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# $E_N^n$ 상의 비선형 퍼지 제어시스템에 대한 제어가능성

# The exact controllability for the nonlinear fuzzy control system in $E_N^n$.

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**Abstract**

This paper we study the exact controllability for the nonlinear fuzzy control system in $E_N^n$ by using the concept of fuzzy number of dimension $n$ whose values are normal, convex, upper semicontinuous and compactly supported surface in $R^n$ .

**Keywords and Phrases** : fuzzy number of dimension $n$ , fuzzy control, nonlinear fuzzy control system, exact controllability

## 1. Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva [3] studied the existence and uniqueness of solution for the fuzzy differential equation on $E^n$ where $E^n$ is normal, convex, upper semicontinuous and compactly supported fuzzy sets in $R^n$ . Seikkala [5] proved the existence and uniqueness of fuzzy solution for the following equation:

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(0) = x_0, \end{cases}$$

where $f$ is a continuous mapping from $R^+ \times R$ into $R$ and $x_0$ is a fuzzy number in $E^1$ . Diamond and Kloeden [2] proved the fuzzy optimal control for the following system:

$$\begin{cases} \dot{x}(t) = a(t)x(t) + u(t), \\ x(0) = x_0 \end{cases}$$

where $x(\cdot)$, $u(\cdot)$ are nonempty compact interval-valued functions on $E^1$ .

We consider the exact controllability for the following nonlinear fuzzy control system:

$$\text{(F.C.S.)} \quad \begin{cases} \dot{x}(t) = a(t)x(t) + f(t, x(t)), \\ x(0) = x_0, \end{cases}$$

where $a: [0, T] \to E_N^n$ is fuzzy coefficient, initial value $x_0 \in E_N^n$ and $f: [0, T] \times E_N^n \to E_N^n$ is nonlinear function and $u(t) \in E_N^n$ is control function.

Let $E_N^n$ be the set of all fuzzy numbers in $R^n$ with edges having bases parallel to axis $X_1, \cdots, X_n$ .

For example, $E_N^2$ be the set of all fuzzy pyramidal numbers in $R^2$ with edges having rectangular bases parallel to the axis $X_1$ and $X_2$ [4].

## 2. Properties of fuzzy numbers

In this section, we give some definitions, properties and notations of the fuzzy number of dimension $n$ .

**Definition 2.1.** We consider a fuzzy graph $G \subset R^n$ that is a functional fuzzy relation in $R^n$ such that its membership function

$\mu_G(x_1, \cdots, x_n) \in [0, 1]$, $(x_1, \cdots, x_2) \in R^n$ has the following properties:

1. For all $x_i \in R$, $(i = 1, \cdots, n)$,

$$\mu_G(x_1, \cdots, x_i, \cdots, x_n) \in [0, 1]$$

is a convex membership function.

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2. For all $\alpha\in[0,1]$,

$$\{(x_1,\cdots,x_n)\in R^n: \mu_G(x_1,\cdots,x_n)=\alpha\}$$

is a convex set.

3. There exists $(x_1,\cdots,x_n)\in R^n$,

$$\mu_G(x_1,\cdots,x_n)=1 .$$

If the above conditions are satisfied, the fuzzy subset $G$ is called a fuzzy number of dimension $n$.

The first projection of $G$ is

$$\vee_{\{x_2,\cdots,x_n\}}\mu_G(x_1,\cdots,x_n)=\mu_{A_1}(x_1),$$

the second projection of $G$ is

$$\vee_{\{x_1,x_3,\cdots,x_n\}}\mu_G(x_1,\cdots,x_n)=\mu_{A_2}(x_2)$$

and the $i$-th projection of $G$ is

$$\vee_{\{x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n\}}\mu_G(x_1,\cdots,x_n)=\mu_{A_i}(x_i),$$

$(i=3,\cdots,n)$.

We denote by fuzzy number in

$$E_N^n\ A=(a_1,a_2,\cdots,a_n),$$

where $a_i$ is projection of $A$ to axis $X_i$ $(i=1,\cdots,n)$, respectively.

And $a_i$ $(i=1,\cdots,n)$ is fuzzy number in $R$.

**Definition 2.2.** The $\alpha$-level set of fuzzy number in $E_N^n$ is defined by

$$[A]^\alpha=\{(x_1,\cdots,x_n)\in R^n: (x_1,\cdots,x_n)\in\prod_{i=1}^n[a_i]^\alpha\},$$

where notation $\prod$ is the Cartesian product of sets.

**Definition 2.3.** Let $A$ and $B$ in $E_N^n$, for all $\alpha\in(0,1]$,

(2.1) $$A=B\Leftrightarrow[A]^\alpha=[B]^\alpha .$$

(2.2) $$[A*_nB]^\alpha=\prod_{i=1}^n[a_i*b_i]^\alpha ,$$

where $*_n$ is operation in $E_N^n$ and $*$ is operation in $E_N$.

**Definition 2.4.** The derivative $x'(t)$ of a fuzzy process $x\in E_N^n$ is defined by

$$[x'(t)]^\alpha=\prod_{i=1}^n[(x_{il}^\alpha)'(t),(x_{ir}^\alpha)'(t)],\ 0<\alpha\le 1$$

provided that is equation defines a fuzzy $x'(t)\in E_N^n$.

The fuzzy integral $\int_a^b x(t)dt$, $a,b\in I$ is defined by

$$[\int_a^b x(t)dt]^\alpha=\prod_{i=1}^n[\int_a^b x_{il}^\alpha(t)\,dt,\ \int_a^b x_{ir}^\alpha(t)\,dt]$$

provided that the Lebesgue integrals on the right exist.

Let $\prod_{i=1}^n[a_i]^\alpha$, $0<\alpha\le1$, be a given family of nonempty areas.

If

(2.3) $$\prod_{i=1}^n[a_i]^\beta\subset\prod_{i=1}^n[a_i]^\alpha \text{ for } 0<\alpha<\beta<1 \text{ and}$$

(2.4) $$\prod_{i=1}^n\lim_{k\to\infty}[a_i]^{\alpha_k}=\prod_{i=1}^n[a_i]^\alpha$$

whenever $(\alpha_k)$ is a nondecreasing sequence converging to $\alpha\in(0,1]$, then the family

$\prod_{i=1}^n[a_i]^\alpha$, $0<\alpha\le1$, represents the $\alpha$-level sets of a fuzzy number $A\in E_N^n$.

Conversely, if $\prod_{i=1}^n[a_i]^\alpha$, $0<\alpha\le1$, are the $\alpha$-level sets of a fuzzy number in $R^n$, then the conditions (2.3) and (2.4) hold true.

We define the metric $d_\infty$ on $E_N^n$.

**Definition 2.5.** Let $A$, $B\in E_N^n$.

$$\begin{aligned} d_\infty(A,B) &= \sup\{d_H([A]^\alpha,[B]^\alpha):\alpha\in(0,1]\} \\ &= \sup\{d_H(\prod_{i=1}^n[a_i]^\alpha,\prod_{i=1}^n[b_i]^\alpha):\alpha\in(0,1]\} \\ &= \sup\{\sqrt{\sum_{i=1}^n(d_H([a_{il}^\alpha],[b_{il}^\alpha]))^2}:\alpha\in(0,1]\} \end{aligned}$$

where $d_H$ is the Hausdorff distance.

The supremum metric $H$ on $C([0,T]:E_N^n)$ is defined by

$$H(x,y)=\sup\{d_\infty(x(t),y(t)):t\in[0,T]\}$$

for all $x,y\in C([0,T]:E_N^n)$.

## 3. The exact controllability

In this section, we show the exact controllability for the following nonlinear fuzzy control system:

$$\text{(F.C.S.)}\begin{cases}\dot{x}(t)=a(t)x(t)+f(t,x(t)),\\ x(0)=x_0,\end{cases}$$

with fuzzy coefficient $a:[0,T]\to E_N^n$, initial value $x_0\in E_N^n$, control $u:[0,T]\to E_N^n$ and inhomogeneous term $f:[0,T]\times E_N^n\to E_N^n$ satisfies a global Lipschitz condition.

The (F.C.S.) is related to the following fuzzy integral system:

$$\text{(F.I.S.)}\begin{cases}x(t)=S(t)x_0+\int_0^t S(t-s)f(s,x(s))ds\\ \qquad+\int_0^t S(t-s)u(s)ds,\\ x(0)=x_0\in E_N^n,\end{cases}$$

where $S(t)$ is fuzzy number of dimension $n$ and

$$[S(t)]^\alpha = \prod_{i=1}^{n}[S_i(t)]^\alpha = \prod_{i=1}^{n}[S_{il}^\alpha(t),\ S_{ir}^\alpha(t)]$$

where $S_{il}^\alpha(t)$ is $\exp\{\int_0^t a_l^\alpha(s)ds\}$ and $S_{ir}^\alpha(t)$ is $\exp\{\int_0^t a_r^\alpha(s)ds\}$. $S_{ij}^\alpha(t)$ $(j=l, r)$ is continuous. That is, there exists a constant $c>0$ such that $|S_{ij}^\alpha(t)| \le c$ for all $t \in [0, T]$.

**Definition 3.1.** The (F.I.S.) is exact controllable if, there exists $u(t)$ such that the fuzzy solution $x(t)$ of (F.I.S.) satisfies

$$x(T) = {}_\alpha x^1 \ \text{( i.e.,}$$
$$[x(T)]^\alpha = \prod_{i=1}^{n}[x_i(T)]^\alpha = \prod_{i=1}^{n}[(x^1)_i]^\alpha = [x^1]^\alpha)$$

where $x^1$ is target set.

We assume that the following linear fuzzy control system with respect to nonlinear fuzzy control system (F.C.S.):

$$\text{(F.C.S. 1)}\begin{cases} \dot{x}(t) = a(t)x(t) + u(t), \\ x(0) = x_0 \in E_N^n \end{cases}$$

is exact controllable. Then

$$x(T) = S(T)x_0 + \int_0^T S(T-s)u(s)ds = {}_\alpha x^1$$

and

$$\begin{aligned} &[x(T)]^\alpha \\ &= \prod_{i=1}^{n}[S_i(T)(x_0)_i + \int_0^T S_i(T-s)u_i(s)ds]^\alpha \\ &= \prod_{i=1}^{n}[S_{il}^\alpha(T)(x_0)_{il}^\alpha + \int_0^T S_{il}^\alpha(T-s)u_{il}^\alpha(s)ds, \\ &\qquad S_{ir}^\alpha(T)(x_0)_{ir}^\alpha + \int_0^T S_{ir}^\alpha(T-s)u_{ir}^\alpha(s)ds] \\ &= \prod_{i=1}^{n}[(x^1)_{il}^\alpha,\ (x^1)_{ir}^\alpha] = [x^1]^\alpha. \end{aligned}$$

Defined the fuzzy mapping $\tilde{g}: \tilde{P}(R^n) \to E_N^n$ by

$$\tilde{g}^\alpha(v) = \begin{cases} \int_0^T S^\alpha(T-s)v(s)ds, & v \subset \overline{\Gamma_u}, \\ 0, & \text{otherwise}. \end{cases}$$

Then there exists

$\tilde{g}_i: \tilde{P}(R) \to E_N (i=1, 2, \cdots, n)$ such that

$$\tilde{g}_i^\alpha(v_i) = \begin{cases} \int_0^T S_i^\alpha(T-s)v_i(s)ds, & v_i(s) \subset \overline{\Gamma_{u_i}}, \\ 0, & \text{otherwise} \end{cases}$$

where $u_i$ is projection of $u$ to axis $X_i$, $(i=1, \cdots, n)$ respectively and

there exists $\tilde{g}_{ij^\alpha}$ $(j=l, r)$

$$\tilde{g}_{il}^\alpha(v_{il}) = \int_0^T S_{il}^\alpha(T-s)v_{il}(s)ds,$$
$$v_{il}(s) \in [u_{il}^\alpha(s),\ u_i^1(s)],$$
$$\tilde{g}_{ir}^\alpha(v_{ir}) = \int_0^T S_{ir}^\alpha(T-s)v_{ir}(s)ds,$$
$$v_{ir}(s) \in [u_i^1(s),\ u_{ir}^\alpha(s)].$$

We assume that $\tilde{g}_{il}^\alpha$, $\tilde{g}_{ir}^\alpha$ are bijective mappings. Hence $\alpha$-level of $u(s)$ are

$$\begin{aligned} [u(s)]^\alpha &= \prod_{i=1}^{n}[u_i(s)]^\alpha = \prod_{i=1}^{n}[u_{il}^\alpha(s), u_{ir}^\alpha(s)] \\ &= \prod_{i=1}^{n}[(\tilde{g}_{il}^\alpha)^{-1}((x^1)_{il}^\alpha - S_{il}^\alpha(T)(x_0)_{il}^\alpha), \\ &\quad (\tilde{g}_{ir}^\alpha)^{-1}((x^1)_{ir}^\alpha - S_{ir}^\alpha(T)(x_0)_{ir}^\alpha)]. \end{aligned}$$

Thus we can be introduced $u(s)$ of nonlinear system

$$\begin{aligned} [u(s)]^\alpha &= \prod_{i=1}^{n}[u_i(s)]^\alpha = \prod_{i=1}^{n}[u_{il}^\alpha(s),\ u_{ir}^\alpha(s)] \\ &= \prod_{i=1}^{n}[(\tilde{g}_{il}^\alpha)^{-1}((x^1)_{il}^\alpha - S_{il}^\alpha(T)(x_0)_{il}^\alpha \\ &\quad - \int_0^T S_{il}^\alpha(T-s)f_{il}^\alpha(s, x_{il}^\alpha(s))ds), \\ &\quad (\tilde{g}_{ir}^\alpha)^{-1}((x^1)_{ir}^\alpha - S_{ir}^\alpha(T)(x_0)_{ir}^\alpha \\ &\quad - \int_0^T S_{ir}^\alpha(T-s)f_{ir}^\alpha(s, x_{ir}^\alpha(s))ds)]. \end{aligned}$$

Then substituting this expression into the (F.I.S.) yields $\alpha$-level of $x(T)$. For each $i=1, \cdots, n$,

$$\begin{aligned} &[x_i(T)]^\alpha \\ &= [S_{il}^\alpha(T)(x_0)_{il}^\alpha + \int_0^T S_{il}^\alpha(T-s)f_{il}^\alpha(s, x_{il}^\alpha(s))ds \\ &\quad + \int_0^T S_{il}^\alpha(T-s)(\tilde{g}_{il}^\alpha)^{-1}((x^1)_{il}^\alpha - S_{il}^\alpha(T)(x_0)_{il}^\alpha \\ &\quad - \int_0^T S_{il}^\alpha(T-s)f_{il}^\alpha(s, x_{il}^\alpha(s))ds)ds, \\ &\quad S_{ir}^\alpha(T)(x_0)_{ir}^\alpha + \int_0^T S_{ir}^\alpha(T-s)f_{ir}^\alpha(s, x_{ir}^\alpha(s))ds \\ &\quad + \int_0^T S_{ir}^\alpha(T-s)(\tilde{g}_{ir^\alpha})^{-1}((x^1)_{ir}^\alpha - S_{ir}^\alpha(T)(x_0)_{ir}^\alpha \\ &\quad - \int_0^T S_{ir}^\alpha(T-s)f_{ir}^\alpha(s, x_{ir}^\alpha(s))ds)ds] \\ &= [(x^1)_{il}^\alpha,\ (x^1)_{ir}^\alpha] = [(x^1)_i]^\alpha \end{aligned}$$

Therefore

$$[x(T)]^\alpha = \prod_{i=1}^{n}[x_i(T)]^\alpha = \prod_{i=1}^{n}[(x^1)_i]^\alpha = [x^1]^\alpha.$$

We now set

$$\begin{aligned} (\Phi x)(t) &= {}_\alpha S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds \\ &\quad + \int_0^t S(t-s)\ \tilde{g}^{-1}(x^1 - S(T)x_0 \\ &\quad - \int_0^T S(T-s)f(s, x(s))ds)ds. \end{aligned}$$

where the fuzzy mappings $\tilde{g}^{-1}$ satisfied above statements.

Notice that $(\Phi x)(T) =_a x^1$, which means that the control $u(t)$ steers the (F.C.S.) from the origine to $x^1$ in time $T$ provided we can obtain a fixed point of the operator $\Phi$.

Assume that the following hypotheses:

(H1) (F.C.S. 1) is exact controllable.

(H2) Inhomogeneous term $f: [0, T] \times E_N^n \to E_N^n$ satisfies a global Lipschitz condition, there exists a finite constant $k_i > 0$ such that

$$d_H([f_i(s, x(s))]^\alpha, [f_i(s, y(s))]^\alpha) \le k_i\, d_H([x_i(s)]^\alpha, [y_i(s)]^\alpha) \tag{3.2}$$

for all $x_i(s), y_i(s) \in E_N$ and $f_i: [0, T] \times E_N \to E_N$ $(i = 1, \cdots, n)$ is the $i$-th projection of $f$.

We denote $k = \max\{k_i | i = 1, \cdots, n\}$.

**Theorem 3.1.** Suppose that hypotheses (H1), (H2) are satisfied. Then the state of the (F.I.S.) can be steered from the initial value $x_0$ to any final state $x^1$ in time $T$.

Proof. The continuous function from $C([0, T] : E_N^n)$ to itself defined by

$$\begin{aligned}(\Phi x)(t) &=_a S(t)x_0 + \int_0^t S(t-s) f(s, x(s))\, ds \\ &+ \int_0^t S(t-s)\, \tilde{g}^{-1}(x^1 - S(T)x_0 \\ &- \int_0^T S(T-s) f(s, x(s)) ds) ds.\end{aligned}$$

There exist $\Phi_i (i = 1, \cdots, n)$ is continuous function from $C([0, T] : E_N)$ to itself.

Let $x, y \in C([0, T] : E_N^n)$ there exist $(i = 1, \cdots, n)$

$x_i, y_i \in C([0, T] : E_N)$.

$$\begin{aligned}&d_H([\Phi_i x_i(t)]^\alpha, [\Phi_i y_i(t)]^\alpha) \\ &= d_H([S_i(t)(x_0)_i + \int_0^t S_i(t-s) f_i(s, x_i(s))\, ds \\ &+ \int_0^t S_i(t-s)\, \tilde{g}_i^{-1}((x^1)_i - S_i(T)(x_0)_i \\ &- \int_0^T S_i(T-s) f_i(s, x_i(s))\, ds) ds]^\alpha, \\ &[S_i(t)(x_0)_i + \int_0^t S_i(t-s) f_i(s, y_i(s))\, ds \\ &+ \int_0^t S_i(t-s)\, \tilde{g}_i^{-1}((x^1)_i - S_i(T)(x_0)_i \\ &- \int_0^T S_i(T-s) f_i(s, y_i(s))\, ds) ds]^\alpha) \le d_H \\ &([\int_0^t S_i(t-s) f_i(s, x_i(s))\, ds]^\alpha, \\ &[\int_0^t S_i(t-s) f_i(s, y_i(s))\, ds]^\alpha) \\ &+ d_H([\int_0^T S_i(T-s)\, \tilde{g}_i^{-1}(\int_0^T S_i(T-s) \\ &f_i(s, x_i(s))\, ds) ds]^\alpha, [\int_0^T S_i(T-s)\, \tilde{g}_i^{-1} \\ &(\int_0^T S_i(T-s) f_i(s, y_i(s))\, ds) ds]^\alpha) \\ &\le d_H(\int_0^t [S_i(t-s) f_i(s, x_i(s))]^\alpha ds, \\ &\int_0^t [S_i(t-s) f_i(s, y_i(s))]^\alpha ds) \\ &+ d_H([\tilde{g}_i(\tilde{g}_i^{-1}(\int_0^T S_i(T-s) f_i(s, x_i(s))\, ds))]^\alpha, \\ &[\tilde{g}_i(\tilde{g}_i^{-1}(\int_0^T S_i(T-s) f_i(s, y_i(s))\, ds))]^\alpha) \\ &\le \int_0^t d_H([S_i(t-s) f_i(s, x_i(s))]^\alpha, \\ &[S_i(t-s) f_i(s, y_i(s))]^\alpha) ds \\ &+ \int_0^T d_H([S_i(T-s) f_i(s, x_i(s))]^\alpha, \\ &[S_i(T-s) f_i(s, y_i(s))]^\alpha) ds \\ &\le c k_i \int_0^t d_H([x_i(s)]^\alpha, [y_i(s)]^\alpha) ds \\ &+ c k_i \int_0^T d_H([x_i(s)]^\alpha, [y_i(s)]^\alpha) ds \\ &\le 2 c k_i T d_H([x_i(s)]^\alpha, [y_i(s)]^\alpha).\end{aligned}$$

Therefore

$$\begin{aligned}&H_1((\Phi x)(t), (\Phi y)(t)) \\ &= \sup_{t \in [0, T]} d_\infty((\Phi x)(t), (\Phi y)(t)) \\ &= \sup_{t \in [0, T], \alpha \in (0, 1]} d_H([(\Phi x)(t)]^\alpha, [(\Phi y)(t)]^\alpha) \\ &= \sup_{t \in [0, T], \alpha \in (0, 1]} \sqrt{\sum_{i=1}^n (d_H([(\Phi_i x_i)(t)]^\alpha, [(\Phi_i y_i)(t)]^\alpha))^2} \\ &\le 2ckT \sup_{t \in [0, T], \alpha \in (0, 1]} \sqrt{\sum_{i=1}^n (d_H([x_i(t)]^\alpha, [y_i(t)]^\alpha))^2} \\ &= 2ckT H_1(x(t), y(t)).\end{aligned}$$

We take sufficiently small $T$, $2ckT < 1$.

Hence $\Phi$ is a contraction mapping. By the Banach fixed point theorem, (F.C.S.) has a unique fixed point $x \in C([0, T] : E_N^n)$.

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