ON A FUNCTIONAL CENTRAL LIMIT THEOREM FOR STATIONARY LINEAR PROCESSES GENERATED BY ASSOCIATED PROCESSES

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ABSTRACT. A functional central limit theorem is obtained for a stationary linear process of the form \( X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j} \), where \( \{\epsilon_t\} \) is a strictly stationary associated sequence of random variables with \( \text{E} \epsilon_t = 0 \), \( \text{E}(\epsilon_t^2) < \infty \) and \( \{a_j\} \) is a sequence of real numbers with \( \sum_{j=0}^{\infty} |a_j| < \infty \). A central limit theorem for a stationary linear process generated by stationary associated processes is also discussed.

1. Introduction and main results

A finite collection of random variables \( \{\epsilon_1, \ldots, \epsilon_m\} \) is said to be associated if for any two coordinatewise nondecreasing functions \( f_1, f_2 \) on \( \mathbb{R}^m \) such that \( \tilde{f}_j = f_j(\epsilon_1, \ldots, \epsilon_m) \) has finite variance for \( j = 1, 2 \), \( \text{cov}(\tilde{f}_1, \tilde{f}_2) \geq 0 \). An infinite collection of random variables is said to be associated if every finite subcollection of random variables is associated. This definition was introduced by Esary, Proschan and Walkup ([2]) as an extension of the bivariate notion of positive quadrant dependence of Lehmann ([7]). A large amount of papers has been concerned with limit theorems for associated processes (see, for example, Newman ([8], [9]).

Let \( \{X_t, t \in \mathbb{Z}^+\} \) be a stationary linear process defined on a probability space \( (\Omega, \mathcal{F}, P) \) of the form

\[
X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j},
\]

Received January 16, 2001.

2000 Mathematics Subject Classification: 60F05, 60F17.

Key words and phrases: central limit theorem, functional central limit theorem, linear process, associated.

†This paper was partially supported by Korea Research Foundation Grant (KRF-2002-042).

†This paper was partially supported by Statistical Research Center for Complex Systems, Seoul National University.
where \( \{a_j\} \) is a sequence of real numbers with \( \sum_{j=0}^{\infty} |a_j| < \infty \) and \( \{\epsilon_t\} \) is a strictly stationary process such that \( E\epsilon_t = 0 \) and \( 0 < E\epsilon_t^2 < \infty \).

The linear processes are special importance in time series analysis and they arise from a wide variety of contexts (see, e.g., Hannan [6] Ch.6). Applications to economics, engineering and physical sciences are extremely broad and a vast amount of literature is devoted to the study of the limit theorems for linear processes under various conditions on \( \epsilon_t \). For the linear processes, Fakhre-Zakeri and Lee ([4]) and Fakhre-Zakeri and Farshidi ([3]) established a central limit theorem (CLT) under the iid assumption on \( \epsilon_t \) and Fakhre-Zakeri and Lee ([5]) proved a functional central limit theorem (FCLT) under the strong mixing condition on \( \epsilon_t \).

Let \( S_n = \sum_{t=1}^{n} X_t \) and \( \tau^2 = \sigma^2(\sum_{j=0}^{\infty} a_j)^2 \). Define, for \( n \geq 1 \), the stochastic process

\[
\xi_n(u) = n^{-1} \tau^{-1} S_{\lfloor nu \rfloor}, \quad u \in [0, 1],
\]

where \( \lfloor x \rfloor \) is the greatest integer not exceeding \( x \).

In this paper, we establish a CLT (FCLT) for a strictly stationary linear process of the form (1), generated by an associated process \( \{\epsilon_t\} \). More precisely, we will prove the following theorems:

**Theorem 1.** Let \( \{X_t\} \) be a stationary linear process of the form (1), where \( \{a_j\} \) is a sequence of constants with \( \sum_{j=0}^{\infty} |a_j| < \infty \) and \( \{\epsilon_t\} \) is a strictly stationary associated process with \( E\epsilon_t = 0 \), \( 0 < E\epsilon_t^2 < \infty \). Assume

\[
0 < \sigma^2 = E\epsilon_t^2 + 2 \sum_{t=2}^{\infty} E(\epsilon_t \epsilon_{t-1}) < \infty.
\]

Then the linear process \( \{X_t\} \) fulfills the CLT.

**Theorem 2.** Let \( \{X_t\} \) be a stationary linear process of the form (1) defined in Theorem 1. If (3) fulfilled then the process \( \{\xi_n\} \) satisfies the FCLT, that is, the process \( \{\xi_n\} \) converges weakly to Wiener measure \( W \) on the space of all functions on \( [0, 1] \), which have left hand limits and are continuous from the right.

2. Proofs

The following lemma needs to prove Theorems 1 and 2 and it is proved by modifying the proof of Lemma 3 in Fakhre-Zakeri and Lee ([5]). Doob’s maximal inequality played important role in their proof. However, in our case, Newman and Wright’s maximal inequality \( E(\max_{1 \leq k \leq n} \)
\(|\epsilon_1 + \cdots + \epsilon_k|^2 \leq n\sigma^2\) (see Theorem 2 of Newman and Wright ([10])) will be used.

**Lemma 1.** Let \(\{\epsilon_t\}\) be a strictly stationary associated process with \(E\epsilon_t = 0, 0 < E\epsilon_t^2 < \infty\). Let \(\hat{X}_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}, \hat{S}_k = \sum_{t=1}^{k} \hat{X}_t, \hat{X}_t = \left(\sum_{j=0}^{\infty} a_j\right) \epsilon_t\) and \(\hat{S}_k = \sum_{t=1}^{k} \hat{X}_t\), where \(\{a_j\}\) is a sequence of real numbers with \(\sum_{j=0}^{\infty} |a_j| < \infty\). If (3) are fulfilled, then

\[
\left(\frac{n^{-\frac{1}{2}}}{1 \leq k \leq n}\right) \max |\hat{S}_k - S_k| \overset{P}{\rightarrow} 0.
\]

**Proof.** See Appendix. \(\square\)

**Proof of Theorem 1.** As in Lemma 1 set

\[
\hat{X}_t = \sum_{j=0}^{\infty} a_j \epsilon_t
\]

and

\[
\hat{S}_n = \sum_{t=1}^{n} \hat{X}_t = \left(\sum_{j=0}^{\infty} a_j\right) \sum_{t=1}^{n} \epsilon_t.
\]

Then

\[
E(\hat{X}_t)^2 = E\left(\sum_{j=0}^{\infty} a_j \epsilon_t\right)^2
\]

\[
= \left(\sum_{j=0}^{\infty} a_j\right)^2 E\epsilon_t^2
\]

\[
\leq \left(\sum_{j=0}^{\infty} |a_j|\right)^2 E\epsilon_t^2 < \infty,
\]

\[
E\hat{X}_1^2 + 2 \sum_{t=2}^{\infty} E(\hat{X}_1 \hat{X}_t) = \left(\sum_{j=0}^{\infty} a_j\right)^2 E\epsilon_t^2 + 2 \left(\sum_{j=0}^{\infty} a_j\right)^2 \sum_{t=2}^{\infty} E(\epsilon_1 \epsilon_t)
\]

\[
= \left(\sum_{j=0}^{\infty} a_j\right)^2 \sigma^2 = \tau^2 < \infty \text{ by (3)}
\]

and \(\hat{X}_t\)'s are stationary associated process (see [2]). Thus \(\{\hat{X}_t, t \in \mathbb{Z}^+\}\) satisfies the CLT by Theorem 12 of [9], that is,

\[
n^{-\frac{1}{2}} \hat{S}_n \overset{D}{\rightarrow} N(0, \tau^2).
\]
According Lemma 1 we also have

\[ n^{-\frac{1}{2}} |\tilde{S}_n - S_n| \xrightarrow{P} 0. \]  

Hence from (7) and (8) the desired conclusion follows. \qed

Proof of Theorem 2. Note that \( \{\tilde{X}_t\} \) is a stationary associated process and that \( \{X_t\} \) satisfies conditions of Theorem 3 of Newman and Wright ([10]) according to (5) and (6). This implies that Theorem 2 holds for the sequence \( \{\xi_n\} \), where we define \( \xi_n \) as in (2), but \( S_{[nu]} \) replacing by \( S_{[nu]} \). By Lemma 1 \( |\tilde{\xi}_n(u) - \xi_n(u)| \xrightarrow{P} 0 \) for all \( 0 \leq u \leq 1 \). Hence, the desired conclusion follows. \qed

Appendix

Proof of Lemma 1. Like in the proof of Lemma 3 of [5] we have

\[
\tilde{S}_k = \sum_{t=1}^{k} \left( \sum_{j=0}^{k-t} a_j \right) \epsilon_t + \sum_{t=1}^{k} \left( \sum_{j=k-t+1}^{\infty} a_j \right) \epsilon_t
\]

\[
= \sum_{t=1}^{k} \left( \sum_{j=0}^{t-1} a_j \epsilon_{t-j} \right) + \sum_{t=1}^{k} \left( \sum_{j=k-t+1}^{\infty} a_j \right) \epsilon_t.
\]

Thus

\[
\tilde{S}_k - S_k = - \sum_{t=1}^{k} \left( \sum_{j=t}^{\infty} a_j \epsilon_{t-j} \right) + \sum_{t=1}^{k} \left( \sum_{j=k-t+1}^{\infty} a_j \right) \epsilon_t
\]

\[
= I + II \text{ (say)}.
\]

It suffices to prove

\[(A.1) \quad n^{-\frac{1}{2}} \max_{1 \leq k \leq n} |I| \xrightarrow{P} 0,\]

and

\[(A.2) \quad n^{-\frac{1}{2}} \max_{1 \leq k \leq n} |II| \xrightarrow{P} 0.\]
First we have for
\[
\begin{align*}
n^{-1}E \max_{1 \leq k \leq n} \left| \sum_{t=1}^{k} \sum_{j=t}^{\infty} a_{j} \epsilon_{t-j} \right|^{2} \\
= n^{-1}E \max_{1 \leq k \leq n} \left( \sum_{j=1}^{\infty} \sum_{t=1}^{j \wedge k} a_{j} \epsilon_{t-j} \right)^{2} \\
\leq n^{-1} \left( \sum_{j=1}^{\infty} |a_{j}| \left\{ E \max_{1 \leq k \leq n} \left| \sum_{t=1}^{j \wedge k} \epsilon_{t-j} \right|^{2} \right\}^{\frac{1}{2}} \right)^{2}
\end{align*}
\]
(A.3) (by Minkowski’s inequality)
\[
\leq n^{-1} \left( \sum_{j=1}^{\infty} |a_{j}| \sigma(j \wedge n)^{\frac{1}{2}} \right)^{2}
\]
(by (3) and Theorem 2 of [10])
\[
= \left( \sum_{j=1}^{\infty} |a_{j}| \sigma((j \wedge n)/n)^{\frac{1}{2}} \right)^{2}
\]
(by the dominated convergence theorem)
\[
= o(1).
\]
Hence (A.1) is proved by Markov inequality. To prove (A.2) write
\[
II = II_{k1} + II_{k2},
\]
where
\[
II_{k1} = a_{1} \epsilon_{k} + a_{2}(\epsilon_{k} + \epsilon_{k-1}) + \cdots + a_{k}(\epsilon_{k} + \cdots + \epsilon_{1})
\]
and
\[
II_{k2} = (a_{k+1} + a_{k+2} + \cdots)(\epsilon_{k} + \cdots + \epsilon_{1}),
\]
and let \(\{p_{n}\}\) be a sequence of positive integers such that
\[
(A.4) \quad p_{n} \to \infty \text{ and } p_{n}/n \to 0 \text{ as } n \to \infty.
\]
Then
\[
\begin{align*}
& n^{-\frac{1}{2}} \max_{1 \leq k \leq n} |I_{I_k2}| \\
& \leq \left( \sum_{j=0}^{\infty} |a_j| \right) n^{-\frac{1}{2}} \max_{1 \leq k \leq p_n} |\epsilon_1 + \cdots + \epsilon_k| \\
& + \left( \sum_{j>p_n} |a_j| \right) n^{-\frac{1}{2}} \max_{1 \leq k \leq n} |\epsilon_1 + \cdots + \epsilon_k| \\
& = III + IV \text{ (say)}.
\end{align*}
\]

It follows from (3) and (A.4) that
\[
\begin{align*}
& \left( \sum_{j=0}^{\infty} |a_j| \right)^2 n^{-1} E \max_{1 \leq k \leq p_n} |\epsilon_1 + \cdots + \epsilon_k|^2 \\
& \leq \left( \sum_{j=0}^{\infty} |a_j| \right)^2 \sigma^2 (p_n/n) = o(1)
\end{align*}
\]
by Theorem 2 of Newman and Wright ([10]) and thus \( III \overset{P}{\rightarrow} 0 \) by Markov inequality. Similarly, by assumption \( \sum_{j=0}^{\infty} |a_j| < \infty \) and Theorem 2 of Newman and Wright ([10])

\[
\begin{align*}
& \left( \sum_{j>p_n} |a_j| \right)^2 n^{-1} E \max_{1 \leq k \leq n} |\epsilon_1 + \cdots + \epsilon_k|^2 \\
& \leq \left( \sum_{j>p_n} |a_j| \right)^2 \sigma^2 = o(1)
\end{align*}
\]
and thus \( IV \overset{P}{\rightarrow} 0 \) by Markov inequality. Hence, \( n^{-\frac{1}{2}} \max_{1 \leq k \leq n} |I_{I_k2}| \overset{P}{\rightarrow} 0 \). It remains to show that \( L_n = n^{-\frac{1}{2}} \max_{1 \leq k \leq n} |I_{I_k2}| \overset{P}{\rightarrow} 0 \). For each \( m \geq 1 \), define \( I_{I_{k1,m}} = b_1 \epsilon_k + b_2 (\epsilon_k + \epsilon_{k-1}) + \cdots + b_k (\epsilon_k + \cdots + \epsilon_1) \), where \( b_k = a_k \) for \( k \leq n \) and \( b_k = 0 \) otherwise and let \( L_{n,m} = n^{-\frac{1}{2}} \max_{1 \leq k \leq n} |I_{I_{k1,m}}| \). Then
\[
(A.6) \quad L_{n,m} \leq n^{-\frac{1}{2}} (|a_1| + \cdots + |a_m|)(|\epsilon_1| + \cdots + |\epsilon_m|) \overset{P}{\rightarrow} 0
\]
as \( n \to \infty \) for each \( m \), and

\[
(A.7) \quad |L_{n,m} - L_n| \leq n^{-\frac{1}{2}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (a_i - b_i)(\epsilon_k + \cdots + \epsilon_{k-i+1}) \right|.
\]

Since

\[
\left| \sum_{i=1}^{k} (a_i - b_i)(\epsilon_k + \cdots + \epsilon_{k-i+1}) \right| = \begin{cases} 
0, & k \leq m \\
\left| \sum_{i=m+1}^{k} a_i(\epsilon_k + \cdots + \epsilon_{k-i+1}) \right|, & \text{otherwise},
\end{cases}
\]
the right-hand side of \((A.7)\)

\[
\leq n^{-\frac{1}{2}} \max_{m < k \leq n} \left( \sum_{i=m+1}^{k} |a_i| |\epsilon_k + \cdots + \epsilon_{k-i+1}| \right)
\]

\[
\leq n^{-\frac{1}{2}} \max_{m < k \leq n} \sum_{i=m+1}^{k} |a_i| \max_{m < l \leq k} |\epsilon_k + \cdots + \epsilon_{k-l}|.
\]

\[
(A.8)
\]

\[
\leq n^{-\frac{1}{2}} \sum_{i > m} |a_i| \max_{m < k \leq n} \max_{m < l \leq k} (|\epsilon_1 + \cdots + \epsilon_k| + |\epsilon_1 + \cdots + \epsilon_{k-l}|)
\]

\[
\leq n^{-\frac{1}{2}} \sum_{i > m} |a_i| \left( \max_{1 \leq i \leq n} |\epsilon_1 + \cdots + \epsilon_{i}| + \max_{1 \leq i \leq n} |\epsilon_1 + \cdots + \epsilon_{i-1}| \right)
\]

\[
= 2n^{-\frac{1}{2}} \sum_{i > m} |a_i| \max_{1 \leq j \leq n} |\epsilon_1 + \cdots + \epsilon_{j}|.
\]

Therefore, by Theorem 2 of Newman and Wright ([10]) it follows from \((A.6), (A.8)\) and Markov inequality that for any \( \delta > 0 \),

\[
\lim_{m \to \infty} \lim_{n \to \infty} \sup P(|L_{n,m} - L_n| > \delta)
\]

\[
\leq \lim_{m \to \infty} 2^2 \delta^2 \left( \sum_{j > m} |a_j| \right)^2 \lim_{n \to \infty} n^{-1} E \max_{1 \leq j \leq n} |\epsilon_1 + \cdots + \epsilon_{j}|^2
\]
\leq \sigma \lim_{m \to \infty} \delta^2 \cdot 2^2 \left( \sum_{j > m} |a_j| \right)^2 \quad (3) \text{ and Theorem 2 of [10]}

(A.9)

= 0 \quad \text{by assumption} \sum_{j=0}^{\infty} |a_j| < \infty.

In view of (A.6) and (A.9) it follows from Theorem 4.2 of Billingsley ([1], p.25) that \( L_n \xrightarrow{P} 0 \) and thus (A.2) is proved. The proof of lemma now completes. \qed

Acknowledgement. The authors thank Young Min Lee who worked as RA in this research.

References


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