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Pointwise Projective Modules and Some Related Modules

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Abstract. Let $\mathcal{R}$ be a commutative ring with 1, and Let M be a (left) R-module. M is said to be pointwise projective if for each epimorphism $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$, where A and $\mathcal{B}$ are any $\mathcal{R}$–modules, and for each homomorphism $\beta : \mathcal{M} \longrightarrow \mathcal{B}$, then for every $m \in \mathcal{M}$, there exists a homomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{A}$, which may depend on m, such that $\alpha \circ \varphi(m) = \beta(m)$. Our mean concern in this paper is to study the relations between pointwise projectivemodules with cancellation modules and their generalization generator modules, multiplication modules and its generalizations.

0. Introduction

Let $\mathcal{R}$ be a commutative ring with 1, and let $\mathcal{M}$ be a (left) $\mathcal{R}$-module. $\mathcal{M}$ is said to be pointwise projective(abbreviated by pwp.) if for each epimorphism $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are any $\mathcal{R}$–modules, and for each homomorphism $\beta : \mathcal{M} \longrightarrow \mathcal{B}$, then for every $m \in \mathcal{M}$, there exists a homomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{A}$, which may depend on m, such that $\alpha \circ \varphi(m) = \beta(m)$. This definition appeared in [2] under the name locally projective. It is clear that every projective module is a pwp. module.

In this paper we study the relations between pwp. modules and cancellation modules and its generalization, generator modules, multiplication modules, quasi-multiplication modules, and weak multiplication modules.

1. Cancellation, generator, and pointwise projective modules

Let $\mathcal{R}$ be a commutative ring with 1, and let $\mathcal{M}$ be an $\mathcal{R}$-module, $\mathcal{M}$ is called a cancellation module if for any ideals $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{R}$, $\mathcal{AM} = \mathcal{BM}$ implies $\mathcal{A} = \mathcal{B}$ ([13]). $\mathcal{M}$ is called weak cancellation module if for each two ideals $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{R}$, $\mathcal{AM} = \mathcal{BM}$ implies $\mathcal{A} + \operatorname{ann}(\mathcal{M}) = \mathcal{B} + \operatorname{ann}(\mathcal{M})$, therefore $\mathcal{M}$ is a cancellation module if and only if it is faithful weak cancellation ([13]). An $\mathcal{R}$ module $\mathcal{M}$ is called a 1/2 weak cancellation if for any ideal $\mathcal{A}$ of $\mathcal{R}$, $\mathcal{AM} = \mathcal{M}$ implies $\mathcal{A}+\operatorname{ann}(\mathcal{M}) = \mathcal{R}$, and $\mathcal{M}$ is 1/2 cancellation if it is faithful 1/2 weak cancellation ([11]).

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An $\mathcal{R}$-module $\mathcal{M}$ is called a generator if for each $\mathcal{R}$-modules N, $\mathcal{B}$ and for each $o \neq f \in \hom(\mathcal{B}, \mathcal{N})$, there exists a homomorphism $g \in \hom(\mathcal{M}, \mathcal{B})$ such that $f \circ g \neq 0$ ([7], p.53). It is known that every generator module is a cancellation module ([6]), but the converse is not true.

There is an example in [11, p.100] of a cancellation module which is not generator. The following results show that a cancellation module is a generator module if it is pointwise projective. But first we recall that an $\mathcal{R}$–module $\mathcal{M}$ is said to be pointwise projective (abbreviated by pwp.) if for each epimorphism $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are any $\mathcal{R}$–modules, and for each homomorphism $\beta : \mathcal{M} \longrightarrow \mathcal{B}$, then for every $m \in \mathcal{M}$, there exists a homomorphism $\varphi : \mathcal{M} \longrightarrow \mathcal{A}$, which may depend on m, such that $\alpha \circ \varphi(m) = \beta(m)$ ([2]). Not that every projective module is a pwp.module. The next result will be useful later ([18]).

Theorem 1.1 (The Dual-Basis Lemma for pointwise projective modules)**.**
Let $\mathcal{M}$ be an $\mathcal{R}$ module. $\mathcal{M}$ is pwp.module if and only if for each $m_1, m_2, \cdots, m_t \in \mathcal{M}$, there are $x_1, x_2, \cdots, x_n \in \mathcal{M}$ and $\varphi_1, \varphi_2, \cdots, \varphi_n \in \mathcal{M}^*$ such that $m_i = \sum_{k=1}^{n} \varphi_k(m_i)\, x_k$, for all i, $1 \leq i \leq t$.

We recall that trace $T(\mathcal{M})$ of an $\mathcal{R}$–module $\mathcal{M}$ (simply T) is defined by $\sum \varphi(\mathcal{M})$, where the sum is taken over all $\varphi \in \mathcal{M}^*$.

Theorem 1.2. *Let $\mathcal{M}$ be a faithful pwp. $\mathcal{R}$-module, then the following statements are equivalent:*

(1) *$\mathcal{M}$ is a generator module.*

(2) *Every simple $\mathcal{R}$-module is a homomorphic image of $\mathcal{M}$.*

(3) *$\mathcal{M}$ is a cancellation module.*

(4) *$\mathcal{M}$ is $1/2$ cancellation module.*

(5) *$\mathcal{M}$ is $1/2$ weak cancellation module.*

(6) *$\mathcal{M}$ is a weak cancellation module.*

(7) *$\mathcal{M}$ contains a finite submodule $\mathcal{W}$ with $\operatorname{ann}(\mathcal{W}) = \operatorname{ann}(\mathcal{M})$.*

(8) *T is generated by idempotent.*

Proof. (1) $\Rightarrow$ (3) ([6]).
(3) $\Rightarrow$ (4) Clear.
(4) $\Rightarrow$ (2) ([11]).
(2) $\Rightarrow$ (1) Assume $T \neq \mathcal{R}$, therefore there exists a maximal ideal $\mathcal{A}$ of $\mathcal{R}$ such that $T \subseteq \mathcal{A}$. Since $\mathcal{R}/\mathcal{A}$ is simple, then there exists an epimorphism $f : \mathcal{M} \longrightarrow \mathcal{R}/\mathcal{A}$. Since $\mathcal{M}$ is pwp., then $\forall m \in \mathcal{M}$, $\exists h : \mathcal{M} \longrightarrow \mathcal{R}$, which may depend on m, such that $\pi \circ h(m) = f(m) = 0$. Thus $f = 0$, contradiction. Hence $T = \mathcal{R}$.

(4) $\Leftrightarrow$ (5) Let $\mathcal{A}$ be an ideal of $\mathcal{R}$ such that $\mathcal{AM} = \mathcal{M}$, $\mathcal{M}$ is $1/2$ cancellation if and only if $\mathcal{A} = \mathcal{R}$ if and only if $\mathcal{A} + \mathrm{ann}(\mathcal{M}) = \mathcal{R}$, because $\mathcal{M}$ is faithful, if and only if $\mathcal{M}$ is $1/2$ weak cancellation.

(3) $\Leftrightarrow$ (6) Its proof is a modification of the proof (4) $\Leftrightarrow$ (5).

(6) $\Rightarrow$ (8) By [14] $T\mathcal{M} = \mathcal{M}$, then $T = \mathcal{R}$, thus T is generated by 1 which is idempotent.

(7) $\Rightarrow$ (8) Let $\mathcal{W} = \{a_1, a_2, \cdots, a_n\}$ be any finite subset of $\mathcal{M}$ By ([14]) for each i, $1 \le i \le n$, there exist $x_i \in \mathrm{ann}(a_i)$, $y_i \in (T(\mathcal{M}))$, such that $x_i + y_i = l$. Let $l - y = \prod_i (l - y_i)$. Then $y \in T(\mathcal{M})$, and $(l-y)a_i = 0\ \forall i$. Now if $\mathrm{ann}(\mathcal{W}) = \mathrm{ann}(\mathcal{M})$, then $l - y \in \mathrm{ann}(\mathcal{M})$ and $ya = a\ \forall a \in \mathcal{M}$. If $\ell \in \mathcal{M}^*$, then $\ell(a) = y\ell(a) \in (y)$. By [14] T is generated by idempotent.

(8) $\Rightarrow$ (7) By assumption $T = \mathcal{R}t$, $t^2 = t$, then $t = \sum_{i=1}^{n} \varphi_i(m_i)$; $\varphi_i \in \mathcal{M}^*$, $m_i \in \mathcal{M}$. Let $\mathcal{W}$ be a submodule of $\mathcal{M}$ generated by $m_1, m_2, \cdots, m_n$. It can easily be proved that $\mathrm{ann}(\mathcal{W}) = \mathrm{ann}(\mathcal{M})$.

(8) $\Rightarrow$ (1) By assumption, $T = \mathcal{R}e$; $e^2 = e$. By [14] $(l - e)\mathcal{M} = (l - e)T\mathcal{M} = 0$, hence $(l - e) \in \mathrm{ann}(\mathcal{M}) = 0$. Thus $e = 1$, therefore $T = \mathcal{R}$. □

Next we seek conditions under which pwp. are $1/2$ weak cancellation. But first we recall the following. Let $\mathcal{M}$ be a pwp. $\mathcal{R}$-module, then $\mathcal{M}_p$ is a pwp. $\mathcal{R}_p$-module for each prime ideal P of $\mathcal{R}$, $J(\mathcal{M}) = J(\mathcal{R})\mathcal{M}$, and $J(\mathcal{M}) = \mathcal{M}$ only if $\mathcal{M} = 0$ ([14]).

Proposition 1.3. *Let M be a non-zero pwp. $\mathcal{R}$-module. If $\mathcal{R}$ is a local ring, then $\mathcal{M}$ is a $1/2$ weak cancellation module.*

Proof. Let $\mathcal{A}$ be an ideal, of $\mathcal{R}$ with $\mathcal{AM} = \mathcal{M}$. Assume $\mathcal{A} + \mathrm{ann}(\mathcal{M}) \ne \mathcal{R}$. Since $\mathcal{R}$ is local, then there exists a unique maximal ideal $\mathcal{B}$ of $\mathcal{R}$ with $\mathcal{A} + \mathrm{ann}(\mathcal{M}) \subseteq \mathcal{B}$, thus $\mathcal{M} = \mathcal{BM}$ It is easily seen that $\mathcal{M}_{\mathcal{B}} = O$. Then there exists $r \in \mathcal{R} - \mathcal{B}$ such that $rm = 0$, i.e. $\mathcal{M} = 0$ and this is a contradiction. □

Proposition 1.4. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$-module. Assume that $\mathcal{M}_p$ is faithful for each maximal ideal P of $\mathcal{R}$, then M is a cancellation module.*

Proof. Let P be a maximal ideal of $\mathcal{R}$, then $\mathcal{M}_p$ is pwp. ([14]), by (1.3) $\mathcal{M}_p$ is $1/2$ weak cancellation, and by (1.2) $\mathcal{M}_p$ is a cancellation module. Now let $\mathcal{A}$ and $\mathcal{B}$ be any two ideals of $\mathcal{R}$ with $\mathcal{AM} = \mathcal{BM}$, then $\mathcal{A}_p\mathcal{M}_p = \mathcal{B}_p\mathcal{M}_p$ for each maximal ideal P of $\mathcal{R}$. Hence $\mathcal{A}_p = \mathcal{B}_p$, and by [1], $\mathcal{A} = \mathcal{B}$. □

A result similar to the following is known for projective modules ([7], p.132).

Proposition 1.5. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$-module, let P be any $\mathcal{R}$-module such that there is an index set I with $\mathcal{M} \oplus \mathcal{M}' \cong \oplus_{i \in I} P_i$; $P_i = P$ for all $i \in I$ and $\mathcal{M}'$ is a direct summand of copies of P, then P is a generator.*

Proof. Let $\mathcal{N}$, $\mathcal{L}$ be two $\mathcal{R}$-modules, and let $0 \ne f \in \hom(\mathcal{N}, \mathcal{L})$. Since every module is an epimorphic image of pwp. (free) module, then there is an epimorphism $h : \mathcal{M} \longrightarrow \mathcal{N}$. Since there is an index set I such that $\mathcal{M} \oplus \mathcal{M}' \cong \oplus_{i \in I} P_i$; $P_i = P$ for all $i \in I$. Thus let $\pi : \oplus P_i \longrightarrow \mathcal{M}$ be the natural epimorphism. Define

$\varphi : P_i \longrightarrow \mathcal{M}$ as $\varphi = \pi|_{p_i}$. Hence $h \circ \varphi : P \longrightarrow \mathcal{N}$ such that $f \circ h \circ \varphi \neq 0$ for some $i \in I$, therefore P is a generator. □

Remark. There exist generator (hence cancellation) modules which are not pwp. modules. For example, let $\mathcal{M} = \mathbb{Z}_2 \oplus \mathbb{Z}$ as $\mathbb{Z}$-module. It is easily seen that $T(\mathcal{M}) = \mathbb{Z}$, hence $\mathcal{M}$ is a generator ([7], p.100) (hence cancellation) module. It is easily seen that $\mathbb{Z}_2$ as $\mathbb{Z}$-module is not pwp., then $\mathcal{M}$ is not a pwp. $\mathbb{Z}$-module.

Next we look at conditions under which generator modules are pwp. modules. First observe the following result.

Proposition 1.6. *Every generator ideal is a pwp. ideal.*

Proof. Let I be a generator ideal of $\mathcal{R}$. By [7, p.100] there exist $x_1, x_2, \cdots, x_n \in I$ and $\varphi_1, \varphi_2, \cdots, \varphi_n \in I^*$ such that $1 = \sum_{j=1}^{n} \varphi_j(x_j)$. Let $a \in I$,

$$a = \sum_{j=1}^{n} \varphi_j(x_j)a = \sum_{j=1}^{n} \varphi_j(a)x_j.$$

Then by the Dual-Basis Lemma for pwp. modules, I is pwp. □

Before stating the next result we introduce the following notation. Let $\mathrm{End}(\mathcal{M})$ (simply $\mathcal{S}$) be the endomorphism ring of an $\mathcal{R}$-module $\mathcal{M}$.

Proposition 1.7. *Let $\mathcal{M}$ be a generator $\mathcal{R}$-module. If $\mathcal{S}$ is commutative, then $\mathcal{M}$ is a pwp. $\mathcal{R}$-module.*

Proof. By [7, p.100], there exist $\ell_1, \ell_2, \cdots, \ell_n \in \mathcal{M}^*$ and $x_1, x_2, \cdots, x_n \in \mathcal{M}$ such that $1 = \sum_{i=1}^{n} \ell_i(x_i)$. Let $m \in \mathcal{M}$, then $m = \sum_{i=1}^{n} \ell_i(x_i)m$, thus $m \in Tm$. Hence by [14] $\mathcal{M}$ is a pwp. module. □

Next we show conditions under which pwp. $\mathcal{S}$-modules are generator $\mathcal{R}$-modules, but before stating the next result we introduce some notation. Let $\mathcal{M}$ be an $\mathcal{R}$-module, define $[,] : \mathcal{M} \times \mathcal{M}^* \longrightarrow \mathcal{S}$ as $[m, f] = f_m$ where $f_m(a) = f(a)m$ for all $a \in \mathcal{M}$. Let Δ be the ideal of $\mathcal{S}$ generated by $\mathrm{Im}[,]$ ([17]). Recall that $\mathcal{M}$ is said to be T-accessible if $T\mathcal{M} = \mathcal{M}$.

Proposition 1.8. *Let $\mathcal{M}$ be a T-accessible $\mathcal{R}$-module that contains an element, which is $\mathcal{R}$-torsion free. If $\mathcal{M}$ is a pwp. $\mathcal{S}$-module, then $\mathcal{M}$ is a generators $\mathcal{R}$-module.*

Proof. Let $x \in \mathcal{M}$ such that $\mathrm{ann}_R(x) = 0$. By the Dual-Basis Lemma for pwp. modules, $x = \sum_{i=1}^{n} \ell_i(x)y_i$; $\ell_i \in \mathcal{M}^*$ and $y_i \in \mathcal{M}$. By [17] $\Delta = {}_ST$, thus $\ell_i(x) \in \Delta$. Therefore there exists $\varphi_{ij} \in \mathcal{M}^*$ such that $\ell_i(x) = \sum_j [x, \varphi_{ij}]$, hence $x = \sum_i \sum_j \varphi_{ij}(y_i)x$. Then $1 - \sum_i \sum_j \varphi_{ij}(y_i) \in \mathrm{ann}(x) = 0$, therefore $\mathcal{M}$ is a generator $\mathcal{R}$-module. □

Remark. The last proposition is false without the condition $\mathcal{M}$ is T-accessible. For example, let $\mathcal{M} = \mathbb{Q}$ as $\mathbb{Z}$-module. It is easily seen that $\mathcal{M}$ is not pwp. $\mathbb{Z}$-module, but $\mathcal{M}$ as End($\mathbb{Q}$)-module is a pwp. module since End($\mathbb{Q}$) $\cong \mathbb{Q}$ ([4]). Note that $T(\mathbb{Q}) = 0$.

2. Multiplication modules and pointwise projective modules

Let $\mathcal{N}$ be a submodule of an $\mathcal{R}$-module $\mathcal{M}$. Put $[\mathcal{N} : \mathcal{M}] = \{r \in \mathcal{R} | r\mathcal{M} \subseteq \mathcal{N}\}$, let $[(a) : \mathcal{M}] = [a : M]$. And put $\Theta(\mathcal{M}) = \sum_{a\in\mathcal{M}}[a : \mathcal{M}]$. We recall that $\mathcal{M}$ is called a multiplication module if for each submodule $\mathcal{N}$ of $\mathcal{M}$ there exists an ideal I of $\mathcal{R}$ such that $\mathcal{N} = I\mathcal{M}$. And an ideal I of $\mathcal{R}$ is a multiplication ideal if I is a multiplication $\mathcal{R}$-module ([3]). Smith ([15]) gave the following characterization for multiplication modules.

Proposition 2.1. *An $\mathcal{R}$-module $\mathcal{M}$ is multiplication if and only if for all $a \in \mathcal{M}$, $\mathrm{ann}(a) + \Theta(\mathcal{M}) = \mathcal{R}$.*

It is known that every projective ideal is a multiplication ideal ([9]), the following is more general;

Proposition 2.2. *Every pwp. ideal is a multiplication ideal.*

Proof. Let I be a pwp. ideal of $\mathcal{R}$. By ([14]), for all $a \in I$, $\mathrm{ann}(a) + T(I) = \mathcal{R}$. To show I is a multiplication ideal it is enough by (2.1) to prove that $T \subseteq \Theta\,(I)$. Let $u \in T$, then $u = \sum_j \ell_j(t)$; $\ell_j \in I^*$, $t_j \in I$. Now, to prove that $u \in \Theta\,(I)$, we must show that $\ell_j(t_j)I \subseteq (t_j)$. Let $x \in \ell_j(t_j)I$, then $x = \ell_j(t_j)z = \ell_j(z)t_j \in (t_j)$. ☐

The following example shows that the last proposition does not hold in pwp. modules in general.

Example 2.3. Let $\mathcal{M} = \mathbb{R} \oplus \mathbb{R}$ be an $\mathbb{R}$-module, hence $\mathcal{M}$ is a pwp. module (Note that $\mathcal{M}$ is a projective module). Assume $\mathcal{M}$ is multiplication, then $\mathcal{S}$ is commutative ([9]). But $\mathcal{S}$ isomorphic to 2×2 matrices on $\mathbb{R}$ which is not commutative, that is contradiction. Thus $\mathcal{M}$ is not a multiplication module.

Next we consider when pwp. $\mathcal{R}$-modules are multiplication modules. Compare with [9].

Proposition 2.4. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$-module with $\mathcal{S}$ a commutative ring, then $\mathcal{M}$ is a multiplication module.*

Proof. By [14], for all $n \in \mathcal{M}$, $\mathrm{ann}(n) + T = \mathcal{R}$. By (2.1) it is enough to prove $T \subseteq \Theta(\mathcal{M})$. Let $u \in T$, by [14] $u \in T^2$. Hence $u = \sum_i \sum_j \sum_k \ell_{ij}\,(m_{ij})\,\alpha_{ik}\,(x_{ik})$ where $\ell_{ij}, \alpha_{ik} \in \mathcal{M}^*, m_{ij}, x_{ik} \in \mathcal{M}$. Now, we show that $\sum_j \sum_k \ell_{ij}\,(m_{ij})\,\alpha_{ik}\,(x_{ik})\,\mathcal{M} \subseteq (x_{ik})$. For all $y \in \mathcal{M}$,

$$\begin{aligned} \ell_{ij}\,(m_{ij})\,\alpha_{ik}\,(x_{ik})\,y &= [y, \alpha_{ik}]\,([x_{ik}, \ell_{ij}]\,(m_{ij})) \\ &= [x_{ik}, \ell_{ij}]\,([y, \alpha_{ik}]\,(m_{ij})) \\ &= \alpha_{ik}\,(m_{ij})\,\ell_{ij}(y)x_{ik} \\ &\in (x_{ik}). \end{aligned}$$

□

The following is an immediate consequence of the proof of the last proposition.

Corollary 2.5. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$-module such that $T \subseteq \Theta(\mathcal{M})$, then $\mathcal{M}$ is a multiplication module.*

It is known that the endomorphism ring of a locally cyclic $\mathcal{R}$-module is commutative ([9]). Thus we have at once, compare with [9].

Proposition 2.6. *Every locally cyclic pwp. $\mathcal{R}$-module is a multiplication module.*

We summarize the last results in the following theorem, compare with [9].

Theorem 2.7. *Let $\mathcal{M}$ be a pwp. $\mathcal{R}$-module, then the following statements are equivalent:*

(1) *$\mathcal{M}$ is a multiplication module.*

(2) *$\mathcal{S}$ is a commutative ring.*

(3) *$\mathcal{M}$ is locally cyclic.*

(4) *$T \subseteq \Theta(\mathcal{M})$.*

Remark. The last theorem fails if the condition that $\mathcal{M}$ is a pwp. $\mathcal{R}$-module is removed. For example, let $\mathbb{Q}$ be the module of rational numbers over the integers $\mathbb{Z}$. It is known that $\mathrm{End}(\mathbb{Q}) \cong \mathbb{Q}$ ([4]), hence commutative. But it is easy to see that $\mathbb{Q}$ is not a multiplication module. Observe that $\mathbb{Q}$ is not a pwp. module ([14]).

It is known that if $\mathcal{M}$ is a finitely generated multiplications $\mathcal{R}$-module, then $\mathcal{M}$ is projective if and only if $\mathrm{ann}(\mathcal{M}) = \mathcal{R}(1-e)$; $e^2 = e$ ([10]). This statement is false if $\mathcal{M}$ is not finitely generated. However, we have the following.

Proposition 2.8. *Let $\mathcal{M}$ be a multiplication $\mathcal{R}$-module with $\mathrm{ann}(\mathcal{M}) = \mathcal{R}(l-e)$; $e^2 = e$. Then $\mathcal{M}$ is a pwp. $\mathcal{R}$-module.*

Proof. Let $a \in \mathcal{M}$, then $\mathcal{R}a$ is a cyclic submodule of $\mathcal{M}$. By [10] $a = \sum_{i=1}^{n} f_i(a) b_i$; $b_i \in \mathcal{M}$, $f_i \in \mathcal{M}^*$. By (1.1) $\mathcal{M}$ is a pwp. module. □

Remark. The last proposition is false without the condition $\mathrm{ann}(\mathcal{M}) = \mathcal{R}(1-e)$; $e^2 = e$. For example, $\mathbb{Z}_2$ as $\mathbb{Z}$-module is multiplication but it is not pwp. Note that $\mathrm{ann}(\mathbb{Z}_2)$ is not generated by an idempotent element.

The following theorem is a generalization of proposition (2.2).

Proposition 2.9. *Every pwp. submodule of a flat multiplication module is a multiplication module.*

Proof. Let $\mathcal{M}$ be a pwp. submodule of the flat multiplication module $\mathcal{N}$. Since $\mathcal{N}$ is a flat module, then for each maximal ideal P of $\mathcal{R}$, $\mathcal{N}_P$ is a flat $\mathcal{R}_p$-module, and by [3] N_p is cyclic. Thus $\mathrm{ann}(\mathcal{N}_p)$ is a pure ideal in $\mathcal{R}_p$. Hence $\mathrm{ann}(\mathcal{N}_p) = 0$ or $\mathrm{ann}(\mathcal{N}_p) \cong \mathcal{R}_p$, thus either $\mathcal{N}_P = (0)$ or $\mathcal{N}_p \cong \mathcal{R}_p$. On the other hand, by [14], $\mathcal{M}_p$ is a pwp. $\mathcal{R}_p$-module and it is contained in $\mathcal{N}_p$, note that $\mathcal{M}_p = (0)$ if $\mathcal{N}_p = (0)$. Hence $\mathcal{M}_p$ is isomorphic to a pwp. ideal of $\mathcal{R}_p$. By (2.2) $\mathcal{M}_p$ is a multiplication module, then $\mathcal{M}_p$ is cyclic. Hence $\mathcal{M}$ is locally cyclic. Therefore by (2.7) $\mathcal{M}$ is multiplication. $\square$

Since every multiplication module with pure annihilator is flat ([8]), we have.

Corollary 2.10. *Every pwp. submodule of a multiplication module with pure annihilator is multiplication.*

3. Quasi-multiplication modules and pointwise projective modules

Let $\mathcal{N}$ be a submodule of an $\mathcal{R}$-module $\mathcal{M}$, put $[\mathcal{N} : \mathcal{M}]_S = \{f \in \mathrm{End}(\mathcal{M}) | f(\mathcal{M}) \subseteq \mathcal{N}\}$, $\Theta_S(\mathcal{M}) = \sum_{a\in\mathcal{M}}[a : \mathcal{M}]_S$, and $\mathrm{ann}_S(a) = \{f \in \mathrm{End}(\mathcal{M})|f(a) = 0\}$; $a \in \mathcal{M}$. An $\mathcal{R}$-module $\mathcal{M}$ is called quasi-multiplication if for all $a \in \mathcal{M}$, $\mathrm{ann}_S(a) + \Theta_S(\mathcal{M}) = S$ ([5]).

It is known that every projective module is quasi-multiplication module ([5]). A similar result holds for pwp. modules.

Proposition 3.1. *Every pwp. module is a quasi-multiplication module.*

Proof. Let $\mathcal{M}$ be an $\mathcal{R}$-module, and let $m \in \mathcal{M}$. By (1.1), $m = \sum_{k=1}^{n} \varphi_k(m)x_k$; $x_k \in \mathcal{M}$, $\varphi_k \in \mathcal{M}^*$. Then $m = \sum_{k=1}^{n}[x_k, \varphi_k](m)$. If $\gamma = \sum_{k=1}^{n}[x_k, \varphi_k]$, then $\gamma \in \Delta$, thus $m = \gamma(m)$. Therefore $1 - \gamma \in \mathrm{ann}_S(m)$. But $\gamma \in \Theta_S(\mathcal{M})$, then $\mathrm{ann}_S(m) + \Theta_S(\mathcal{M}) = S$. $\square$

The following example shows that there are quasi-multiplication modules, which are not pwp. modules

Example 3.2. Let $\mathcal{R} = \mathbb{Z}_8$. Consider the $\mathcal{R}$-module $\mathcal{M} = \left(\overline{0}, \overline{4}\right) \oplus \mathbb{Z}_8$. It is Clear that $\left(\overline{0}, \overline{4}\right)$ and $\mathbb{Z}_8$ are quasi- multiplication modules. By [5] $\mathcal{M}$ is a quasi-multiplication module. Since the only non-zero homomorphism $f : (\overline{0}, \overline{4}) \longrightarrow \mathbb{Z}_8$ is $f(\overline{x}) = \overline{x}$, then by (1.1) $(\overline{0}, \overline{4})$ is not pwp. Therefore $\mathcal{M}$ is not pwp. ([14]).

Next we consider when quasi-multiplication modules are pwp. modules. It is known that if $\mathcal{M}$ is a finitely generated quasi multiplication $\mathcal{R}$-module such that for all $a \in \mathcal{M}$, $\mathrm{ann}(a)$ is generated by an idempotent element, then $\mathcal{M}$ is a projective module ([5]). We generalize this result in theorem (3.4). But first we recall the following.

Proposition 3.3 ([18]). *Let $\mathcal{M}$ be an $\mathcal{R}$-module, then $\mathcal{M}$ is a pwp. module if and only if for all column-finite matrices $\{a_{ij}|i \in I;\ j \in J\}$; $a_{ij} \in \mathcal{R}$ (i.e. each column has only a finite number of non-zero elements), and all families $\{m_i|i \in I\}$; $m_i \in \mathcal{M}$, with $\sum_i a_{ij}m_i = 0$ for all $j \in J$, and for each finite subset I_o of I, there are finite families $\{x_k|k \in K\}$; $x_k \in \mathcal{M}$, $\{r_{ki}|k \in K, i \in I\}$; $r_{ki} \in \mathcal{R}$ such that:*

a. $\sum_{i\in I_0} r_{ki}a_{ij} = 0$.

b. $m_i = \sum_k r_{ki} x_k, \quad \forall i \in I_0$.

Theorem 3.4. *Let $\mathcal{M}$ be a quasi-multiplication $\mathcal{R}$-module such that for all $a \in \mathcal{M}$, $\operatorname{ann}(a) = \mathcal{R}(l-e)$; $e^2 = e$ and e depends on a, $\mathcal{M}$ is a pwp. module.*

Proof. Let $\{a_{kj} | k \in K,\ j \in J\}$; $a_{kj} \in \mathcal{R}$, and let $\{m_k | k \in K\}$; $m_k \in \mathcal{M}$, such that:

$$\sum_{k \in K} a_{kj} m_k = 0, \quad \forall j \in J \tag{1}$$

Since $\operatorname{ann}_S(m_k) + \Theta_S(\mathcal{M}) = S$, then there exist $\varphi \in \operatorname{ann}_S(m_k)$, $g = \sum_{i \in I} \gamma_i \in \Theta_S(\mathcal{M})$, with:

$$\varphi + g = I \tag{2}$$

Since $\gamma_i(\mathcal{M}) \subseteq (m_i)$, then $\gamma_i(m_k) = r_{ik} m_i$. Let $K = \{l, 2, \cdots, n\}$ be an arbitrary subset of K, and let $\{r_{ik} e_i |\ k \in K,\ i \in I\}$; $r_{ik} e_i \in \mathcal{R}$, where e_i is an idempotent element such that $\operatorname{ann}(m_i) = \mathcal{R}(l - e_i)$. Thus $m_i = e_i m_i$. By (2), $\sum_{i \in I} \gamma_i(m_k) = m_k$. Therefore $\sum_{i \in I} r_{ik} m_i = m_k$, hence $m_k = \sum_{i \in I} r_{ik} e_i m_i$. By (1), $0 = \gamma_i \left(\sum_{k \in K'} a_{kj} m_k \right) = m_i \left(\sum_k a_{kj} r_{ik} \right)$. Then

$$\sum_k a_{kj} r_{ik} e_i = \left(\sum_k a_{kj} r_{ik} \right) e_i = 0.$$

By (3.3), $\mathcal{M}$ is a pwp. module. □

It was proved in [5] that if $\mathcal{M}$ is a finitely generated quasi multiplication $\mathcal{R}$-module with $\mathcal{R}/\operatorname{ann}(\mathcal{M}) = S$, then $\mathcal{M}$ is multiplication. Besides if $\operatorname{ann}(\mathcal{M}) = \mathcal{R}(l-e)$; $e^2 = e$, then $\mathcal{M}$ is a projective module ([15]). We generalize this result as follows.

Proposition 3.5. *Let $\mathcal{M}$ be a quasi-multiplication $\mathcal{R}$-module with $\mathcal{R}/\operatorname{ann}(\mathcal{M}) = S$ and $\operatorname{ann}(\mathcal{M}) = \mathcal{R}(l-e)$; $e^2 = e$, then $\mathcal{M}$ is a pwp. module.*

Proof. Let $0 \neq m \in \mathcal{M}$. Since $\operatorname{ann}_S(m) + \Theta_S(\mathcal{M}) = S$, there exist $\varphi \in \operatorname{ann}_S(m)$ and $g = \sum \gamma_j \in \Theta_S(\mathcal{M})$ such that $\varphi + g = I$. Since $S = \mathcal{R}/\operatorname{ann}(\mathcal{M})$, then there exists $r_j \in \mathcal{R}$ such that $\gamma_j(x) = r_j x$ for all $x \in \mathcal{M}$. Because $\gamma_j(\mathcal{M}) \subseteq (a_j)$; $a_j \in \mathcal{M}$, thus $r_j \mathcal{M} \subseteq (a_j)$. Hence for all $x \in \mathcal{M}$, $r_j \in \Theta(\mathcal{M})$. Now $m = g(m) = \sum_j r_j m$, hence $1 - \sum r_j \in \operatorname{ann}(m)$. Therefore by (2.1) $\mathcal{M}$ is multiplication. By (2.8) $\mathcal{M}$ is pwp. □

4. Weak multiplication modules and pointwise projective modules

An $\mathcal{R}$-module $\mathcal{M}$ is said to be a weak multiplication module if for each submodule $\mathcal{N}$ of $\mathcal{M}$, $\mathcal{N} = \sum \varphi(\mathcal{M})$, where the sum is taken over all $\varphi \in \hom(\mathcal{M}, \mathcal{N})$ ([12]).

It is proved in [12] that a T-accessible $\mathcal{R}$-module $\mathcal{M}$ is weak multiplication if and only if for all $m \in \mathcal{M}$, $m \in Tm$. Since for every pwp. $\mathcal{R}$-module $\mathcal{M}$, for all $m \in \mathcal{M}$, $m \in Tm$ ([14]), so we have at once.

Proposition 4.1. *Every pwp. $\mathcal{R}$-module is weak multiplication.*

Remark. The converse of the last proposition is false. In fact, the $\mathbb{Z}$-module Z_2 is weak multiplication ([12]), but it is not pwp. ([14]).

Therefore we will look at conditions under which weak multiplication modules are pwp. modules.

Proposition 4.2. *Let $\mathcal{M}$ be weak multiplication T-accessible $\mathcal{R}$-module. If S is commutative, then $\mathcal{M}$ is a pwp. module.*

Proof. By assumption, for all $m \in \mathcal{M}$, $m \in Tm$. By [14] $\mathcal{M}$ is pwp. □

Remark. Let $\mathcal{M}$ be an $\mathcal{R}$-module. It is proved in [12] that if $T = \mathcal{R}$, then $\mathcal{M}$ is weak multiplication, therefore there exist weak multiplication modules which are not pwp. modules. For examples, let $\mathcal{M} = \mathbb{Z}_2 \oplus \mathbb{Z}$ as $\mathbb{Z}$-module, hence $T(\mathcal{M}) = \mathbb{Z}$, that means $\mathcal{M}$ is weak multiplication. But $\mathcal{M}$ is not pwp., because $\mathbb{Z}_2$ as $\mathbb{Z}$-module is not pwp. ([14]). Note that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_2) \cong \mathbb{Z}_2$, $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}$ and $\hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_2) \neq 0$ hence by [16] $\mathrm{End}_{\mathbb{Z}}(\mathcal{M})$ is not commutative.

It is observed in [14] that a pwp. $\mathcal{R}$-module is not necessary a pwp. S-module. The next proposition shows when this statement holds. First we need the following proposition ([6]).

Proposition 4.3.

(1) *Let $\mathcal{M}$ be a weak multiplication $\mathcal{R}$-module such that S is a commutative ring, then $\mathcal{M}$ is a weak multiplication S-module.*

(2) *Let $\mathcal{M}$ be a T-accessible $\mathcal{R}$-module. If $\mathcal{M}$ is a weak multiplications S-module and S is a commutative ring, then $\mathcal{M}$ is a weak multiplication $\mathcal{R}$-module.*

Proposition 4.4. *Let $\mathcal{M}$ be an $\mathcal{R}$-module such that S is commutative. Then $\mathcal{M}$ is a pwp. $\mathcal{R}$-module if and only if $\mathcal{M}$ is a pwp. $\mathcal{S}$-module.*

Proof. ($\Rightarrow$) By (4.1), $\mathcal{M}$ is a weak multiplication $\mathcal{R}$-module, then by (4.3) $\mathcal{M}$ is a weak multiplication $\mathcal{S}$-module. By [17] $\Delta\mathcal{M} = \mathcal{M}$, and ${}_ST\mathcal{M} = M$, thus by (4.2) $\mathcal{M}$ is a pwp. S-module.

($\Leftarrow$) By [17], $\Delta\mathcal{M} = \mathcal{M}$, hence $T\mathcal{M} = \mathcal{M}$ and $\mathcal{M}$ is a weak multiplication S-module. By (4.3) $\mathcal{M}$ is a weak multiplication $\mathcal{R}$-module. Therefore by (4.2) $\mathcal{M}$ is a pwp. $\mathcal{R}$-module. □

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