A PRODUCT FORMULA OF SEIBERG-WITTEN INVARIANTS

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ABSTRACT. Let $X$ be a 4-manifold obtained by gluing two symplectic 4-manifolds $X_i$, $i = 1, 2$, along embedded surfaces. Using the gradient flow of a functional on 3-dimensional Seiberg-Witten theory along the cylindrical end, we study the Seiberg-Witten equations on $X$ and have a product formula of Seiberg-Witten invariants on $X$ from the ones on $X_i$, $i = 1, 2$.

Suppose that $X_1$ and $X_2$ are a closed, oriented 4-manifolds. Let $\Sigma$ be a closed, oriented Riemann surface with genus $g(\Sigma) > 1$. And let $\Sigma \to X_i$ for $i = 1, 2$ be smooth embeddings representing homologies of infinite order. Suppose that the self-intersections $\Sigma \cdot \Sigma$ in $X_i$ are zero. Then there is a regular neighborhood of $\Sigma$ diffeomorphic to $D^2 \times \Sigma$. We have $X_1^0$ and $X_2^0$ by removing the interiors of these regular neighborhoods. Denote by $Y$ the boundary $S^1 \times \Sigma$. There is an orientation reversing diffeomorphism $\partial X_1^0 \to \partial X_2^0$ which is identity on $\Sigma$ and is complex conjugation on $S^1$ factor. We denote by $Z = X_1 \cup \Sigma \cup X_2$ manifold obtained by gluing $X_1^0$ and $X_2^0$ along the boundary $Y$ into $Z$ with $Z - Y = X_1^0 \bigsqcup X_2^0$. Fix metrics on $X_1^0$ and $X_2^0$ which have cylindrical ends with orientation preserving isometric to $[-1, \infty) \times Y$ and $[-1, \infty) \times \bar{Y}$ respectively. Let $Z_s$ be the compact manifold obtained truncating the ends of $X_1^0$ and $X_2^0$ at $s \times Y$ and $s \times \bar{Y}$, respectively and then identifying two truncated manifolds along their common boundary $Y$. Thus the diffeomorphism type of $Z_s$ is independent of $s$.

From now on we fix Spin$^c$ structures $\tilde{P}_1$ and $\tilde{P}_2$ on $X_1^0$ and $X_2^0$ whose determinant line bundles restricted to $Y$ are both isomorphic to the...
pullback from $\Sigma$ of a line bundle of degree $2 - 2g(\Sigma_1)$ on $\Sigma$. We choose small perturbation of monopole equations for $X_1^0$ and $X_2^0$ so that the equations on $X_1$ are
\[
\begin{cases}
F_+^A = q(\psi) + \sqrt{-1}\varphi_1(*n + dt \wedge n) + \sqrt{-1}\eta_1^+,
D_A\psi = 0,
\end{cases}
\]
where $n$ is a harmonic 1-form on $\Sigma$ and $\eta_1^+$ is a compactly supported self-dual 2-form and $\varphi_1$ is a smooth function which is 1 on $[0, \infty) \times Y$ and vanishes off of $[-1, \infty) \times Y$.

Let $M_{d_1}(\bar{P}_1, n, \eta_1^+)$ be the moduli space of finite energy solutions to the perturbed equations with dimension $2d_1$. In a similar way, we define the moduli spaces $M_{d_2}(\bar{P}_2, n, \eta_2^+)$ of finite energy solutions on $X_2^0$ with dimension $2d_2$. Let $S_d$ be the set of isomorphism classes of Spin$^c$ structures $\bar{P}$ on $Z$ with the property that $\bar{P}_{|X_1^0} = \bar{P}_1$, $\bar{P}_{|X_2^0} = \bar{P}_2$ and $\frac{1}{4}(c_1(det\bar{P})^2[Z] - (2\chi + 3\sigma)(Z)) = 2d$. If $\bar{P} \in S_d$, for each $s \geq 0$ we have the corresponding Spin$^c$ structure $\bar{P}_s$ over $Z_s$. For sufficiently large $s$, let $\eta^+ = \eta_1^+ + \eta_2^+$ and define the moduli space $M_d(\bar{P}_s, n, \eta^+)$ of solutions to the perturbed Seiberg-Witten equations
\[
\begin{cases}
F_+^A = q(\psi) + \sqrt{-1}\varphi_2(*n + dt \wedge n) + \sqrt{-1}\eta_1^+,
D_A\psi = 0,
\end{cases}
\]
where $\varphi_2 : Z_s \to [0, 1]$ is the function which is $\varphi_1$ on $X_1^0(s)$ and $\varphi_2$ on $X_2^0(s)$.

For simplicity, we shall write $M_{d_1}(\bar{P}_1)$ and $M_{d_2}(\bar{P}_2)$ for $M_{d_1}(\bar{P}_1, n, \eta_1^+)$ and $M_{d_2}(\bar{P}_2, n, \eta_2^+)$, respectively.

Now we will define the moduli space of solutions to the monopole equations on non-compact 4-manifolds with ends isometric to $[-1, \infty) \times Y$. We consider only solutions to the equations with finite energy on the cylindrical end. Let $Y = S^1 \times \Sigma$ and let $X^0$ be a Riemannian 4-manifold whose end is orientation-preserving isometric to $[-1, \infty) \times Y$. Let a Spin$^c$ structure $\bar{P}$ on $X^0$ be given and denote the restriction of $\bar{P}$ to $Y$ by $\bar{P}_Y$. Then for any solution $(A, \psi)$ to the Seiberg-Witten equations with respect to this Spin$^c$ structure, there is a temporal gauge for $\bar{P}$ restricted to the cylindrical end so that the flow line $\gamma$ satisfies the gradient flow equation. A finite energy solution is a solution for which an associated flow line $\gamma : [0, \infty) \to C^*(\bar{P}_Y)$ satisfies $\lim_{t \to \infty}(f(\gamma(t)) - f(\gamma(0))) < \infty$. Here $C^*(\bar{P}_Y)$ is the space of pairs of a connection on the determinant line.
bundle of \( \tilde{P} \) and a section of spinor bundle on \( Y \). And \( f \) is the function defined by

\[
f(A, \psi) = \int_Y (F_A \wedge A + \langle \psi, \partial_A \psi \rangle).
\]

And the difference is

\[
\begin{align*}
f(\gamma(t)) - f(\gamma(0)) & = \int_0^t \frac{\partial}{\partial t} (f \circ \gamma) \, dt \\
& = \int_0^t \frac{\partial}{\partial t} \int_Y (F_A \wedge A + \langle \psi, \partial_A \psi \rangle) \\
& = \int_0^t \int_Y \left( \frac{\partial}{\partial t} \langle *F_A, A \rangle + \frac{\partial}{\partial t} \langle \psi, \partial_A \psi \rangle \right) \\
& = \int_0^t \int_Y \left( 2 \langle *F_A, \frac{\partial A}{\partial t} \rangle + 2 \langle \frac{\partial \psi}{\partial t}, \partial_A \psi \rangle - 2 \langle \frac{\partial A}{\partial t}, q(\psi) \rangle \right) \\
& = \int_0^t \int_Y \left( 2 \left| \frac{\partial A}{\partial t} \right|^2 + 2 \left| \frac{\partial \psi}{\partial t} \right|^2 \right) < \infty.
\end{align*}
\]

Intuitively, the finiteness of energy implies that \( (A, \psi) \) approaches to the static solution as \( t \to \infty \). Taubes obtains the following:

**Theorem 1.** [17]. Let \( X^0, Y, \tilde{P} \) be given as above. Let \( (A, \psi) \) be a finite energy solution to the Seiberg-Witten equations associated \( \tilde{P} \). Then there is a \( C^\infty \)-product structure for \( \tilde{P}|_{[0, \infty) \times Y} \) such that in this product structure \( (A, \psi) \) converges exponentially fast to a static solution.

Let \( \mathcal{M}_d(\tilde{P}) \) be the moduli space of all finite energy solutions to the Seiberg-Witten equations.

\[\mathcal{M}_d(\tilde{P}) = \{(A, \psi)| (i) \ (A, \psi) \text{ is in a temporal gauge on } [0, \infty) \times Y, \]

(ii) \( \gamma(t) = (A(t), \psi(t)) \) satisfies the gradient flow equation,

(iii) \( \lim_{t \to \infty} f(\gamma(t)) - f(\gamma(0)) < \infty \}. \]

Then \( \mathcal{M}_d(\tilde{P}) \) is a smooth compact manifold of dimension \( 2d \) except singularity.

Theorem 1 shows the existence of limit of gauge equivalence classes

\[r(A, \psi) = \lim_{t \to \infty} (A_t, \psi_t).\]
The limit defines a continuous map
\[ r : \mathcal{M}_d(\tilde{P}) \rightarrow \mathcal{R}(Y) \]
to the moduli space of solutions of the Seiberg-Witten equations on \( Y \).

Suppose that \((A_1, \psi_1) \in \mathcal{M}_d(\tilde{P}_1)\) and \( r_1(A_1, \psi_1) = r_2(A_2, \psi_2) = \rho \in \mathcal{R}(Y) \). For sufficiently large \( s \), we can make a small modification to \((A_1, \psi_1)\) so that it is equivalent to \( \rho \in \mathcal{R}(Y) \) on the end \([s, \infty) \times Y\) of \( X_1^s \). Making similar modification to \((A_2, \psi_2)\) allows us to join two solutions to form \((A_1, \psi_1) \# (A_2, \psi_2)\) on \( Z_s \) which may not satisfy the equations on the neck region. But this can be deformed to a solution on \( Z_s \). In fact, we have the following result.

**Theorem 2.** Suppose that \((A_1, \psi_1)\) and \((A_2, \psi_2)\) are regular points of their moduli spaces and suppose that \( r_1, r_2\) are transverse at \(((A_1, \psi_1), (A_2, \psi_2))\). Then gluing and deforming determine a diffeomorphism
\[
\bigsqcup_{d_1 + d_2 - g(\Sigma_2) = d} \mathcal{M}_{d_1}(\tilde{P}_1) \times_r \mathcal{M}_{d_2}(\tilde{P}_2) \rightarrow \bigsqcup_{\tilde{P}_s \in S_d} \mathcal{M}_d(\tilde{P}_s).
\]

Here \( d_1 \) or \( d_2 \) are greater than or equal to \( g(\Sigma_2) \) and \( \mathcal{M}_{d_1} \times_r \mathcal{M}_{d_2} \) is the fiber product
\[
\mathcal{M}_{d_1} \times_r \mathcal{M}_{d_2} = \{ ((A_1, \psi_1), (A_2, \psi_2)) | r_1(A_1, \psi_1) = r_2(A_2, \psi_2) \}.
\]

This theorem shows that the formal dimension of the moduli space is as follows:

\[
\dim \mathcal{M}_d(\tilde{P}_s) = \dim \mathcal{M}_{d_1} + \dim \mathcal{M}_{d_2} - \dim \mathcal{R}(\Sigma).
\]

Let \( X = X^0 \cup (D^2 \times \Sigma) = X^0 \cup W^0 \) be a compact symplectic 4-manifold. The condition on \( \tilde{P}X^0 \rightarrow X^0 \) implies that it has an extension to a Spin\(^c\) structure over \( X \). The extended Spin\(^c\) structure differs by an even multiple of \( PD[\Sigma] \), which is Poincare dual of \([\Sigma] \in H_2(\tilde{X}, \mathbb{Z})\).

Now we will consider only 0-dimensional moduli spaces with respect to an extended Spin\(^c\) structure \( \tilde{P} \) over \( X \). Applying the above Theorem 2 to \( X = X^0 \cup W^0 \) we obtain a diffeomorphism
\[
\mathcal{M}_{d_1}(X^0) \times_r \mathcal{M}_{d_2}(W^0) \rightarrow \mathcal{M}_0(\tilde{P}).
\]

Here \((d_1, d_2)\) is one of \((0, g(\Sigma_2))\) or \((g(\Sigma_2), 0)\). It follows from the fact that \( d_1 \) or \( d_2 \) is greater than or equal to \( g(\Sigma_2) \) and \( d_1 + d_2 = g(\Sigma_2) \). Otherwise, the fiber product \( \mathcal{M}_{d_1} \times_r \mathcal{M}_{d_2} \) is empty for a Spin\(^c\) structure \( \tilde{P} \) with nonzero Seiberg-Witten invariant.
COROLLARY 3. Let $X_1$ and $X_2$ be symplectic manifolds and $Z = X_1 \times_{\Sigma} X_2$. Let $\tilde{P}_{X_1}$ and $\tilde{P}_{X_2}$ be Spin structures on $X_1^0$ and $X_2^0$ such that $c_1(\det \tilde{P}_{X_i})(\Sigma) = 2 - 2g(\Sigma_i)$ for $i = 1, 2$. Then there is a diffeomorphism

$$\mathcal{M}_{d_1} (X_1^0, \tilde{P}_{X_1}) \times_r \mathcal{M}_{d_2} (X_2^0, \tilde{P}_{X_2}) \rightarrow \mathcal{M}_{d_1 + d_2 - g(\Sigma_1)} (Z, \tilde{P}),$$

where $(d_1, d_2) = (0, g(\Sigma_2)), (g(\Sigma_2), 0)$ or $(g(\Sigma_2), g(\Sigma_2))$.

THEOREM 4. For $d=0$, there is a diffeomorphism

$$\mathcal{M}_0 (X_1^0, \tilde{P}_{X_1}) \times_r \mathcal{M}_{g(\Sigma_2)} (X_2^0, \tilde{P}_{X_2}) \rightarrow \mathcal{M}_0 (Z, \tilde{P}),$$

and we have a relation

$$SW_Z(\tilde{P}) = SW_{X_i^0} (\tilde{P}_{X_i}) \cdot p,$$

where $p$ is the degree of $r : \mathcal{M}_{g(\Sigma_2)} (X_2^0, \tilde{P}_{X_2}) \rightarrow \mathcal{R}(\Sigma)$.

Proof. Let $\{s_i\}$ be a sequence approaching to infinity and let $(A_i, \psi_i)$ be a sequence in $\mathcal{M}_0 (Z_{s_i}, \tilde{P})$ with respect to the metric $g_{s_i}$. After passing to a subsequence, $(A_i, \psi_i)$ converges to $((A_0, \psi_0), (B_0, \phi_0))$ in compact topology. If $f(A_0, \psi_0) = f((A_i, \psi_i |_{X_1(s_i)}))$, then $(A_0, \psi_0)$ is a solution to the SW-equations and is in $\mathcal{M}_0 (X_1^0, \tilde{P}_{X_1})$. But if $f(A_0, \psi_0) < f((A_i, \psi_i |_{X_1(s_i)}))$, then there are a sequence of $t_i \rightarrow \infty$ and $\epsilon > 0$ such that

$$\left| \int_{t_i}^{t_i + \epsilon} F_{A_i} \wedge F_{A_i} \right| \geq \epsilon.$$

Since the solution $(A_i, \psi_i)$ decays exponentially to a static solution, we can assume that

$$\left| \int_{t_i}^{t_i + \epsilon} F_{A_i} \wedge F_{A_i} \right| \geq \epsilon.$$

Take a sequence $r_i < t_i$ with $\lim_{i \rightarrow \infty} r_i = \infty$ and an embedding $I_i : (-r_i, r_i) \times Y \rightarrow X_i^0$ by $(t, x) \rightarrow (t + t_i, x)$. After taking a subsequence $I_i^* (A_i, \psi_i)$ converges to $(\tilde{A}_i, \tilde{\psi}_i)$ on $\mathbb{R} \times Y$ in compact topology, which satisfies a gradient flow equation and

$$\left| \int_{0}^{\epsilon} F_{\tilde{A}_i} \wedge F_{\tilde{A}_i} \right| \geq \epsilon.$$

Repeating the same process we obtain finite numbers $(\tilde{A}_i, \tilde{\psi}_i), i = 1, 2, \ldots, k$, such that $\lim_{t \rightarrow -\infty} (A_0, \psi_0) = \lim_{t \rightarrow -\infty} (\tilde{A}_1, \tilde{\psi}_1)$ and $\lim_{t \rightarrow -\infty} (\tilde{A}_2, \tilde{\psi}_2)$.
\((\tilde{A}_{i-1}, \tilde{\psi}_{i-1}) = \lim_{t \to -\infty} (A_t, \psi_t)\) in \(\mathcal{R}(Y)\) for all \(i = 2, \ldots, k\). Then \(f(A_i, \psi_i)[X^0_i] = f(A_0, \psi_0) + \sum_{k=1}^i f(A_k, \psi_k) = (2\chi + 3\sigma)(X^0_i)\).

Since \((A_0, \psi_0) - (2\chi + 3\sigma)(X^0_i) \geq 0\) and \(f(A_i, \psi_i) \geq 0\) for all \(i\), we have \(k=0\) and \(f(A_0, \psi_0) = (2\chi + 3\sigma)(X^0_1)\).

Similarly, on \(X^0_2\) we have a limit of subsequence \((A_i, \psi_i)\) of solutions restricted to \(X_2(s_i)\) with respect to the metric \(g_{s_i}\), of the form \((B_0, \phi_0) \equiv (\tilde{B}_1, \tilde{\phi}_1) \cdots \equiv (\tilde{B}_i, \tilde{\phi}_i)\), where \(\lim_{t \to -\infty} (B_0, \phi_0) = \lim_{t \to -\infty} (\tilde{B}_1, \tilde{\phi}_1)\), \(\lim_{t \to -\infty} (\tilde{B}_i, \tilde{\phi}_i) = \lim_{t \to -\infty} (\tilde{B}_i, \tilde{\phi}_i)\) for all \(i = 2, \ldots, k\) and \(\lim_{t \to -\infty} (\tilde{B}_i, \tilde{\phi}_i) = \lim_{t \to -\infty} (A_0, \phi_0)\).

Therefore we have

\[
\begin{align*}
f(B_0, \phi_0) + \sum_{i=1}^l f(\tilde{B}_i, \tilde{\phi}_i) &= f(A_i, \psi_i)[Z] - f(A_0, \psi_0)[X^0_i] \\
&= (2\chi + 3\sigma)(X^0_2),
\end{align*}
\]

and so \(l = 0\). Then up to sign \(SW_Z(\tilde{P})\) equals to \(p \cdot SW_{X^0_1}(\tilde{P}_{x^1_1})\).

\[\square\]

**Theorem 5.** For the canonical Spin\(^c\) structure \(K_{X_0}^{\pm 1}\) on \(X\), \(\dim \mathcal{M}_{X_0}(K_{X_0}^{\pm 1}|_{X_0})\) is zero and the Seiberg-Witten invariant \(SW_{X^0_1}(K_{X}^{\pm 1}|_{X_0})\) is nonzero.

**Proof.** We show that the dimension of moduli space \(\mathcal{M}_{W_0}(K_{X}^{\pm 1}|_{W_0})\) is \(2g(\Sigma_2)\). For a general Spin\(^c\) structure \(L = K_{W_0}^{\pm 1} \otimes F^2\), by the index theorem

\[
\dim \mathcal{M}_{W_0}(L) = \frac{1}{4} (c_1(L)^2[W^0] - (2\chi + 3\sigma)(W^0)).
\]

We assume that \(c_1(L)(\Sigma) = 2 - 2g(\Sigma_1)\). A finite energy solution \((A, \psi) = (A, \alpha, 0) \in \mathcal{M}_{W_0}(L)\) exponentially decays to a static solution in an appropriate gauge. Since \(F_B^* \wedge F_B = 0\) for a static solution \(B\),

\[
c_1(L)^2[W^0] = -\frac{1}{4\pi^2} \int_{W_0} F_A \wedge F_A
\]

is finite. Let \(\lim_{t \to -\infty} (A_t, \psi_t) = (A_0, \alpha_0, 0) \in \mathcal{R}(Y)\). Then the limit \((A_0, \alpha_0)\) is pull back of a solution \((A_\Sigma, \alpha_\Sigma)\) which is represented by \(\alpha_0^{-1}(0) = \cup_{i=1}^{2g(\Sigma_2)} \{x_i\} \times S^1\). Let \(N(x_i)\) be a neighborhood of \(x_i\) in \(\Sigma\).

We can assume that

\[
\int_{N(x_i)} \frac{\sqrt{-1}}{2\pi} F_A = 1,
\]

by a choice of metric in $\Sigma$. The holomorphic section $\alpha$ converges exponentially fast to a section which is defined by zero at end of \( \{x_i\} \times D^2 \) and nonzero at end of \( \{y\} \times D^2 \) for $y \not\in \{x_1, \ldots, x_g\}$ and so $\alpha$ can be extended to a holomorphic section over $\Sigma \times S^2$. Also $A_0|_{D^2}$ can be extended to a holomorphic connection on a line bundle over $S^2$. Then we have

$$\int_{y \times D^2} \frac{\sqrt{-1}}{2\pi} F_{A_0} = \text{the number of zeros of } \alpha \text{ over } D^2(=n),$$

and

$$\int_{x_i \times D^2} \frac{\sqrt{-1}}{2\pi} F_{A_0} = n + 1.$$

So we have

$$c_1(L)^2[W^0]$$

$$= 2 \int_{\Sigma} \frac{\sqrt{-1}}{2\pi} F_A \int_{D^2} \frac{\sqrt{-1}}{2\pi} F_A$$

$$= 2 \left( \int_{\Sigma - \bigcup_{i=1}^{g(\Sigma_2)} N(x_i)} \frac{\sqrt{-1}}{2\pi} F_A + \sum_{i=1}^{g(\Sigma_2)} \int_{N(x_i)} \frac{\sqrt{-1}}{2\pi} F_A \right)$$

$$\times \left( 1 + 2 \cdot \frac{\sqrt{-1}}{2\pi} \int_{D^2} F_{A_0} \right)$$

$$= 2(2 - 2g(\Sigma_1)) + 2 \int_{\Sigma - \bigcup_{i=1}^{g(\Sigma_2)} N(x_i)} \frac{\sqrt{-1}}{2\pi} F_A \int_{D^2} \frac{\sqrt{-1}}{2\pi} F_{A_0}$$

$$+ \sum_{i=1}^{g(\Sigma_2)} \int_{N(x_i)} \frac{\sqrt{-1}}{2\pi} F_A \int_{D^2} \frac{\sqrt{-1}}{2\pi} F_{A_0}$$

$$= 2[2 - 2g(\Sigma_1) + 2(2 - 2g(\Sigma_1) - g(\Sigma_2))n + 2g(\Sigma_2)(n + 1)]$$

$$= 4 - 4g(\Sigma_1) + 4n(2 - 2g(\Sigma_1)) + 4g(\Sigma_2).$$

Thus we have

$$\dim \mathcal{M}_{W^0}(L)$$

$$= \frac{1}{4}(4 - 4g(\Sigma_1) + 4n(2 - 2g(\Sigma_1)) + 4g(\Sigma_2) - (4 - 4g(\Sigma)))$$

$$= 2g(\Sigma_2) + n(2 - 2g(\Sigma_1)), $$
where
\[ n = \int_{\gamma \times D^2} \frac{\sqrt{-1}}{2\pi} F_{A_0} = \# \text{ of zeros of } \alpha \text{ over } D^2. \]

For a Spin\(^c\) structure \( L = K_{X^{-1}}^{-1} = K_{W_0}^{-1} \otimes F^2 \) with \( c_1(F)(\Sigma) = g(\Sigma_2), \)
\[ 2c_1(F)[\{y\} \times D^2] = c_1(K_{X^{-1}}^{-1})[\{y\} \times D^2] - c_1(K_{W_0}^{-1})[\{y\} \times D^2] = 0. \]

Then \( n = 0 \) and \( \dim \mathcal{M}_{W_0}(K_{X^{-1}}^{-1}) = 2g(\Sigma_2). \) From Theorem 4 \( \dim \mathcal{M}_X(K_{X^{-1}}^{-1}) = 0 \) and \( \dim \mathcal{M}_{X^0}(L) = 0, \) \( SW_X(K_{X^{-1}}^{-1}) \) is equal to \( SW_{X^0} \cdot p \) up to sign.

Therefore \( SW_{X^0} \) is not trivial. \( \square \)

Using the results of [3] and [17] we can prove the following.

**Theorem 6.** There is a Spin\(^c\) structure \( L \) on \( X^0 \) with \( c_1(L)(\Sigma) = 2 - 2g(\Sigma_1) \) such that the degree of \( r \) is nonzero.

**Theorem 7.** Let \( X \) be a symplectic manifold with an embedded submanifold \( \Sigma_1 \) of minimal genus \( \geq 1 \) in its homology class \( [\Sigma_1] \in H_2(X, \mathbb{Z}) \) and of self-intersection number \( \geq 0. \) Let \( Z = X^0 \Sigma_2 X. \) Then there is a Spin\(^c\) structure \( \hat{L} \) on \( Z \) with nonzero Seiberg-Witten invariant and degree \( 2 - 2g(\Sigma_1) \) on \( \Sigma. \)

**Proof.** Let \( \hat{L} \) be a Spin\(^c\) structure on \( Z = X^0 \Sigma_2 X \) obtained by gluing restriction \( K_{X^{-1}}^{-1}|_{X^0} \) of canonical Spin\(^c\) structure on \( X \) to \( X^0 \) as in Theorem 6. Applying Theorem 4 to \( Z = X^0 \Sigma_2 X, \) we have
\[ SW_Z(\hat{L}) = \pm SW_{X^0}(K_{X^{-1}}^{-1}|_{X^0}) \cdot p, \]
where \( p \) is the degree of map \( r : \mathcal{M}_{X^0}(L|_{X^0}) \to \mathcal{R}(\Sigma). \) Then the Seiberg-Witten invariant on \( Z \) with respect to a Spin\(^c\) structure \( L \) is nonzero. \( \square \)

**References**


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