8c_1^2c_2b^2 + 5c_1^2)\alpha^4\beta^6 \\
+ 8c_1b^2(2c_2 - 3c_1^2 - 2c_2b^2)\alpha^6\beta^4 - 8c_1b^2(3c_1^2b^2 + 8c_2b^2 + 1)\alpha^8\beta^2 - 80c_1b^4\alpha^{10}, \]

\[ E_1 = - c_1^2c_2(5c_1^2 - 2c_2)\beta^{12} + 4c_1^2(4c_2b^2 - c_1^2 + 8c_2\alpha^2\beta^10 \\
- 2c_1^2(12c_2b^4 + 4c_1^2b^2 + 99c_2b^2 - c_1^2 - 1)\alpha^4\beta^8. \]
\[ + 2c_1b^2\{c_1b^2(72c_2 - c_2^2 + 9c_1) + 67c_1 - 72\}\alpha^6\beta^6 \\
+ 8b^4\{10c_1^2c_2b^2 + 14c_1^2 + 9c_1 - 40c_2\}\alpha^8\beta^4 \\
- 8b^2\{(7c_1^2 - 44c_2)b^4 - 44b^2 + 2c_1^2 + 2c_1\}\alpha^{10}\beta^2 - 320b^6\alpha^{12}, \\
E_2 \equiv c_1^3c_2^3\beta^{12} - c_1^3c_2(2c_2b^2 + c_2 - 6)\alpha^2\beta^{10} \\
+ c_1(-31c_1^2c_2b^2 + 40c_2^3b^2 + c_1^2c_2 - 3c_1^2)\alpha^4\beta^8 \\
+ c_1b^2(40c_1c_2b^2 - 48c_2^3b^2 - 16c_1^2 + 13c_1 - 27c_2 - 36)\alpha^6\beta^6 \\
+ 4b^2\{c_1b^2(3c_1^2 + 18c_1 - 4c_2) + 54c_1 - 36\}\alpha^8\beta^4 \\
+ 4c_1b^2\{(84c_2 + c_1^2)b^4 + 84b^2 - 2c_1\}\alpha^{10}\beta^2 - 32c_1b^2(9b^4 + 1)\alpha^{12}, \\
F_1 = - c_1^2c_2^2\beta^{14} + c_1^2c_2^2(2c_2b^2 + c_2 - 15)\alpha^2\beta^{12} + c_1^2c_2(29c_2b^2 - 2c_2 + 15)\alpha^4\beta^{10} \\
- c_1\{c_2b^2(40c_1c_2b^2 + 13c_1 - 36) - c_1(-31c_2b^2 + c_2 + 1)\}\alpha^6\beta^8 \\
+ b^2\{-4c_1b^2(-6c_1^2 + 9c_1 + 40c_2) + c_1(13c_1 - 36)\}\alpha^8\beta^6 \\
- 4b^2\{c_1(44c_2 + c_1^2)b^4 + (84c_2 - c_1^2 - 9c_1)b^2 - 2c_1c_2\}\alpha^{10}\beta^4 \\
+ 4b^2\{(84c_2 + c_1^2)b^4 + 44b^2 - 2c_1\}\alpha^{12}\beta^2 - 160b^6\alpha^{14}, \\
F_2 = 5c_1^3c_2\beta^{12} - c_1c_2(32c_2^3b^2 + 3c_1^3)\alpha^2\beta^{10} \\
+ 2c_1\{c_2b^2(6c_2^3b^2 + 2c_1^2 + 34c_2 + 5) - c_1^2\}\alpha^4\beta^8 \\
- b^2\{c_1c_2b^2(60c_2 - c_1^2 + 9c_1) + 8c_1^3 + 190c_1c_2 - 36c_2\}\alpha^6\beta^6 \\
- 2b^2\{c_1b^2(40c_2^3b^2 + 4c_2 + c_1^2 - 9c_1) + 54c_1 - 36\}\alpha^8\beta^4 \\
+ 16c_1b^2(9c_2b^4 + 5b^2 + c_2)\alpha^{10}\beta^2 - 16c_1b^2(4b^4 + 1)\alpha^{12}. \\
\]

which yield two equations as follows:

(4.11) \[ \alpha^2\beta^2D_1(r_0 + s_0) + \beta E_1r_000 + 2F_1s_0 = 0, \]

(4.12) \[ \alpha^2\beta^2D_2(r_0 + s_0) + \beta E_2r_000 + 2\alpha^2F_2s_0 = 0. \]

From (4.12) we obtain

(4.13) \[ c_1^3c_2^3\beta^{13}r_000 \equiv 0 \pmod{\alpha^2}. \]

If \( c_2 \neq 0 \), then there exists a function \( f(x) \) such that \( r_00 = \alpha^2f(x) \). Thus we have

(4.13') \[ r_{ij} = a_{ij}f(x). \]

Transvection by \( b^iy^j \) leads to

(4.13'') \[ r_0 = \beta f(x); \quad r_j = b_jf(x). \]
Elimination \((r_0 + s_0)\) from (4.11) and (4.12), from (4.13') we have
\[(4.14) \quad f(x)\beta\alpha^2(D_2E_1 - D_1E_2) + 2(D_2F_1 - \alpha^2D_1F_2)s_0 = 0.\]

From \(\alpha^2 \not\equiv 0 \pmod{\beta}\) it follows that there exists a function \(g(x)\) satisfying \(s_0 = g\beta\).
Hence (4.14) is reduced to
\[(4.14') \quad \alpha^2\{f(x)(D_2E_1 - D_1E_2) - 2g(x)D_1F_2\} + 2g(x)D_2F_1 = 0.\]

Since only the term \(-4c_1^5c_2^5g(x)\beta^{24}\) of \(2g(x)D_2F_1\) seemingly does not contain \(\alpha^2\), we must have \(hp(22)\) \(V_{22}\) such that \(\beta^{24} = \alpha^2V_{22}\). Thus it is a contradiction because of \(\alpha^2 \not\equiv 0 \pmod{\beta}\), that is, \(D_2F_1\) does not contain \(\alpha^2\) as a factor. Hence from (4.14') we have \(g(x) = 0\), which leads to \(s_0 = 0\) and \(s_1 = 0\). Further, substituting \(g(x) = 0\) into (4.14'), we obtain
\[(4.14'') \quad f(x)(D_2E_1 - D_1E_2) = 0.\]

If \((D_2E_1 - D_1E_2) = 0\), then the term of \(D_2E_1 - D_1E_2\) which does not contain \(\alpha^2\) as a factor is \(-4c_1^5c_2^5(3c_1^2 + 2c_2)\beta^{22}\). If \(3c_1^2 + 2c_2 \neq 0\), then there exists \(hp(20)\) \(V_{20}\) such that \(\beta^{22} = \alpha^2V_{20}\). From \(\alpha^2 \not\equiv 0 \pmod{\beta}\) and \(b^2 \neq 0\) we have \(V_{22} = 0\). It is a contradiction, which leads to \(D_2E_1 - D_1E_2 \neq 0\). Thus from (4.14'') we have \(f(x) = 0\). From (4.13') we get \(r_{ij} = 0\).

In each exceptional case where \(c_2 = 0\) or \(3c_1^2 + 2c_2 = 0\), we have the same conclusion similarly.

Summarizing up, we obtain \(r_{ij} = 0\) and \(s_1 = 0\), that is,
\[(4.15) \quad b_{i;j} + b_{j;i} = 0, \quad b^rb_{r;i} = 0.\]

Therefore \(b_1(x)\) is the so-called Killing vector field with a constant length.

According to Hashiguchi, Hōjō & Matsumoto [4], the condition (4.15) is equivalent to \(b_{i;j} = 0\). So we have

**Theorem 4.1.** Let \(F^2\) be a two-dimensional Finsler space with a special \((\alpha, \beta)\)-metric (3.1) satisfying \(b^2 \neq 0\). If \(F^2\) is a Landsberg space, then \(F^2\) is a Berwald space.

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