퍼지 구의 수렴성질

Convergent Properties of Fuzzy Spheres

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요 약

본 논문에서는 퍼지 구의 성질을 조사하고, 이에 대한 수렴정리를 수립하였다. 퍼지 구에 대한 수렴정리가 컴퓨터 그래픽이나 영상인식에 응용이 되길 기대한다.

Abstract

In this paper, we investigate the properties of fuzzy spheres and estabilish convergence theorems of them. We expect that convergence theorems of fuzzy spheres may be useful for computer graphics and patten recognition.

I. Introduction

The fuzzy logic was introduced in 1965 by Professor Lotfi A Zadeh[1]. In a series of papers[2-5], Rosenfeld introduced certain ideas in fuzzy plane geometry, area, height, width, diameter, and perimeter of a fuzzy subset of a plane. The results developed in these papers have applications to pattern recognition. In [6], Gupta and Ray introduced fuzzy plane projective geometry. Their approach was axiomatic and involved fuzzy singletons. In this paper, we experiment with developing fuzzy geometry by limiting process of the notion of fuzzy sphere whose degree of circularity function is measured by a fuzzy set. The concept of a circularity function is also used in defining the union, intersection, and complement of fuzzy spheres. The circularity function of a fuzzy sphere converges to one of a crisp sphere as the fuzzy sphere shapes itself like a crisp sphere.

We recall some definitions and results used in this paper. Let X be a nonempty crisp set. A fuzzy subset μ of X is a function from X into the closed unit interval $[0,1]$, that is, $\mu: X\to[0,1]$. In the literature, a fuzzy set may also be written as a set of ordered pairs: $\{(x, \mu(x)): x\in X\}$, where $\mu(x)$ is referred to as the membership function or grade of membership. If the range of the fuzzy set μ contains only two values 0 or 1, then μ is identical to the characteristic function of a subset of X.

Motivated by the concept of equations of motion we can specify surfaces in the three-dimensional xyz-space by using equations, $x=x(s,t)$, $y=y(s,t)$, $z=z(s,t)$, to express the coordinates of a point (x,y,z) on the surface as functions of auxiliary variables s and t. These are called parametric equations for the surface, and the variables s and t are called parameters. The parameters s and t should be viewed as independent variables the ordered pairs of which vary over a two dimensional rectangular region. Let S be the surface consisting of all ordered triplets $(x(s,t), y(s,t), z(s,t))$ on the xyz-

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space, where $x(s,t), y(s,t)$, and $z(s,t)$ are continuous real valued functions defined on a closed two dimensional rectangular region R.

Let $P(s,t)=(x(s,t), y(s,t), z(s,t))$ for $(s,t)\in R$. The surface S is called a closed surface if for any two points on S, there exists a space curve C that connects the two points such that $C\subset S$. If a closed surface S does not intersect itself at any other point on the xyz-space, then S is called a simple closed surface. Spheres and ellipsoids are typical examples of simple closed surfaces.

II. Fuzzy Spheres

Let S be a simple closed surface that is given parametrically by $(x(s,t), y(s,t), z(s,t))$ for $(s,t)\in R$, where $x(s,t)$, $y(s,t)$, and $z(s,t)$ are real valued continuous functions on a closed two dimensional rectangular region R. And let $\mathbb{R}$ denote the set of all real numbers.

Definition 2.1. A function $\mu_{\tilde{s}}: R\rightarrow[0,1]$ is called a circularity function of S if

(1) there exits a function $f:\mathbb{R}^3\rightarrow[0,1]$ such that $\mu_{\tilde{s}}(s,t) = f(x(s,t), y(s,t), z(s,t))$ for all $(s,t)\in R$,

(2) $\mu_{\tilde{s}}(s,t)=1$ for all $(s,t)\in R$ when S is a sphere.

Clearly, $\mu_{\tilde{s}}(s,t)$ is a fuzzy subset of R. Intuitively, the circularity function $\mu_{\tilde{s}}(s,t)$ can be thought of as a numerical measure of the degree of circularity for simple closed surface on the xyz-space.

Definition 2.2. Let S be a simple closed surface on the xyz-space defined as above. A fuzzy sphere on the xyz-space is given by

$$\tilde{S}=\{((x(s,t), y(s,t), z(s,t)), \mu_{\tilde{s}}(s,t))|(s,t)\in R\}$$

where

(1) $x(s,t)$, $y(s,t)$, and $z(s,t)$ are continuous parametric functions on R that define S.

(2) $\mu_{\tilde{s}}(s,t)$ is a circularity function on S.

Roughly speaking, a fuzzy sphere is formed by a simple closed surface S together with a circularity function $\mu_{\tilde{s}}(s,t)$, for all $(s,t)\in R$. For the sake of simplicity, we shall use the symbol $\tilde{S}=\langle (x(s,t), y(s,t), z(s,t)), \mu_{\tilde{s}}(s,t)\rangle$ for a fuzzy sphere

$$\tilde{S}=\{((x(s,t), y(s,t), z(s,t)), \mu_{\tilde{s}}(s,t))|(s,t)\in R\}.$$

If S is a crisp sphere, then the corresponding fuzzy sphere must have the maximum degree of circularity, that is, $\mu_{\tilde{s}}(s,t)=1$, for $(s,t)\in R$. So we can write

$$\tilde{S}=\langle (x(s,t), y(s,t), z(s,t)), 1\rangle.$$

The follwing example shows various fuzzy spheres.

Example 2.3.

Let $S=\{((x(s,t), y(s,t), z(s,t))\mid(s,t)\in R\}$ be a simple closed sphere on $\mathbb{R}^3$. Then we can take C the smallest closed cuboid including S. Let P_0 be the center point of C. Define the function $\mu_{\tilde{s}}(s,t)$ as follows :

$$\mu_{\tilde{s}}(s,t) = \begin{cases} \dfrac{|P(s,t)-P_0|}{Max\{|P(s,t)-P_0|: P\in S\}}, & \text{if } P_0\in I(S) \\ 0, & otherwise \end{cases}$$

, where $|P(s,t)-P_o|$ is the distance from P_0 to $P(s,t)$, and $I(S)$ is the set of all interior points of S. Then $\mu_{\tilde{s}}(s,t)$ is a circularity function of S.

So we can take $\mu_{\tilde{s}}(s,t)$- fuzzy sphere $\tilde{S}$ such that

$$\tilde{S}=\langle (x(s,t), y(s,t), z(s,t)), \mu_{\tilde{s}}(s,t)\rangle$$ □

We can consider geometrically the above Example 2.3. If S is a crisp sphere, then the parametric equations for the surface, S, is given by $x(s,t) = r\cos s\sin t$,

$y(s,t) = r \sin s \sin t$ and $z(s,t) = r \cos t$, $(s,t) \in [0, 2\pi] \times [0, \pi]$.

Since S is a crisp sphere, it is clear that for all $(s,t) \in [0, 2\pi] \times [0, \pi]$,

$$|P(s,t) - P_0| = Max\{|P(s,t) - P_0| : P \in S\}$$

So, the calculated $\mu_{\tilde{s}}(s,t)$ is 1. Hence the fuzzy sphere $\tilde{S}$ is

$$\tilde{S} = \langle (x(s,t), y(s,t), z(s,t)), 1 \rangle$$
$$= \langle r\cos s \sin t,\ r\sin s \sin t,\ r\cos t,\ 1 \rangle$$

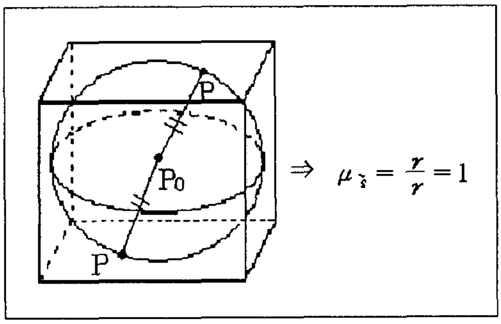

Fig. 1. 1- fuzzy sphere

Similarly, if S is an ellipsoid, then the parametric equations for the surface, S, is given by

$x(s,t) = a \cos s \sin t$, $y(s,t) = b \sin s \sin t$ and $z(s,t) = c \cos t$, $(s,t) \in [0, 2\pi] \times [0, \pi]$ then we can calculate the circularity function of S as follow

$$\mu_{\tilde{s}}(s,t) = \frac{|P(s,t) - P_0|}{Max\{|P(s,t) - P_0| : P \in S\}}.$$

In particular, if $|P(s,t) - P_0| = l(s,t)$ and $Max\{|P(s,t) - P_0| : P \in S\} = |P_2 - P_o| = m$, then the ellipsoid is $\frac{l(s,t)}{m}$- fuzzy sphere, such that

$$\tilde{S} = \langle (x(s,t), y(s,t), z(s,t)),\ \mu_{\tilde{s}}(s,t) \rangle$$
$$= \langle a \cos s \sin t,\ b \sin s \sin t,\ c \cos t,\ \frac{l(s,t)}{m} \rangle$$

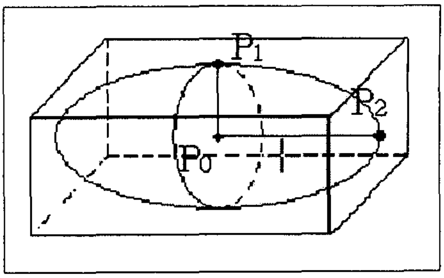

Fig. 2. $\frac{l(s,t)}{m}$- fuzzy sphere.

And also the circularity function $\mu_{\tilde{s}}(s,t)$ can also be used as a measure in comparing two fuzzy spheres. Let $\tilde{S}_1$ and $\tilde{S}_2$ be two fuzzy spheres with circularity functions $\mu_{\tilde{s}_1}(s,t)$ and $\mu_{\tilde{s}_2}(s,t)$, respectively. Then $\tilde{S}_1$ is called a fuzzy subsphere of $\tilde{S}_2$, written as $\tilde{S}_1 \leqslant \tilde{S}_2$, if $\mu_{\tilde{s}_1}(s,t) \le \mu_{\tilde{s}_2}(s,t)$ for all $(s,t) \in R$. And $\tilde{S}_1$ is said to be equal to $\tilde{S}_2$, written as $\tilde{S}_1 \sim \tilde{S}_2$, if $\tilde{S}_1 \leqslant \tilde{S}_2$, $\tilde{S}_2 \leqslant \tilde{S}_1$, that is, $\mu_{\tilde{s}_1}(s,t) \le \mu_{\tilde{s}_2}(s,t)$ and $\mu_{\tilde{s}_2}(s,t) \le \mu_{\tilde{s}_1}(s,t)$ for all $(s,t) \in R$. From the definition of equal relation, we can see that the relation $\sim$ is an equivalence relation[7].

III. Convergence of Fuzzy Spheres

In this section, we discuss several convergent theorems of fuzzy sphere sequence. First of all we introduce concepts of the union, the intersection, and the complement of fuzzy spheres.

For $n = 1, 2, \cdots$, let S_n be a simple closed surface given parametrically in terms of ordered triplets $(x_n(s,t), y_n(s,t), z_n(s,t))$ for $(s,t) \in R$, where $x_n(s,t)$, $y_n(s,t)$, and $z_n(s,t)$ are real valued continuous functions on R, and let $\tilde{S}_n$ be the fuzzy sphere that is formed by S_n with the circularity function $\mu_{\tilde{s}_n}(s,t)$, such that

$\widetilde{S_n} = \langle (x_n(s,t), y_n(s,t), z_n(s,t)), \mu_{\widetilde{S_n}}(s,t) \rangle$

Also, let

$\widetilde{S} = \langle (x(s,t), y(s,t), z(s,t)), \mu_{\widetilde{S}}(s,t) \rangle$

be a fuzzy sphere with the circularity function $\mu_{\widetilde{S}}(s,t)$.

For two fuzzy spheres

$\widetilde{S_1} = \langle (x_1(s,t), y_1(s,t), z_1(s,t)), \mu_{\widetilde{S_1}}(s,t) \rangle$

$\widetilde{S_2} = \langle (x_2(s,t), y_2(s,t), z_2(s,t)), \mu_{\widetilde{S_2}}(s,t) \rangle$,

the union of $\widetilde{S_1}$ and $\widetilde{S_2}$, denoted by $\widetilde{S_1} \cup \widetilde{S_2}$, is defined as follows :

Let

$$\mu_{\widetilde{S_1}\cup\widetilde{S_2}}(s,t) = (\mu_{\widetilde{S_1}} \vee \mu_{\widetilde{S_2}})(s,t).$$
$$= \mu_{\widetilde{S_1}}(s,t) \vee \mu_{\widetilde{S_2}}(s,t)$$

and

$$S_2^{*2} = \langle (x_2(s,t), y_2(s,t), z_2(s,t)),$$
$$\mu_{\widetilde{S_1}\cup\widetilde{S_2}}(s,t) = \mu_{\widetilde{S_2}}(s,t) \rangle \subset S_2$$

If we can construct a simple closed sphere

$$S_2^* = S_2^{*1} \cup S_2^{*2},$$

where

$$S_2^{*1} = \langle (x_1(s,t), y_1(s,t), z_1(s,t)),$$
$$\mu_{\widetilde{S_1}\cup\widetilde{S_2}}(s,t) = \mu_{\widetilde{S_1}}(s,t) \rangle$$

then we define

$$\widetilde{S_1} \cup \widetilde{S_2} = \widetilde{S_2^*}$$
$$= \langle (x^*(s,t), y^*(s,t), z^*(s,t)),$$
$$\mu_{\widetilde{S_1}\cup\widetilde{S_2}}(s,t) \rangle$$

Roughly speaking, the simple closed sphere $\widetilde{S_2^*}$ is reconstructed by S_2 with the circularity function

$$\mu_{\widetilde{S_1}\cup\widetilde{S_2}}(s,t) = \mu_{\widetilde{S_1}}(s,t) \vee \mu_{\widetilde{S_2}}(s,t).$$

We can consider that the geometric significance of $\widetilde{S_1} \cup \widetilde{S_2}$ is given Fig. 3

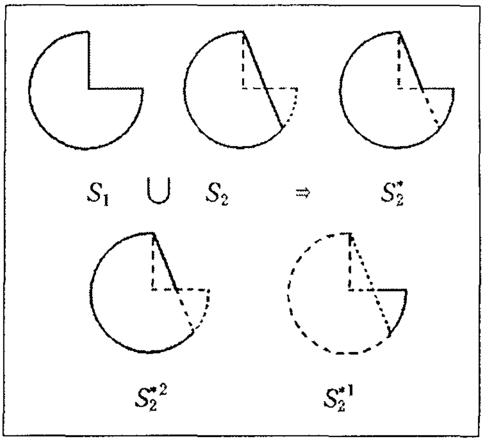

Fig. 3. the union of fuzzy spheres

For a given sequence $\{\widetilde{S_n}\}$ of fuzzy spheres, the fuzzy sphere $\bigcup_{n=1}^{\infty} \widetilde{S_n}$ is given by

$$\bigcup_{n=1}^{\infty} \widetilde{S_n} = \{((x^*(s,t), y^*(s,t), z^*(s,t)), \mu_{\cup\widetilde{S}}(s,t) \mid (s,t) \in R\}$$

with the circularity function

$$\mu_{\bigcup_{n=1}^{\infty}\widetilde{S_n}}(s,t) = (\bigvee_{n=1}^{\infty} \mu_{\widetilde{S_n}})(s,t)$$
$$= \bigvee_{n=1}^{\infty} \mu_{\widetilde{S_n}}(s,t).$$

if we can construct a simple closed sphere

$$\{((x^*(s,t), y^*(s,t), z^*(s,t)) \mid (s,t) \in R\}$$

by $\bigcup_{n=1}^{\infty} S_n$ satisfying $\mu_{\cup\widetilde{S_n}}(s,t) = \bigvee_{n=1}^{\infty} \mu_{\widetilde{S_n}}(s,t)$.

From now on we assume that $\bigcup_{n=1}^{\infty} S_n$ exists for a fuzzy sequence $\{\widetilde{S_n}\}$.

Remark. In the problem of drawing a sphere by computer, we first consider a sequence of fuzzy spheres with their circularity functions in a non-decreasing order. As the circularity function gets closer and closer to 1, we would expect that the fuzzy sphere gets closer and closer to a crisp sphere.

Also the intersection of the fuzzy spheres $\widetilde{S_1}$ and $\widetilde{S_2}$, denoted by $\widetilde{S_1} \cap \widetilde{S_2}$, and the intersection of a sequence $\{\widetilde{S_n}\}$ of fuzzy spheres, denoted by $\bigcap_{n=1}^{\infty} \widetilde{S_n}$, are defined as the fuzzy spheres similarly defined to the union of the fuzzy sphere exchanging the symbols of union ($\bigcup$), join ($\bigvee$), and meet ($\bigwedge$) for intersection ($\bigcap$), meet ($\bigwedge$) and join ($\bigvee$), respectively.

The complement of $\widetilde{S}$, denoted by $\widetilde{S}^c$, is defined to be the fuzzy sphere

$$\widetilde{S}^c = \langle\ (x(s,t), y(s,t), z(s,t)), \mu_{\widetilde{S}^c}(s,t)\ \rangle$$

with the circularity function $\mu_{\widetilde{S}^c}(s,t)$ given by $\mu_{\widetilde{S}^c}(s,t) = 1 - \mu_{\widetilde{S}}(s,t)$.

The following proposition gives the most elementary properties of the fuzzy spheres.

Proposition. 3.1

(1) $\widetilde{S_1} \le \widetilde{S_1} \bigcup \widetilde{S_2}$

(2) $\widetilde{S_1} \le \bigcup_{n=1}^{\infty} \widetilde{S_n}$

(3) $\widetilde{S_1} \ge \widetilde{S_1} \bigcup \widetilde{S_2}$

(4) $\widetilde{S_1} \ge \bigcup_{n=1}^{\infty} \widetilde{S_n}$

(5) $\widetilde{S} \le \widetilde{S} \bigcup \widetilde{S}^c$, $\widetilde{S}^c \le \widetilde{S} \bigcup \widetilde{S}^c$

(6) $\widetilde{S} \bigcup \widetilde{S}^c \le \widetilde{S}$, $\widetilde{S} \bigcup \widetilde{S}^c \le \widetilde{S}^c$

(7) $\widetilde{S} \bigcup \widetilde{S}^c \ne \widetilde{S_0}$, where $\widetilde{S_0}$ is formed by a crisp simple closed surface S_0 together with a circularity function $\mu_{\widetilde{S_0}}(s,t) = 1$, $(s,t) \in R$.

(8) $\widetilde{S} \bigcap \widetilde{S}^c \ne \emptyset$

Properties (1)-(6) are similar to those of the classical set theory, but (7) and (8) are somewhat different from the classical ones.

Since the fuzzy sphere has been defined with the circularity function, we can introduce a concept of the convergence in the sense of ciecularity function for fuzzy spheres.

Definition 3.2 *The sequence $\{\widetilde{S_n}\}$ of fuzzy spheres is* said to converge in the sense of circularity function to the fuzzy sphere $\widetilde{S}$ if $\mu_{\widetilde{S_n}}(s,t)$ converges to $\mu_{\widetilde{S}}(s,t)$ for all $(s,t) \in R$.

The following theorem may be called the " Convergence Theorem of fuzzy sphere sequence for the sequence of monotone circularity functions," or the monotone convergence theorem.

Theorem 3.3.[7] Let $\widetilde{S_1} \leqslant \widetilde{S_2} \leqslant \cdots \leqslant \widetilde{S_n} \leqslant \cdots$ be a nondecreasing sequence of fuzzy spheres. Then $\{\widetilde{S_n}\}$ converges in the sence of circularity to the fuzzy sphere $\bigcup_{n=1}^{\infty} \widetilde{S_n}$.

Proof. For each $(s,t) \in R$, it is easily seen that $\{\mu_{\widetilde{S_n}}(s,t)\}$ is a nondecreasing sequence of real numbers bounded above and

$$\sup\{\mu_{\widetilde{S_n}}(s,t) \mid n = 1, 2, \cdots\} = \bigvee_{n=1}^{\infty} \mu_{\widetilde{S_n}}(s,t).$$

Thus, it follows from Theorem 2.5 of [8] that

$$\lim_{n\to\infty} \mu_{\widetilde{S_n}}(s,t) = \bigvee_{n=1}^{\infty} \mu_{\widetilde{S_n}}(s,t)$$

for all $(s,t) \in R$. Since the circularity function of $\bigcup_{n=1}^{\infty} \widetilde{S_n}$ is given by $\bigvee_{n=1}^{\infty} \mu_{\widetilde{S_n}}(s,t)$, the theorem follows. □

The following property can be obtained in the same way as in Theorem 3.2.

Corollary 3.4 Let $\widetilde{S_1} \geqslant \widetilde{S_2} \geqslant \cdots \geqslant \widetilde{S_n} \geqslant \cdots$ be a nonincreasing sequence of fuzzy spheres. Then $\{\widetilde{S_n}\}$ converges in the sence of circularity to the fuzzy sphere

$\bigcap_{n=1}^{\infty} \widetilde{S_n}$, that is, $\lim_{n\to\infty} \widetilde{S_n} = \bigcap_{n=1}^{\infty} \widetilde{S_n}$ in the sense of circularity function.

Lemma 3.5 Let R is a two-dimensional rectangular region, and $\{\mu_{\tilde{S}_n}(s,t)\}$, $\{\mu_{\tilde{T}_n}(s,t)\}$ are sequences of real-valued functions defined on R. Let $\mu_{\tilde{S}_n}(s,t) \to \mu_{\tilde{S}}(s,t)$ and $\mu_{\tilde{T}_n}(s,t) \to \mu_{\tilde{T}}(s,t)$ on R. Then $\mu_{\tilde{S}_n}(s,t) \vee \mu_{\tilde{T}_n}(s,t)$ converges to $\mu_{\tilde{S}}(s,t) \vee \mu_{\tilde{T}}(s,t)$ on R.

Proof. Let $\varepsilon > 0$ be given, and $(s,t) \in R$ be fixed. Since $\mu_{\tilde{S}_n}(s,t) \to \mu_{\tilde{S}}(s,t)$ and $\mu_{\tilde{T}_n}(s,t) \to \mu_{\tilde{T}}(s,t)$ there exist N such that

$$k, l \geq N \Rightarrow |\mu_{\tilde{S}_k}(s,t) - \mu_{\tilde{S}}(s,t)| < \varepsilon$$

and $|\mu_{\tilde{T}_l}(s,t) - \mu_{\tilde{T}}(s,t)| < \varepsilon$.

Let $M = \max\{k, l\}$. Then there exist $m \geq M$ such that $|\mu_{\tilde{S}_m}(s,t) - \mu_{\tilde{S}}(s,t)| < \varepsilon$

and

$$|\mu_{\tilde{T}_m}(s,t) - \mu_{\tilde{T}}(s,t)| < \varepsilon.$$

And then

$$\mu_{\tilde{S}}(s,t) - \varepsilon < \mu_{\tilde{S}_m}(s,t) < \mu_{\tilde{S}}(s,t) + \varepsilon$$

and

$$\mu_{\tilde{T}}(s,t) - \varepsilon < \mu_{\tilde{T}_m}(s,t) < \mu_{\tilde{T}}(s,t) + \varepsilon .$$

Hence

$$\begin{aligned}&(\mu_{\tilde{S}}(s,t) - \varepsilon) \vee (\mu_{\tilde{T}}(s,t) - \varepsilon)\\ &< \mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t) \qquad \cdots\cdots\cdots\cdots \\ &< (\mu_{\tilde{S}}(s,t) + \varepsilon) \vee (\mu_{\tilde{T}}(s,t) + \varepsilon).\end{aligned}$$

(1)

Consider

$$|(\mu_{\tilde{S}_m} \vee \mu_{\tilde{T}_m}(s,t)) - (\mu_{\tilde{S}}(s,t) \vee \mu_{\tilde{T}}(s,t))|$$

I. The case $\mu_{\tilde{S}}(s,t) = \mu_{\tilde{T}}(s,t)$.

By (1), we obtain

$$\begin{aligned}&(\mu_{\tilde{S}}(s,t) - \varepsilon) \vee (\mu_{\tilde{T}}(s,t) - \varepsilon)\\ &< \mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t)\\ &< (\mu_{\tilde{S}}(s,t) + \varepsilon) \vee (\mu_{\tilde{T}}(s,t) + \varepsilon).\end{aligned}$$

Since $\mu_{\tilde{S}}(s,t) = \mu_{\tilde{T}}(s,t)$, we can get

$$\begin{aligned}&\mu_{\tilde{S}}(s,t) - \varepsilon\\ &< \mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t)\\ &< \mu_{\tilde{S}}(s,t) + \varepsilon.\end{aligned}$$

From

$$|(\mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t)) - \mu_{\tilde{S}}(s,t)| < \varepsilon$$

We have the conculsion

$$|(\mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t)) - (\mu_{\tilde{S}}(s,t) \vee \mu_{\tilde{T}}(s,t))| < \varepsilon.$$

II. The case $\mu_{\tilde{S}}(s,t) < \mu_{\tilde{T}}(s,t)$.

By (1) we can gets

$$\begin{aligned}&(\mu_{\tilde{S}}(s,t) - \varepsilon) \vee (\mu_{\tilde{T}}(s,t) - \varepsilon)\\ &< \mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t)\\ &< (\mu_{\tilde{S}}(s,t) + \varepsilon) \vee (\mu_{\tilde{T}}(s,t) + \varepsilon).\end{aligned}$$

Since $\mu_{\tilde{S}}(s,t) < \mu_{\tilde{T}}(s,t)$, it follows that

$$\begin{aligned}&\mu_{\tilde{T}}(s,t) - \varepsilon\\ &< \mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t)\\ &< \mu_{\tilde{T}}(s,t) + \varepsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}&(\mu_{\tilde{S}}(s,t) \vee \mu_{\tilde{T}}(s,t)) - \varepsilon\\ &< \mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t)\\ &< (\mu_{\tilde{S}}(s,t) \vee \mu_{\tilde{T}}(s,t)) + \varepsilon.\end{aligned}$$

Consequently, we have

$$|(\mu_{\tilde{S}_m}(s,t) \vee \mu_{\tilde{T}_m}(s,t)) - (\mu_{\tilde{S}}(s,t) \vee \mu_{\tilde{T}}(s,t))| < \varepsilon.$$

III. The case $\mu_{\tilde{S}}(s, t) > \mu_{\tilde{T}}(s, t)$.

In a similar way, we can prove that the case of $\mu_{\tilde{S}}(s, t) > \mu_{\tilde{T}}(s, t)$. Therefore, the proof is now complete by the case I, II, III. □

When the symbol of $\bigvee$ and $\bigwedge$ are exchanged, we can get the folliwing result similar to Lemma 3.4, and the proof is similar to the above.

Lemma 3.6. Let R is a two-dimensional rectangular region, and $\{\mu_{\tilde{S}_n}(s, t)\}$, $\{\mu_{\tilde{T}_n}(s, t)\}$be a sequence of real -valued functions defined on R. Let $\mu_{\tilde{S}}(s, t)$, $\mu_{\tilde{T}}(s, t)$: $R \to \mathbb{R}$ be a real-valued functions such that

$$\lim_{n\to\infty} \mu_{\tilde{S}_n}(s, t) = \mu_{\tilde{S}}(s, t)$$
$$\lim_{n\to\infty} \mu_{\tilde{T}_n}(s, t) = \mu_{\tilde{T}}(s, t)$$

on R.

Then $\mu_{\tilde{S}_n}(s, t) \bigwedge \mu_{\tilde{T}_n}(s, t)$ converges to $\mu_{\tilde{S}}(s, t) \bigwedge \mu_{\tilde{T}}(s, t)$ on R .

The scalar multiple of $\tilde{S}$, denoted by $\alpha\tilde{S}$, is defined to be the fuzzy sphere

$$\alpha\tilde{S} = \langle (x(s, t), y(s, t), z(s, t)), \mu_{\alpha\tilde{S}}(s, t) \rangle$$

with the circularity function

$$\mu_{\alpha\tilde{S}}(s, t) = (\alpha, \mu_{\tilde{S}})(s, t)$$
$$= \alpha \cdot \mu_{\tilde{S}}(s, t).$$

Theorem 3.7. Let $\{\tilde{S}_n\}$ and $\{\tilde{T}_n\}$ converge in the since of circularity to the fuzzy sphere $\tilde{S}$, $\tilde{T}$ respectly.
Then

(1) $\{\tilde{S}_n \bigcup \tilde{T}_n\}$ converges in the sense of circularity to the fuzzy sphere $\tilde{S} \bigcup \tilde{T}$.

(2) $\{\tilde{S}_n \bigcap \tilde{T}_n\}$ converges in the sense of circularity to the fuzzy sphere $\tilde{S} \bigcap \tilde{T}$.

(3) $\{\alpha\tilde{S}_n\}$ converges in the sense of circularity to the fuzzy sphere $\alpha\tilde{S}$.

Proof. We need to prove (3) ; the remaing properties can be obtained directly from the Lemma 3.5 and Lemma 3.6 For (3), since $\{\tilde{S}_n\}$ converges in the sense of circularity to $\tilde{S}$, for each $(s, t) \in R$

$$\lim_{n\to\infty} \mu_{\alpha\tilde{S}_n}(s, t) = \alpha \lim_{n\to\infty} \mu_{\tilde{S}_n}(s, t)$$
$$= \alpha \mu_{\tilde{S}}(s, t)$$

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