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Hyers-Ulam-Rassias Stability of Popoviciu's Functional Equation in Banach Modules

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Abstract. In this paper we study the Hyers-Ulam-Rassias stability of Popoviciu's functional equation in Banach modules over a Banach algebra.

1. Introduction

In 1940, S. M. Ulam ([13]) raised the following question: *Under what conditions does there exist an additive mapping near an approximately additive mapping ?*

In 1941, for Banach spaces the Ulam problem was first solved by D. H. Hyers ([7]) by proving that if $\delta > 0$ and $f : E_1 \to E_2$ is a mapping with E_1, E_2 Banach spaces, such that $||f(x+y)-f(x)-f(y)|| \le \delta$ for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \to E_2$ such that $||f(x) - T(x)|| \le \delta$ for all $x \in E_1$.

In 1978, Th. M. Rassias ([11]) gave a generalization of the Hyers' result in the following way: Let E_1 and E_2 be a normed space and a Banach space, respectively, and $f : E_1 \to E_2$ a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ (the real field) for each fixed $x \in E_1$. Assume that there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that $\|f(x+y)-f(x)-f(y)\| \le \theta(||x||^p + ||y||^p)$ for all $x, y \in E_1$. Then there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that $\|f(x) - T(x)\| \le \frac{2\theta}{2-2^p}||x||^p$ for all $x \in E_1$.

In connection with the facts above, the stability problems of functional equations have been extensively investigated by many mathematicians (see, for example, [2], [3], [4], [5], [6], [8], [9]).

Recently T. Trif ([12]) studied the Hyers-Ulam-Rassias stability of the Popoviciu's functional equation (from [10]) for normed spaces which is the Jensen type

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functional equation

$$3f\Big(\frac{x+y+z}{3}\Big)+f(x)+f(y)+f(z)=2\Big[f\big(\frac{x+y}{2}\big)+f\big(\frac{y+z}{2}\big)+f\big(\frac{z+x}{2}\big)\Big].$$

We here extend the Hyers-Ulam-Rassias stability of the Popoviciu's functional equation to Banach modules over a Banach algebra, and obtain some related results.

2. Results

Throughout this section, let B be a unital normed algebra with norm $|\cdot|$ over the complex field $\mathbb{C}$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be a left normed B-module and a left Banach B-module with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

Note that a mapping $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is called B-linear if $f(ax) = af(x)$ for all $a \in B$ and all $x \in {}_B\mathbb{B}_1$.

Given a function $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, we set

$$\begin{aligned} Df(x,y,z) \quad := \quad & 3f\left(\frac{ax+ay+az}{3}\right)+af(x)+af(y)+af(z) \\ & -2\Big[af\big(\frac{x+y}{2}\big)+f\big(\frac{ay+az}{2}\big)+f\big(\frac{az+ax}{2}\big)\Big] \end{aligned}$$

for all $a \in B$ and all $x, y, z \in {}_B\mathbb{B}_1$.

Theorem 1. *Assume that $\delta,\ \theta \in [0,\ \infty)$ and that $p \in (0,1)$. Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping such that*

$$\|Df(x,y,z)\| \le \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{1}$$

for all $a \in B$ with $|a| = 1$ and all $x, y, z \in {}_B\mathbb{B}_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x)-f(0)-A(x)\| \le \frac{\delta}{3}+\frac{\theta}{2^{1-p}-1}\|x\|^p \quad \textit{for all} \quad x \in {}_B\mathbb{B}_1.$$

Proof. By [12, Theorem 3.1], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the condition given in the statement. The additive mapping A given in the proof of [12, Theorem 3.1] is similar to the additive mapping given in the proof of [11, Theorem].

Using the same reasoning as in the proof of [11, Theorem] and the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, it follows that the additive mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is $\mathbb{R}$-linear.

Let $a \in B$ with $|a| = 1$. Setting $y = x$ and $z = -2x$ in (1), we get

$$\|3f(0)+af(-2x)-4f(-\frac{a}{2}x)\| \le \delta+\theta(2+2^p)\|x\|^p \quad \text{for all} \quad x \in {}_B\mathbb{B}_1. \tag{2}$$

Put $\varepsilon := \delta + 3\|f(0)\|$. From (2) we have

$$\|af(-2x) - 4f(-\frac{a}{2}x)\| \leq \varepsilon + \theta(2 + 2^p)\|x\|^p \quad \text{for all} \quad x \in {}_B\mathbb{B}_1.$$

Replacing x by $-2x$ in the above relation yields

$$\|af(4x) - 4f(ax)\| \leq \varepsilon + \theta 2^{2p}(1 + 2^{1-p})\|x\|^p \quad \text{for all} \quad x \in {}_B\mathbb{B}_1. \tag{3}$$

Using induction on n with (3), we see that

$$\|af(2^{2n}x) - 4f(2^{2(n-1)}ax)\| \leq \varepsilon + \theta 2^{2np}(1 + 2^{1-p})\|x\|^p \tag{4}$$

for all $x \in {}_B\mathbb{B}_1$ and all positive integers n. Note that there exists a $K > 0$ such that $\|az\| \leq K|a|\ \|z\|$ for all $a \in B$ and all $z \in {}_B\mathbb{B}_2$ by the definition of a normed module.

Now letting $a = 1$ in (4) and then replacing x by ax in the result, we obtain

$$\begin{aligned} \|f(2^{2n}ax) - 4f(2^{2(n-1)}ax)\| &\leq \varepsilon + \theta 2^{2np}(1 + 2^{1-p})\|ax\|^p \\ &\leq \varepsilon + \theta 2^{2np}(1 + 2^{1-p})K^p\|x\|^p \quad \text{for all} \quad x \in {}_B\mathbb{B}_1. \end{aligned} \tag{5}$$

On account of (4) and (5), we get

$$\begin{aligned} \|f(2^{2n}ax) - af(2^{2n}x)\| &= \|f(2^{2n}ax) - 4f(2^{2(n-1)}ax) \\ &\quad +4f(2^{2(n-1)}ax) - af(2^{2n}x)\| \\ &\leq \|f(2^{2n}ax) - 4f(2^{2(n-1)}ax)\| \\ &\quad +\|af(2^{2n}x) - 4f(2^{2(n-1)}ax)\| \\ &\leq 2\varepsilon + (K^p + 1)2^{2np}(1 + 2^{1-p})\|x\|^p \end{aligned}$$

for all $x \in {}_B\mathbb{B}_1$. So $2^{-2n}\|f(2^{2n}ax) - af(2^{2n}x)\| \to 0$ as $n \to \infty$ for all $x \in {}_B\mathbb{B}_1$. Hence we conclude that

$$A(ax) = \lim_{n\to\infty} 2^{-2n}f(2^{2n}ax) = \lim_{n\to\infty} 2^{-2n}af(2^{2n}x) = aA(x)$$

for all $a \in B$ with $|a| = 1$ and all $x \in {}_B\mathbb{B}_1$. Since A is $\mathbb{R}$-linear and $A(cx) = cA(x)$ for each element $c \in B$ with $|c| = 1$, we have

$$\begin{aligned} A(ax + by) &= A(ax) + A(by) \\ &= A(|a|\frac{a}{|a|}x) + A(|b|\frac{b}{|b|}y) \\ &= |a|A(\frac{a}{|a|}x) + |b|A(\frac{b}{|b|}y) \\ &= |a|\frac{a}{|a|}\ A(x) + |b|\frac{b}{|b|}A(y) \\ &= aA(x) + bA(y) \end{aligned}$$

for all $a, b \in B \setminus \{0\}$ and all $x, y \in {}_B\mathbb{B}_1$. Thus the unique $\mathbb{R}$-linear mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is a B-linear mapping, as desired. □

Corollary 1. *Let E_1 and E_2 be a complex normed space and a complex Banach space, respectively. Let $f : E_1 \to E_2$ be a mapping such that*

$$\|Df(x, y, z)\| \leq \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for $a = 1$, i and all $x, y \in E_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-linear mapping $A : E_1 \to E_2$, where $\mathbb{C}$ is the complex field, such that

$$\|f(x) - f(0) - A(x)\| \leq \frac{\delta}{3} + \frac{\theta}{2^{1-p} - 1}\|x\|^p \quad \textit{for all} \quad x \in {}_B\mathbb{B}_1.$$

Proof. Since $\mathbb{C}$ is a complex Banach algebra, we see that E_1 and E_2 are considered as a normed $\mathbb{C}$- module and a Banach $\mathbb{C}$- module, respectively. By Theorem 1, there exists a unique $\mathbb{C}$-linear mapping $A : E_1 \to E_2$ satisfying the condition given in the statement. □

Theorem 2. *Let $\theta \in [0, \infty)$ and $p \in (1, \infty)$. Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping satisfying $f(0) = 0$ and*

$$\|Df(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p) \tag{6}$$

for all $a \in B$ with $|a| = 1$ and all $x, y, z \in {}_B\mathbb{B}_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2^{p-1}}{2^{p-1} - 1}\theta\|x\|^p \quad \textit{for all} \quad x \in {}_B\mathbb{B}_1.$$

Proof. By [12, Theorem 3.3], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfying the condition given in the statement. The additive mapping A given in [12, Theorem 3.3] is similar to the additive mapping given in the proof of [11, Theorem].

Using the same reasoning as in the proof of [11, Theorem] and the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, it follows that the additive mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is $\mathbb{R}$-linear.

Let $a \in B$ with $|a| = 1$. Putting $y = x$ and $z = -2x$ in (6) we see, as in the proof of Theorem 1, that

$$\|af(-2x)\| - 4f(-\frac{a}{2}x)\| \leq \ \theta(2 + 2^p)\|x\|^p \quad \text{for all} \quad x \in {}_B\mathbb{B}_1.$$

Replacing x by $-\frac{x}{2}$ in the above relation yields

$$\|af(x) - 4f(2^{-2}ax)\| \leq \ \theta(1 + 2^{p-1})2^{1-p}\|x\|^p \quad \text{for all } x \in {}_B\mathbb{B}_1. \tag{7}$$

Starting from (7) it is easy to prove that

$$\|af(2^{-2n}x) - 4f(2^{-2(n+1)}ax)\| \leq \theta(1+2^{p-1})2^{1-(2n+1)p}\|x\|^p$$

for all $x \in {}_B\mathbb{B}_1$ and all positive integers n.

Following the similar method as in the proof of Theorem 1, we have

$$\|f(2^{-2n}ax) - af(2^{-2n}x)\| \leq (K^p+1)\theta(1+2^{p-1})2^{1-(2n+1)p}\|x\|^p$$

for all $x \in {}_B\mathbb{B}_1$ and some $K > 0$. So $2^n\|f(2^{-2n}ax) - af(2^{-2n}x)\| \to 0$ as $n \to \infty$ for all $x \in {}_B\mathbb{B}_1$. The rest of the proof is similar to the corresponding part of the proof of Theorem 1. □

Corollary 2. *Let E_1 and E_2 be a complex normed space and a complex Banach space, respectively. Let $f : E_1 \to E_2$ be a mapping such that*

$$\|Df(x,y,z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for $a = 1,\ i$ and all $x, y \in E_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-linear mapping $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2^{p-1}}{2^{p-1}-1}\theta\|x\|^p \quad \textit{for all} \quad x \in {}_B\mathbb{B}_1.$$

Proof. The proof is similar to the one of Corollary 1 by using Theorem 2. □

Theorem 3. *Assume that $\delta,\ \theta \in [0,\ \infty)$ and that $p \in (0,1)$. Let B be a unital Banach $*$-algebra, and B^+ the set of positive elements of B. Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping such that*

$$\|Df(x,y,z)\| \leq \delta + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $a \in B^+$ with $|a| = 1$ or $a = i$, and all $x,\ y,\ z \in {}_B\mathbb{B}_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - f(0) - A(x)\| \leq \frac{\delta}{3} + \frac{\theta}{2^{1-p}-1}\|x\|^p \quad \textit{for all} \quad x \in {}_B\mathbb{B}_1.$$

Proof. By the same reasoning as in the proof of Theorem 1, there exists a unique $\mathbb{R}$-linear mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - f(0) - A(x)\| \leq \frac{\delta}{3} + \frac{\theta}{2^{1-p}-1}\|x\|^p \quad \text{for all } x \in {}_B\mathbb{B}_1.$$

By the same method as the proof of Theorem 2.1, we see that

$$A(ax) = \lim_{n\to\infty} 2^{-2n}f(2^{2n}ax) = \lim_{n\to\infty} 2^{-2n}af(2^{2n}x) = aA(x)$$

for all $a \in B^+$ with $|a| = 1$ or $a = i$, and all $x \in {}_B\mathbb{B}_1$, and so

$$\begin{aligned} A(ax+by) &= aA(x)+bA(y), \\ A(ix) &= iA(x) \end{aligned}$$

for all $a, b \in B^+ \setminus \{0\}$ and all $x, y \in {}_B\mathbb{B}_1$. For any element $a \in B$, $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = {a_1}^+ - {a_1}^- + i{a_2}^+ - i{a_2}^-$, where ${a_1}^+$, ${a_1}^-$, ${a_2}^+$, and ${a_2}^-$ are positive elements (see [1], Lemma 38.8). Therefore,

$$\begin{aligned} A(ax) &= A({a_1}^+ x {a_1}^- x + i{a_2}^+ x - i{a_2}^- x) \\ &= {a_1}^+ A(x) {a_1}^- A(x) + {a_2}^+ A(ix) - {a_2}^- A(ix) \\ &= {a_1}^+ A(x) - {a_1}^- A(x) + i{a_2}^+ A(x) - i{a_2}^- A(x) \\ &= ({a_1}^+ - {a_1}^- + i{a_2}^+ - i{a_2}^-)A(x) \\ &= aA(x) \end{aligned}$$

for all $a \in B$ and all $x \in {}_B\mathbb{B}_1$. Hence $A(ax+by) = A(ax) + A(by)aA(x + bA(y)$ for all $a, b \in B$ and all $x, y \in {}_B\mathbb{B}_1$. Thus there exists a unique B-linear mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - f(0) - A(x)\| \le \frac{\delta}{3} + \frac{\theta}{2^{1-p}-1}\|x\|^p \quad \text{for all} \quad x \in {}_B\mathbb{B}_1.$$

We complete the proof of the theorem. □

Theorem 4. *Assume that $\theta \in [0, \infty)$ and $p \in (1, \infty)$. Let B be a unital Banach $*$-algebra over $\mathbb{C}$, and B^+ the set of positive elements of B. Let $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ be a mapping satisfying $f(0) = 0$ such that*

$$\|Df(x,y,z)\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $a \in B^+$ with $|a| = 1$ or $a = i$, and all $x, y, z \in {}_B\mathbb{B}_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-linear mapping $A : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ such that

$$\|f(x) - A(x)\| \le \frac{2^{p-1}}{2^{p-1}-1}\theta\|x\|^p \quad \textit{for all} \quad x \in {}_B\mathbb{B}_1.$$

Proof. The proof is similar to the one of Theorem 3. □

Remark. In Theorem 1, 2, 3 and 4, when the difference

$$\begin{aligned} Df(x,y,z) \;:=\; & 3f\left(\frac{ax+ay+az}{3}\right) + af(x) + af(y) + af(z) \\ & -2\left[af\left(\frac{x+y}{2}\right) + f\left(\frac{ay+az}{2}\right) + f\left(\frac{az+ax}{2}\right)\right] \end{aligned}$$

is replaced by

$$\begin{aligned} Df(x,y,z) &:= 3f\left(\frac{ax+ay+az}{3}\right)+f(ax)+f(ay)+af(z) \\ &\quad -2\left[f\left(\frac{ax+ay}{2}\right)+f\left(\frac{ay+az}{2}\right)+f\left(\frac{az+ax}{2}\right)\right] \end{aligned}$$

or

$$\begin{aligned} Df(x,y,z) &:= 3f\left(\frac{ax+ay+az}{3}\right)f(ax)+f(ay)+f(az) \\ &\quad -2\left[f\left(\frac{ax+ay}{2}\right)+af\left(\frac{y+z}{2}\right)+af\left(\frac{z+x}{2}\right)\right], \end{aligned}$$

the results do also hold.

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