PROPERTIES OF A \textit{kth} ROOT
OF A HYPERSONAL OPERATOR

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Abstract. In this paper, we study some properties of \((\sqrt[k]{H})\) (defined below). In particular we show that an operator \(T \in (\sqrt[k]{H})\)
satisfying the translation invariant property is hyponormal and an
invertible operator \(T \in (\sqrt[k]{H})\) and its inverse \(T^{-1}\) have a common
nontrivial invariant closed set. Also we study some cases which
have nontrivial invariant subspaces for an operator in \((\sqrt[k]{H})\).

Let \(\mathcal{H}\) and \(\mathcal{K}\) be separable, complex Hilbert spaces and \(L(\mathcal{H}, \mathcal{K})\) denote
the space of all bounded linear operators from \(\mathcal{H}\) to \(\mathcal{K}\). If \(\mathcal{H} = \mathcal{K}\),
we write \(L(\mathcal{H})\) in place of \(L(\mathcal{H}, \mathcal{K})\).

An operator \(T\) is called hyponormal if \(T^*T \geq TT^*\), or equivalently,
if \(\|T^*h\| \geq \|T^*h\|\) for all \(h \in \mathcal{H}\). Let \((\mathcal{H})\) denote the class of hyponormal
operators. We say that an operator \(T \in L(\mathcal{H})\) is a \textit{kth} root of a hyponormal
operator if \(T^k\) is hyponormal for some positive integer \(k \geq 2\). We
denote this class by \((\sqrt[k]{H})\). In particular the class \((\sqrt[2]{H})\) consists of square roots of hyponormal operators.

In this paper, we study some properties of \((\sqrt[k]{H})\) (defined below). In particular we show that an operator \(T \in (\sqrt[k]{H})\) satisfying the translation
invariant property is hyponormal and an invertible operator \(T \in (\sqrt[k]{H})\)
and its inverse \(T^{-1}\) have a common nontrivial invariant closed set. Also
we study some cases which have nontrivial invariant subspaces for an
operator in \((\sqrt[k]{H})\).

1. Some properties

We start this section with some examples of \textit{kth} roots of hyponormal
operators.

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EXAMPLE 1.1. If $T \in \mathcal{L}(\mathcal{H})$ is any nilpotent operator of order $k-1$, then by Halmos characterization $T$ is unitarily equivalent to the following operator matrix

$$A = \begin{pmatrix} 0 & A_{12} & \cdots & A_{1k} \\ 0 & \cdots & \cdots & A_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \end{pmatrix}.$$ 

Since $A \in (\sqrt[k]{\mathcal{H}})$ and $k$th roots of hyponormal operators are unitarily invariant, $T \in (\sqrt[k]{\mathcal{H}})$.

The following are the straightway conclusions about shifts.

**Proposition 1.2.** Let $T$ be a weighted shift with nonzero weights $\{\alpha_n\}_{n=0}^{\infty}$. Then $T \in (\sqrt[k]{\mathcal{H}})$ if and only if $|\alpha_{n-k}| \cdots |\alpha_{n-1}| \leq |\alpha_n| \cdots |\alpha_{n+k-1}|$ for $n = k, k+1, \cdots$.

**Proof.** Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Since $T^k e_n = \alpha_n \cdots \alpha_{n+k-1} e_{n+k}$ and $T^* e_n = \bar{\alpha}_n^{-1} \cdots \bar{\alpha}_{n-k} e_{n-k}$, it is easy to calculate that $T^k$ is hyponormal if and only if $|\alpha_{n-k}| \cdots |\alpha_{n-1}| \leq |\alpha_n| \cdots |\alpha_{n+k-1}|$ for $n = k, k+1, \cdots$. 

**Corollary 1.3.** Let $T$ be a weighted shift with nonzero weights $\{\alpha_n\}_{n=0}^{\infty}$. If $T$ is hyponormal, then $T \in (\sqrt[k]{\mathcal{H}})$ for every $k \in \mathbb{N}$.

Next we give another example of $k$th roots of hyponormal operators.

**Example 1.4.** Let $T_x$ be the weighted shift with nonzero weights $\alpha_0 = x, \alpha_1 = \sqrt{\frac{2}{3}}, \alpha_2 = \sqrt{\frac{3}{4}}, \cdots$. Then it is an easy calculation from Proposition 1.2 that $T_x \in (\sqrt[k]{\mathcal{H}})$ if and only if $0 < x \leq \sqrt{\frac{(k+1)^2}{4k+2}}$.

We observe that $T_x$ is a $(k+1)$th root of a hyponormal operator, but is not a $k$th root of a hyponormal operator if $\sqrt{\frac{(k+1)^2}{4k+2}} < x \leq \sqrt{\frac{(k+2)^2}{4k+6}}$. In particular, $T_x$ is a $k$th root of a hyponormal operator, but is not a hyponormal operator if $\sqrt{\frac{2}{3}} < x \leq \sqrt{\frac{(k+1)^2}{4k+2}}$.

Next we state some properties of an operator in $(\sqrt[k]{\mathcal{H}})$.

**Proposition 1.5.** Let $T \in (\sqrt[k]{\mathcal{H}})$. Then

(a) $\alpha T \in (\sqrt[k]{\mathcal{H}})$ for all scalar $\alpha$.

(b) If $T$ is invertible, then $T^{-1}$ is a $k$th root of a hyponormal operator.

(c) If $M \in \text{Lat}(T)$, then $T|_M$ is a $k$th root of a hyponormal operator.
(d) The set of all kth roots of hyponormal operators is closed in the
norm topology.

Proof. (a) It is obvious.
(b) If T is invertible, then $T^k$ is invertible and hyponormal. Hence
$T^{-k} = (T^{-1})^k$ is hyponormal. Thus $T^{-1} \in (\sqrt[k]{H})$.
(c) If $\mathcal{M} \in \text{Lat}(T)$, then $(T|_{\mathcal{M}})^k = T^k|_{\mathcal{M}}$. Since $T^k|_{\mathcal{M}}$ is hyponormal,
$T|_{\mathcal{M}} \in (\sqrt[k]{H})$.
(d) If $T_n \to T$, then $T_n^k \to T^k$. Since the set of all hyponormal
operators is closed in the norm topology and $T_n^k$ are hyponormal, $T^k$
is hyponormal. Thus $T \in (\sqrt[k]{H})$. □

PROPOSITION 1.6. $(\sqrt[k]{H})$ is a proper subclass of $\mathcal{L}(\mathcal{H})$.

Proof. Since $T^k$ is hyponormal, ker $T^k = \ker T^{2k}$. Hence ker $T^k$
= ker $T^{k+1}$. Let $U^*$ be any unilateral backward shift on $l^2(\mathbb{N})$. Since
ker($U^*^k$) \neq ker($U^*^{k+1}$) for any $k \in \mathbb{N}, U^* \notin (\sqrt[k]{H})$. □

Next we characterize a matrix on 2-dimensional complex Hilbert
space which is in $(\sqrt[k]{H})$. Since every matrix on a finite dimensional
complex Hilbert space is unitarily equivalent to a upper triangular ma-
trix and a kth root of a hyponormal operator is unitarily invariant, it
suffices to characterize a upper triangular matrix $T$. From the direct
calculation, we get the following characterization.

PROPOSITION 1.7. For $k \geq 2$ we have
$$T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in (\sqrt[k]{H}) \iff b(a^{k-1} + a^{k-2}c + \cdots + c^{k-1}) = 0.$$

We remark here that Proposition 1.7 offers the convenient criterion to
find some examples of operators in $(\sqrt[k]{H})$. Also we observe that $(\sqrt[k]{H})$
is not necessarily normal on a finite dimensional space.

EXAMPLE 1.8. If $k = 3$ in Proposition 1.7, then $T \in (\sqrt[3]{H})$ if and
only if $b(a^2 + ac + c^2) = 0$. Take $a = 2, b = 1$, and $c = -1 + \sqrt{3}i$. Then
$$T = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 + \sqrt{3}i \end{pmatrix} \in (\sqrt[3]{H}),$$
but $T$ is not a normal operator.

It is known that hyponormal operators have translation-invariant
property. On the other hand, the class of square roots of hyponormal
operators may not have the translation-invariant property. For example, if \( T \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \) is defined as
\[
T = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},
\]
then \( T \) is a square root of a hyponormal operator. But
\[
[(T - \lambda)^{2}, (T - \lambda)^{2}] = \begin{pmatrix} -4|\lambda|^{2}AA^{*} & 0 \\ 0 & 4|\lambda|^{2}A^{*}A \end{pmatrix},
\]
which is not positive. Hence \((T - \lambda)^{2}\) is not necessarily hyponormal.

In light of the above statement, it is natural to ask the following question: What is the class of operators in \((\sqrt[\phi]{\mathcal{H}})\) satisfying the translation invariant property?

**Theorem 1.9.** If \( T - \lambda \) is in \((\sqrt[\phi]{\mathcal{H}})\) for every \( \lambda \in \mathbb{C} \), then \( T \) is hyponormal.

**Proof.** If \((T - \lambda)^{k}\) is hyponormal for every \( \lambda \in \mathbb{C} \), then
\[
[(T^{*} - \bar{\lambda})^{k}, (T - \lambda)^{k}] \geq 0.
\]

Therefore, we have
\[
0 \leq [(T^{*} - \bar{\lambda})^{k}, (T - \lambda)^{k}]
= (T^{*} - \bar{\lambda})^{k}(T - \lambda)^{k} - (T - \lambda)^{k}(T^{*} - \lambda)^{k}
= \left[ \sum_{r=0}^{k} \binom{k}{r} (T^{*})^{k-r}(-\bar{\lambda})^{r} \right] \left[ \sum_{s=0}^{k} \binom{k}{s} T^{k-s}(-\lambda)^{s} \right]
- \left[ \sum_{r=0}^{k} \binom{k}{r} (T^{*})^{k-r}(-\bar{\lambda})^{r} \right] \left[ \sum_{s=0}^{k} \binom{k}{s} (T^{*})^{k-s}(-\lambda)^{s} \right].
\]

(1)

Set \( \lambda = \rho e^{\theta} \) for every \( 0 \leq \theta < 2\pi \) and \( \rho > 0 \). Then we get
\[
(1) = \sum_{r=0}^{k} \sum_{s=0}^{k} (-1)^{r+s} \binom{k}{r} \binom{k}{s} \rho^{r+s} e^{i(s-r)\theta} (T^{*})^{k-r}T^{k-s}
- \sum_{r=0}^{k} \sum_{s=0}^{k} (-1)^{r+s} \binom{k}{r} \binom{k}{s} \rho^{r+s} e^{i(s-r)\theta} (T^{*})^{k-s}(T^{*})^{k-r}.
\]

Since terms in (1) are eliminated when \( r = s = k, r = k, \) and \( s = k \), we do eliminate these terms and then divide by \( \rho^{2k-2} \). Then we obtain
\[
0 \leq \binom{k}{k-1} \binom{k}{k-1} [T^{*}T - TT^{*}] + \frac{1}{\rho} \text{(the other terms)}.
\]
Letting $\rho \to \infty$, we get $T^*T \geq TT^*$.

We remark that the converse of Theorem 1.9 may not hold. For example, let $U \in \mathcal{L}(l^2(\mathbb{N}))$ denote the unilateral shift. Then it is known that $T = 2U + U^*$ is hyponormal, but $T^2$ is no longer hyponormal. Hence the class of operators in $(\sqrt{\mathcal{H}})$ with the translation-invariant property forms a proper subclass of hyponormal operators.

Recall that if $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, then $\{T^n x\}_{n=0}^\infty$ is called the orbit of $x$ under $T$, and is denoted by $\text{orb}(T, x)$. If $\text{orb}(T, x)$ is dense in $\mathcal{H}$, then $x$ is called a hypercyclic vector for $T$.

**Theorem 1.10.** If $T \in (\sqrt{\mathcal{H}})$ is invertible, then $T$ and $T^{-1}$ have a common nontrivial invariant closed set.

**Proof.** Since $T^k$ is hyponormal, it follows from [6] that $T^k$ has no hypercyclic vector. Then $T$ has no hypercyclic vector from [1]. [6, Theorem 2.15] implies that $T$ and $T^{-1}$ have a common nontrivial invariant closed set. \(\square\)

**Corollary 1.11.** If $T \in (\sqrt{\mathcal{H}})$ is invertible, then $T^{-1}$ has no hypercyclic vector.

**Proof.** Since $T^{-1}$ is hyponormal by Proposition 1.5, it follows from the proof of Theorem 1.10 that $T^{-1}$ has no hypercyclic vector. \(\square\)

**Lemma 1.12.** ([6, Theorem 2.1]) Let $\mathcal{L}(\mathcal{H})$. Then $T$ has a hypercyclic vector if and only if for any non-empty open subsets $V$ and $W$ of $\mathcal{H}$ there exists a non-negative integer $n$ with $T^{-n}(V) \cap W \neq \emptyset$.

**Theorem 1.13.** Let $T = U|T|$ (polar decomposition) be invertible in $(\sqrt{\mathcal{H}})$. Then the Aluthge transform of $T$, $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ has no hypercyclic vector.

**Proof.** Assume $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ has a hypercyclic vector. Since $T$ has no hypercyclic vector from the proof of Theorem 1.10, by Lemma 1.12 there exist non-empty open subsets $V$ and $W$ of $\mathcal{H}$ such that $T^{-n}(V) \cap W = \emptyset$ for all non-negative integer $n$. Hence for all non-negative integer $n$, $T^{-n}(V) \subset W_c$ where $W_c = \mathcal{H} \setminus W$. Thus $V \subset T^n(W_c)$ for all non-negative integer $n$. Since $T^n = U|T|^{1/2} \tilde{T}^{n-1}|T|^{1/2}$, we get that for all non-negative integer $n$

$$V \subset U|T|^{1/2} \tilde{T}^{n-1}|T|^{1/2}(W_c),$$

i.e.,

$$|T|^{1/2}(V) \subset \tilde{T}^n|T|^{1/2}(W_c).$$
Hence we have
\[ \hat{T}^{-n}[[T]^{1/2}(V)] \cap [[T]^{1/2}(W^c)]^c = \phi \]
for all non-negative integer \( n \). Since \([|T|^{1/2}(W^c)]^c = |T|^{1/2}(W)\), we obtain
\[ \hat{T}^{-n}[[T]^{1/2}(V)] \cap [[T]^{1/2}(W)] = \phi \]
for all non-negative integer \( n \). Since \([|T|^{1/2}(V)] \) and \([|T|^{1/2}(W)]\) are open, we have the contradiction, because \( \hat{T} \) has a hypercyclic vector. \( \square \)

2. Subscalarity

A bounded linear operator \( S \) on \( \mathcal{H} \) is called scalar of order \( m \) if it possesses a spectral distribution of order \( m \), i.e., if there is a continuous unital morphism,
\[ \Phi : C_0^m(\mathbb{C}) \longrightarrow \mathcal{L}(\mathcal{H}) \]
such that \( \Phi(z) = S \), where \( z \) stands for the identity function on \( \mathbb{C} \) and \( C_0^m(\mathbb{C}) \) for the space of compactly supported functions on \( \mathbb{C} \), continuously differentiable of order \( m \), \( 0 \leq m \leq \infty \). An operator is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

Next we study some cases with subscalarity.

**Theorem 2.1.** Let \( T \in \mathcal{L}(\mathcal{H}) \) be a square root of a hyponormal operator. If one of the following conditions holds:

1. \( T \) is compact,
2. \( T^{2n} \) is normal for some integer \( n \),
3. \( T^* \) is a square root of a hyponormal operator, and
4. \( m(\sigma(T)) = 0 \) where \( m \) is the planar Lebesgue measure, then \( T \) is subscalar.

**Proof.** (1) If \( T \) is compact, then \( T^2 \) is compact and hyponormal. By [3, Corollary 4.9], \( T^2 \) is normal. (2) If \( (T^2)^n \) is normal for some integer \( n \), \( T^2 \) is normal from [14]. (3) If \( T^* \) is a square root of a hyponormal operator, \( T^2 \) is normal. Also (4) if \( m(\sigma(T)) = 0 \) where \( m \) is the planar Lebesgue measure, then \( T^2 \) is normal by [12].

Since \( T^2 \) is normal in any cases, by [13, Theorem 1]
\[ T = A \oplus \begin{pmatrix} B & C \\ 0 & -B \end{pmatrix}, \]
where \( A \) and \( B \) are normal and \( C \) is a positive one-to-one operator commuting with \( B \). By [8, Theorem 4.5], \( T \) is subscalar. \( \square \)
Corollary 2.2. Let \( T \) be a square root of a hyponormal operator. Suppose that \( T \) is compact, or \( T^{2n} \) is normal for some integer \( n \), or \( T^* \) is a square root of a hyponormal operator. If \( \sigma(T) \) has the property that there exists some non-empty open set \( U \) such that \( \sigma(T) \cap U \) is dominating for \( U \), then \( T \) has a nontrivial invariant subspace.

Proof. The proof follows from Theorem 2.1 and [4]. \( \square \)

It is known that a hyponormal and compact operator is normal. But we observe from Theorem 2.1 that a square root of a hyponormal operator, which is compact, is not necessary a normal operator. For example,

\[
T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

is a square root of a hyponormal operator and is a compact operator, but is not necessary a normal operator.

Theorem 2.3. Let \( T \) be in \( \sqrt{\mathcal{H}} \). If \( T \) is quasinilpotent, then \( T \) is subscalar.

Proof. Since \( \sigma(T) = \{0\} \), by the spectral mapping theorem \( \sigma(T^k) = \sigma(T)^k = \{0\} \). Since \( T^k \) is quasinilpotent and hyponormal, \( T^k = 0 \). Since \( T \) is nilpotent, \( T \) is subscalar by [8]. \( \square \)

Recall that an \( X \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) is called a quasiaffinity if it has trivial kernel and dense range. An operator \( A \in \mathcal{L}(\mathcal{H}) \) is said to be a quasiaffine transform of an operator \( T \in \mathcal{L}(\mathcal{K}) \) there exists a quasiaffinity \( X \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \) such that \( XA = TX \).

Corollary 2.4. Let \( T \) be a square root of a hyponormal operator. Suppose that \( T \) is compact, quasinilpotent, or \( T^{2n} \) is normal for some integer \( n \). If \( A \) is any quasiaffine transform of \( T \), then \( \sigma(T) \subset \sigma(A) \).

Proof. It is clear from Theorem 2.1, Theorem 2.3, and [9]. \( \square \)

References


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