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# A Commutativity Theorem for Rings

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Abstract. The aim of the present paper is to establish for commutativity of rings with unity 1 satisfying one of the properties $(xy)^{k+1} = x^k y^{k+1} x$ and $(xy)^{k+1} = yx^{k+1}y^k$, for all $x, y$ in $R$, and the mapping $x \to x^k$ is an anti-homomorphism where $k \geq 1$ is a fixed positive integer.

## 1. Introduction

Throughout this paper, $R$ will represent an associative ring, $Z(R)$ denotes the center of $R$ and for any pair of ring elements $x, y$ in $R$, the symbol $[x, y]$ stands for the commutator $xy - yx$.

There are numerous results in the existing literature concerning commutativity of rings satisfying various special cases of the following properties:

($P_1$) Let $k \geq 1$ be a fixed integer such that $(xy)^{k+1} = x^k y^{k+1} x$, for all $x, y \in R$.

($P_2$) Let $k \geq 1$ be a fixed integer such that $(xy)^{k+1} = yx^{k+1}y^k$, for all $x, y \in R$.

In most of the cases, the underlying polynomial identities in ($P_1$) and ($P_2$) are particularly assumed for $k = 1$ (see [1] and [5]).

In an attempt to prove commutativity of rings satisfying such conditions, Abujabal and Khan ([1]) have shown that a ring $R$ with 1 is commutative if, for all $x, y$ in $R$, such that $(xy)^2 = xy^2x$ or $(xy)^2 = yx^2y$. In the same paper, it is remarked that the example 3 of [2] would demonstrate that each of the conditions ($P_1$) and ($P_2$) does not assure commutativity for any choice of $k > 1$.

We present the same example in a slight different way which is rather easy to appreciate.

**Example 1.1.** Let

$$R = \left\{ \alpha I + A \,\middle|\, A = \begin{pmatrix} 0 & \beta & \gamma \\ 0 & 0 & \delta \\ 0 & 0 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ where } \alpha, \beta, \gamma, \delta \in \mathbf{Z}_p \right\},$$

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where $p$ is a prime and $Z_p$ is the ring of integers modulo $p$. There is no prime $p$ such that $p$ divides $n$ if $n$ is odd and $p$ divides $2p/n$ if $n$ is even. It can easily be checked that $R$ is not commutative.

**Remark 1.2.** The non-commutative ring of $3 \times 3$ strictly upper triangular matrices over a ring of integers provide an example to show that the above properties $(P_1)$ and $(P_2)$are not valid for arbitrary rings.

The ring of the above Example 1.1 and Remark 1.2 shows that neither of the properties $(P_1)$ nor $(P_2)$ guarantees the commutativity of arbitrary rings. It is natural to ask: What additional conditions are needed to force the commutativity for arbitrary ring which satisfies $(P_1)$ or $(P_2)$?

To investigate the commutativity of a ring $R$ with the property $(P_1)$ or $(P_2)$, we need some extra conditions on $R$ such as the property:

(*) Define a map $x \to x^k$ by an anti-homomorphism in $R$ as follows:

$$(xy)^k = y^k x^k \text{ and } (x+y)^k = x^k + y^k \text{ for all } x, y \in R$$

where $k > 1$ is a fixed positive integer.

One of the most beautiful result in Ring Theory is a theorem due to Herstein ([3]) which states that a ring $R$ in which the mapping $x \to x^n$ for a fixed integer $n > 1$ is an onto homomorphism, must be commutative.The objective of this note is to generalize above result when the map $x \to x^k$ is an anti-homomorphism and prove the following:

## 2. Main result

**Theorem 2.1.** *Let $R$ be a ring with unity 1 satisfying $(P_1)$ or $(P_2)$. If $R$ satisfies the property $(*)$, then $R$ is commutative.*

*Proof.* Assume that $k > 1$, in our hypothesis, we have

$$x(yx)^k y = x^k y^{k+1} x \text{ for all } x, y \in R.$$

By $(*)$ we get

$$x^k \left[x, y^{k+1}\right] = 0, \text{ for all } x, y \text{ in } R. \tag{1}$$

Replace $x$ by $1 + x$ in (1) and using $(1 + x)^k = 1 + x^k$ to get

$$\left[x, y^{k+1}\right] = 0. \tag{2}$$

Again replacing $1 + y$ for $y$ in (2) we obtain

$$y^k + y \in Z(R), \text{ for all } x \text{ in } R. \tag{3}$$

Combining (3) with $y^{k^2} + y^k \in Z(R)$, we have $y^{k^2} - y \in Z(R)$, for all $y \in R$. Hence the commutativity of $R$ follows by an application of Herstein's Theorem 18 of [4]. □

Similar arguments may be used if $R$ satisfies the property $(P_2)$. The following is an immediate consequence of Theorem 2.1.

**Corollary 2.2.** *Let $R$ satisfies the hypothesis of Theorem 2.1 with $(P_1)$ be replaced by $(xy)^{k+1} = y^{k+1}x^{k+1}$. Then $R$ is commutative.*

*Proof.* By hypothesis we have

$$xy(xy)^k = y^{k+1}x^{k+1}. \text{ Using}(*), \text{ to get } \left[x, y^{k+1}\right] x^k = 0.$$

Now the rest of the proof carries over almost verbatim as above Theorem 2.1. We omit the proof to avoid repetition. □

**Remark 2.3.** The following example demonstrates that anti-homomorphism cannot be replaced by homomorphism in the Theorem 2.1 and Corollary 2.2.

**Example 2.4.** Consider the non-commutative ring

$$R = \left\{ aI + B \middle| B = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} a, b, c, d \in GF(p) \right\}.$$

It can be easily seen that $R$ satisfies $(xy)^p = x^p y^p, (xy)^{p+1} = x^{p+1}y^{p+1}, (x + y)^p = x^p + y^p$ for an odd prime $p$ and $(xy)^4 = x^4y^4, (xy)^5 = x^5y^5, (x+y)^4 = x^4+y^4$, for $p = 2$.

**Remark 2.5.** Existence of unity 1 in the hypothesis of Theorem 2.1 and Corollary 2.2 may be justified by the following:

**Example 2.6.** Let $D_m$ be the ring of $m \times m$ matrices over a division ring $D$, and $A_m = \{(a_{ij}) \in D_m | a_{ij} = 0 \text{ when } i \leq j\}$. Then $A_m$ is necessarily non-commutative ring for any positive integer $m > 2$. But $A_3$ satisfies $(P_1)$ or $(P_2)$ and $(*)$.

We conclude our discussion with the following:

**Problem 2.7.** Let $R$ be a ring with 1 satisfying the condition $(P_1)$ or $(P_2)$. Is $R$ commutative?

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## References

[1] H. A. S. Abujabal and M. A. Khan, *Some elementary commutativity theorems for associative rings*, Kyungpook Math. J., **33**(1993), 49-51.

[2] E. C. Johnson, E. C. Outcalt and A. Yaqub, *An elementary commutativity theorem for rings*, Amer. Math. Monthly, **75**(1968), 288-289.

[3] I. N. Herstein, *Power maps in rings*, Michigan Math. J., **8**(1961), 29-32.

[4] I. N. Herstein, *A generalization of theorem of Jacobson*, Amer. J. Math., **73**(1951), 756-762.

[5] G. Yuanchun, *Some commutativity theorems for associative rings*, Acta Sci. Natur. Univ.: Jilin, **3**(1982), 13-18.