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A Study on Determination of Shear Center of Beam with Arbitrary Cross Section

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ABSTRACT

It is important to find the shear center of beam with arbitrary cross-section in structures. In this study, it is introduced to determine the shear center that gets the equivalent stiffness matrix representing arbitrary cross section of beam and applies concepts of equivalent energy. This method shows the results of applying on examples that the exact and approximate solution of open and cross section of beam is known. The shear center of composite rotor blade by the experiment and by the suggested method was compared in this study.

Key Words : Shear Center, Equivalent Stiffness Matrix, Concept of Equivalent Energy, Arbitrary Cross Section, Reduction

1. Introduction

The shear center is defined as a specific point in a section that causes a bending moment but prevent a twist moment. The shape of cross section decides the specific point. It has been known that the intersection point of each resultant force of the shear stress from paralleled principal direction of a cross section decides the shear center. However, it has not been simple to apply the definition to get the shear center. It is important to find the location of shear center and consider the influence of the torsion because the beam with thin open cross section is weak at twist.

The Shear center was first found in 1913 by Timoshenko, but it is hard to find out the shear center of built-up beam or beams made up of two or more different materials except for simple sections where the exact solution or approximate solution is known.

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Especially, rotor blade for modern helicopters is a structure of typical composite materials. It is anisotropic and non-homogeneous. This advanced rotor blade helped the construction of hub with no bearings, and provided the structural couple for the improvement of aero-elastic stability. But tension-shear-torsion-bending deformation could be coupled, and the effect of warping would be serious. This complexity made the theory of Euler's beam ineffective. But there is no general and appropriate theory that can explain the rotor blade with this complicated cross section. Even when the rotor blade is supposed to be an 1-dimensional beam, the description on proper motion geometry of beam would just be an approximated description on constitutive equation of 3-dimensional elastic coefficients. In this case, minimizing the errors in 1-dimensional beam modeling is the key to the solution. 3-dimensional finite element model could be a solution, but it would cost too much and the analysis of the result would be hard. The length is longer than the width of axis direction of rotor blade. This is why the 1-dimensional modeling is used in computer-based analysis.

Reissner[1,2] solved the 2-dimensional differential equation with the minimum complementary energy analysis, and Kismatka[3] with power series solution and saint-venant's semi-inverse method. They had to try to solve those complex equations and develop special finite elements to get the solution of shear center[4].

Reduction was introduced to reduce the excessive memory and amount of storage as the numerical analysis developed. From Guyan[5]'s study as a start, many researchers[6,7,8] including Irons have announced treatises on reduction. In this study, Guyan's reduction was applied to get the equivalent stiffness matrix of beams with arbitrary cross sections. Based on the definition of the shear center, the shear center was found by excluding coupled terms. To verify the result, the simple closed section with exact solution and the open cross section with approximate solution was compared. Also, this and Jujin[9]'s experimental solution of the shear center of rotor blade with composite material was compared.

2. Concept of equivalent energy

As shown in Fig.1, the shear center of arbitrary cross section at free end of cantilevered beam is origin O. When origin O' distance is *a* in y-direction, *b* is z-direction from origin O, displacement vector can be expressed as below at origin O.

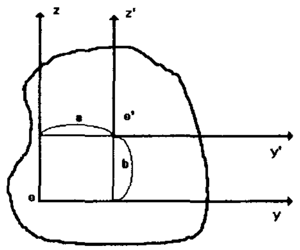


Fig. 1 Coordinate system

$$\{u\}^T=(x\ y\ z\ \theta_x\ \theta_y\ \theta_z) \tag{1}$$

Assume that the coordinate systems expressed by origin O and O', and displacement {u} and {u'} are in principal direction the displacement vector {u'} can be expressed as below at origin O'. Oh[10]'s study was used for the determination of the principal direction.

$$\{u'\}^T=(x'\ y'\ z'\ \theta_x'\ \theta_y'\ \theta_z') \tag{2}$$

The relationship between load vector {f}, and displacement vector {u} and {u'} is as below.

$$\{f\}=[A]\{u\}=[B]\{u'\}$$
$$=\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & a_{26} \\ 0 & 0 & a_{33} & 0 & a_{35} & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & a_{53} & 0 & a_{55} & 0 \\ 0 & a_{62} & 0 & 0 & 0 & a_{66} \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix}$$
$$=\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix}\begin{Bmatrix} x' \\ y' \\ z' \\ \theta_x' \\ \theta_y' \\ \theta_z' \end{Bmatrix} \tag{3}$$

Stiffness matrix [A] is of shape as above, because they are in principal direction shown as coordinate system of Fig. 1. The relation between {u} and {u'} can be expressed as below.

$$\begin{aligned} x' &= x + b\theta_y - a\theta_z && (a) \\ y' &= y - b\theta_x && (b) \\ z' &= z + a\theta_x && (c) \\ \theta_x' &= \theta_x && (d) \\ \theta_y' &= \theta_y && (e) \\ \theta_z' &= \theta_z && (f) \end{aligned} \tag{4}$$

To determine the shear center, the equations related to the shear center of force and displacement are written as below. Here, *a* and *b* are the distance of shear center.

$$\{f_s\} = [A_s]\{u_s\} = [B_s]\{u'_s\}$$

$$= \begin{bmatrix} a_{22} & 0 & 0 \\ 0 & a_{33} & 0 \\ 0 & 0 & a_{44} \end{bmatrix} \begin{Bmatrix} y \\ z \\ \theta_x \end{Bmatrix} \qquad (5)$$

$$= \begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{Bmatrix} y' \\ z' \\ \theta'_x \end{Bmatrix}$$

From the concepts of the energy conservation,

$$\frac{1}{2}\{u_s\}^T[A_s]\{u_s\} = \frac{1}{2}\{u'_s\}^T[B_s]\{u'_s\} \qquad (6)$$

Expanding the above, a and b become as below.

$$a = -\frac{b_{34}}{b_{33}}, \quad b = \frac{b_{24}}{b_{22}} \qquad (7)$$

3. Equivalent Stiffness Matrix

To extract the equivalent stiffness matrix of the arbitrary cross section, non-structural node to the expected section should be given to 3-dimensional finite element model, as shown in Fig. 2. And the node is given the degree of freedom that shows the feature of the whole section, considering the stiffness and the direction of each node. Next, to represent the movement of the section,

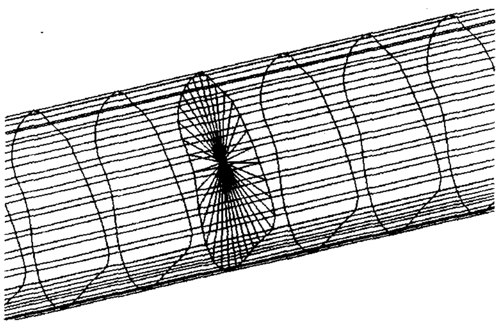


Fig. 2 Illustration of reduction technique

reduction is applied to the non-structural node. Equivalent stiffness matrix is given by outputting stiffness matrix.

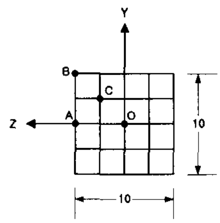
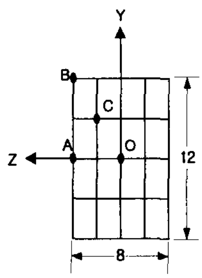
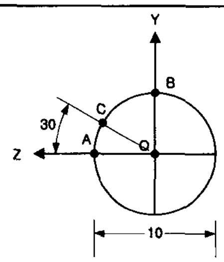
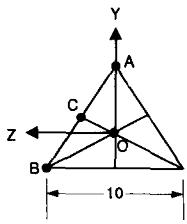
In this study, traditional Guyana's reduction[5] which generally results in an approximate solution for the mass and damping matrix was used. But in case of stiffness matrix, the exact equivalent matrix can be found. This equivalent stiffness matrix decides the shear center by using simple matrix operation.

In this process, commercial finite element analysis software without any other special programming procedure was performed. In case of complex feature and built-up beam, it is possible to perform 3-dimensional beam modeling and reduction of common finite element analysis program to get the equivalent stiffness matrix easily.

4. Results and Discussion

To verify the validity of the solution on the location of the shear center with the application of energy equivalent theory, the case on uniform beams with closed cross section of already known the exact solution and with open cross section of known the approximate solution was studied. The location of the experimental

Table 1 Types of closed cross section of beam (mm)

	
(a) square	(b) rectangle
	
(c) circle	(d) regular triangle

shear center and the result of this study were also compared.

Each stiffness matrix by using a 1,000mm cantilever beam with square, rectangle, circle, regular triangle cross section was extracted as the (a) (b) (c) (d) (e) of Table 1 shows, and was confirmed the location of the shear center. In each case, the shear center was located on point O, and the distance of the location of shear center that was calculated from stiffness matrix that was inferred from each different point A~C from O become the exact solution.

The result of the analysis is shown on Table 2, which shows that the exact solution and the result are almost identical.

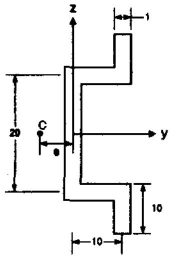
On open cross section, Boresi's approximate solutions[13] and the result of this study was compared as shown on Table 3. The result of the analysis is shown in Table 4.

The ratio(θ_x / θ_y) of loaded shear center with approximate solution and the result of this study was compared.(θ_x : angle of twist, θ_y : angle of deflection)

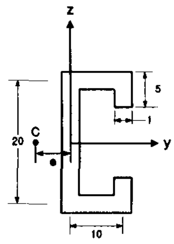
The result tells that there is less torsion at shear center of this study than Boresi's solution .

This result is because Boresi's is also an approximate solution. Fig. 3 shows the result of Boresi's solution approximate one and the shear center abstracted by this study.

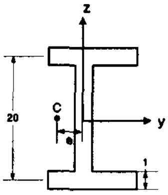
Table 3 Types of open cross section (mm)



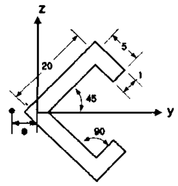
Type 1



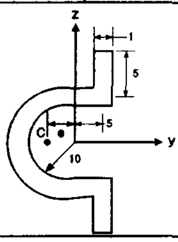
Type 2



Type 3



Type 4



Type 5

As the suggested method did not assume any particular shape and material of the cross sections, this can be applied to build-up beams as well.

Table 2 Results of closed cross section of beam for shear center (mm)

Type of cross section \ Center		Y			Z		
		Exact	In this study	Error (%)	Exact	In this study	Error (%)
Square	A	0	0	0	5	5.009	0.18
	B	5	5.009	0.18	5	5.009	0.18
	C	2.5	2.5	0	2.5	2.5	0
Rectangle	A	0	0	0	4	4.001	0.025
	B	6	6.002	0.033	4	4.001	0.025
	C	3	3.001	0.033	2	2.0005	0.025
Circle	A	0	0	0	5	5.0003	0.006
	B	5	5.0003	0.006	0	0	0
	C	2.5	2.50003	0.001	4.3301	4.3302	0.002
Regular Triangle	A	5.773	5.774	0.017	0	0	0
	B	2.887	2.888	0.035	5	5	0
	C	1.4435	1.4435	0	2.5	2.5	0

Next, the validity examination by comparing Jujin[9]'s composite rotor blade featuring NACA-0012 and this study's analytical shear center was performed.

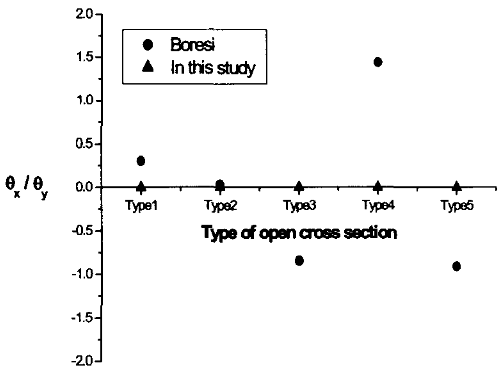
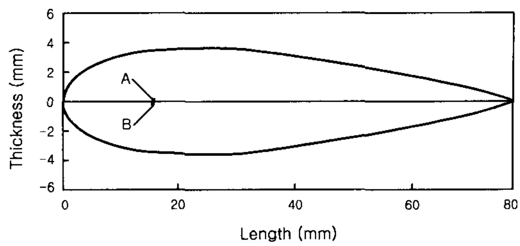


Fig. 3 Result of θ_x / θ_y for open cross section

Table 4 Results of open cross section of beam for shear center

Type		e(mm)	θ_x / θ_y
Type 1	Boresi	3.1494	0.299472
	In this study	3.0397	-0.00069
Type 2	Boresi	5.6355	0.030557
	In this study	5.5981	0.0003
Type 3	Boresi	2.5714	-0.84471
	In this study	3.0082	-0.00012
Type 4	Boresi	1.4002	1.437164
	In this study	1.3544	-0.00028
Type 5	Boresi	15.0275	-0.90696
	In this study	14.7816	-0.00037

Fig. 4 shows both Jujin[9]'s experiment result and our analytical shear center. The difference of two shear centers seems to be little, considering the experimental and analytical errors.



A : Shear Center (Experiment) B : Shear Center (In this study)

Fig. 4 Shear center of the NACA-0012 composite blade

5. Conclusion

Getting the shear center has important meanings to beams on composite materials like rotor blades. In this study, the method that extracts the equivalent stiffness matrix of arbitrary cross section by using reduction on finite element analysis and that gets the shear center by this stiffness matrix was suggested. The result of open and closed cross section with exact solution and approximate solution with this method was compared, and the results were turned out to be identical. This method could be applied to built-up beams that have different materials, as the location of the shear center does not assume on any premise. Jujin's experimental solution on composite rotor blade was also compared. The suggested method prevents waste of developing new programs as it extracts the stiffness matrix easily by using commercial finite element analysis programs, e.g. DMAP of MSC/NASTRAN. It also provides easy calculations, as the result is extruded from the simple modeling of particular section.

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