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On the Sum of Two Radical Classes

Dedicated to the memory of Prof. A. M. Zaidi.

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Abstract. Let $\wp_1$, $\wp_2$ be the radical classes of rings. Y. Lee and R. E. Propes have defined their sum by $\wp_1 + \wp_2 = \{R \in \omega : \wp_1(R) + \wp_2(R) = R\}$. They have shown that $\wp_1 + \wp_2$ is not a radical class in general. In this paper, a few results of Lee and Propes are generalized and also new conditions are investigated under which this sum becomes a radical class.

1. Introduction

Lee and Propes ([2]) introduced the concept of the sum of two radical classes. They have shown that 'Sum' is not a radical class in general. In the present paper we generalize a few results of [2] and investigate new conditions under which 'Sum' becomes a radical class. In the following we shall be working within the class of all associative rings. Let $\wp_1$, $\wp_2$ be radical classes of rings, then 'Sum' is defined as

$$\wp_1 + \wp_2 = \{R \in \omega : \wp_1(R) + \wp_2(R) = R\}.$$

The following main results were proved by Lee and Propes ([2]).

Theorem 1 ([2]). *If $S(\alpha) \cap \tau = 0$, $S(\tau) \cap \alpha = 0$, and $\alpha \cap \tau = 0$, then $\alpha + \tau$ is a radical class. Recall that $S(\alpha)$ and $S(\tau)$ are the semi-simple classes of the radicals α and τ, respectively.*

Theorem 2 ([2]). *Assume that $S(\tau) \cap \alpha = 0$ and $S(\alpha) \cap \tau = 0$ and that (I, $\mathcal{M} \leq R$, $I/\mathcal{M} \in \alpha \cap \tau$, $\mathcal{M} \supset \alpha(R)$) implies $I \in \alpha \cap \tau$. Then $\alpha + \tau$ is a radical class.*

Theorem 3 ([2]). *If $S(\tau) \cap \alpha = 0$ and $S(\alpha) \cap \tau = 0$ and $\alpha + \tau = \alpha \cup \tau$, then $\alpha + \tau$ is a radical class.*

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Let ω be the universal class of all associative rings and $\mathcal{M}$ be a sub-class of ω and let $\mathcal{M}_0$ be the homomorphic closure of $\mathcal{M}$ in ω. For each $R \in \omega$, let $D_1(R)$ be the set of all ideals of R. Inductively we define

$$D_{n+1}(R) = \{I : I \text{ is an ideal of some ring in } D_n(R)\}.$$

Let $D(R) = \cup D_n(R),\ \ n = 1, 2, 3, \cdots$. By [1], we have

$$\mathcal{LM} = \{R \in \omega : D(R/I) \cap \mathcal{M}_0 \neq 0, \text{ for each proper ideal } I \text{ of } R\},$$

is the construction for lower radical determined by $\mathcal{M}$, and $\mathcal{M} \subseteq \mathcal{LM}$.

Definition 4. A sub-class $\wp$ of ω is said to be a radical class if

(1) $\wp$ is homomorphically closed.

(2) For each $R \in \omega$ has maximum $\wp$-ideal, $\wp(R)$.

(3) For each $R \in \omega$, we have $R/\wp(R) \in S\wp$ i.e. $\wp(R/\wp(R)) = 0$.

In [2], it was shown that $\wp_1 + \wp_2$ always satisfied (1), (2). For undefined terms, we may refer [3] and [4].

Theorem 5. *If* $S\wp_1 \cap \wp_2 = 0$ *then* $\wp_1 + \wp_2$ *is a radical class.*

Proof. By [2], $\wp_1 + \wp_2$ is homomorphically closed, therefore $\wp_1 + \wp_2 \subseteq \mathcal{L}(\wp_1 + \wp_2)$ (see [4], theorem 9.3, page 40), for the reverse inclusion let $R \in \omega$ such that $R \notin \wp_1 + \wp_2$, therefore $(\wp_1 + \wp_2)(R) \neq R$, $\wp_1(R) + \wp_2(R) \neq R$ and consequently we have $\wp_1(R) \neq R$, $R/\wp_1(R) \neq 0$. Let $K \in D(R/\wp_1(R)) \cap (\wp_1 + \wp_2)$. Now $K \in D(R/\wp_1(R))$ implies that there is a finite chain of ideals such that

$$K = K_n/\wp_1(R) \leq K_{n-1}/\wp_1(R) \leq \cdots \leq K_1/\wp_1(R) \leq R/\wp_1(R).$$

Since $R/\wp_1(R) \in S\wp_1$ and $S\wp_1$ is hereditary (see [4], theorem 8.1, page 29). Hence $K \in S\wp_1$ and consequently $\wp_1(K) = 0$. Since $K \in \wp_1 + \wp_2$, therefore $\wp_1(K) + \wp_2(K) = K$ and hence we have $\wp_2(K) = K$. This implies that $K \in \wp_2$ and hence we have $K \in \wp_2 \cap S\wp_1 = 0$ and hence $K = 0$.

Thus $D(R/\wp_1(R)) \cap (\wp_1 + \wp_2) = 0$. This mean that $R \notin \mathcal{L}(\wp_1 + \wp_2)$ and consequently we have $\mathcal{L}(\wp_1 + \wp_2) \subseteq \wp_1 + \wp_2$. This implies that $\mathcal{L}(\wp_1 + \wp_2) = \wp_1 + \wp_2$. $\square$

Remarks that theorems 1, 2 and 3, of [2] can be obtained as a special case of our theorem 5.

Theorem 6. *If* $\wp_1$ *and* $\wp_2$ *are radical classes such that* $S\wp_1$ *and* $S\wp_2$ *are homomotphically closed classes, then* $\wp_1 + \wp_2$ *is radical class.*

Proof. Let $R \in \omega$, then by an isomorphism theorem,

$$R/(\wp_1(R) + \wp_2(R)) \cong R/\wp_1(R)/(\wp_1(R) + \wp_2(R))/(\wp_1(R)).$$

Since $R/\wp_1(R) \in S\wp_1$ and $S\wp_1$ is homomorphically closed, we have $R/(\wp_1(R) + \wp_2(R)) \in S\wp_1$. Similarly we have $R/(\wp_1(R) + \wp_2(R)) \in S\wp_2$. Consequently we have $R/(\wp_1(R) + \wp_2(R)) \in S\wp_1 \cap S\wp_2$. By [2], $S(\wp_1 + \wp_2) = S\wp_1 \cap S\wp_2$. Therefore $R/(\wp_1(R) + \wp_2(R)) \in S(\wp_1 + \wp_2)$.

Now $\wp_1 + \wp_2$ satisfies (3) of definition 4, and by [2] $\wp_1 + \wp_2$ satisfies (1), (2) of definition 4. Therefore $\wp_1 + \wp_2$ is a radical class. $\square$

We provide here an example to show that such a result is quite natural.

Example. It was shown by Szasz ([3]) that the classes

$$\delta_n = \{A \in \omega : a^n = a, \ \ \forall a \in A\}$$

are homomorphically closed. Later in [4], R. Wiegandt has shown that these classes are all homomorphically closed, semi-simple classes. Let $\wp_1 = \cup\delta_{n1}$(upper radical generated by δ_{n1}) and $\wp_2 = \cup\delta_{n2}$, by theorem 6, $\wp_1 + \wp_2$ is a radical class.

References

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