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Structure of the Double Four-spiral Semigroup

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Abstract. In this paper, we first give an alternative description of the fundamental orthodox semigroup $\overline{A}(1, 2)$. We then use this to represent the double four-spiral semigroup DSp_4 as a regular Rees matrix semigroup over $\overline{A}(1, 2)$.

1. Introduction

In [2], [3] Byleen, Meakin and Pastijn introduced the four-spiral semigroup Sp_4 and the double four-spiral semigroup DSp_4 and studied their properties in detail. These regular semigroups play an important role in the theory of idempotent generated bismple but not completely simple semigroups. The semigroup $\overline{A}(1, 2)$ was introduced in [3] as a tool to analyse the structure of DSp_4. In this paper we give an alternative description of $\overline{A}(1, 2)$ and we show that the bicyclic semigroup $C(p, q)$ is an inverse transversal of $\overline{A}(1, 2)$. We then represent DSp_4 as a regular Rees matrix semigroup over $\overline{A}(1, 2)$ which is analogous to the Byleen's representation of Sp_4 as a regular Rees matrix semigroup over the bicyclic semigroup $C(p, q)$ ([1]).

First we will introduce the terminologies which are used in this paper. We use whenever possible the notation of Clifford and Preston ([4]).

Given a nonempty set A we denote by F_A the *free semigroup* on A. The elements of F_A are the nonempty finite words $a_1a_2 \cdots a_m$, $a_i \in A$, $1 \leq i \leq m$. The multiplication on F_A is given by

$$(a_1a_2 \cdots a_m)(b_1b_2 \cdots b_n) = a_1a_2 \cdots a_mb_1b_2 \cdots b_n.$$

If 1 denotes the empty word, then $F_A^1 = F_A \cup \{1\}$ is called the *free monoid on* A. For any word $a = a_1a_2 \cdots a_m$ in F_A, the integer m is called the length of a and the length of 1 is by definition 0.

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Let $F = F^1_{\{x,y\}} = F_{\{x,y\}} \cup \{1\}$ be the free monoid on $\{x, y\}$. For each $m \geq 0$, let F^m be the subset F consisting of words of length m. Thus, for example, $F^0 = \{1\}$, $F^1 = \{x, y\}$ and $F^2 = \{x^2, xy, yx, y^2\}$. Define $\alpha : F \to F$ by $(1)\alpha = 1$, $(x)\alpha = 1$, $(y)\alpha = 1$ and $(a)\alpha = a_2 a_3 \cdots a_m$ for any word $a = a_1 a_2 \cdots a_m$ of length $m \geq 2$. For each $r > 0$ let α^r denote the composite of α $r-$times and let $\alpha^0 = id$ on F. Thus for any $a = a_1 a_2 \cdots a_m \in F$,

$$(a)\alpha^r = \begin{cases} 1 & \text{if } r \geq m \\ a_{r+1} a_{r+2} \cdots a_m & \text{if } r < m\ . \end{cases}$$

Lemma 1. *If* $a = a_1 a_2 \cdots a_m$ *is a word of length* m *and* $b = b_1 b_2 \cdots b_n$ *is a word of length* n, *then*

$$(ab)\alpha^r = (a)\alpha^r (b)\alpha^{r-\min(r,m)} \tag{1}$$

and

$$(a(b)\alpha^s)\alpha^r = (a)\alpha^r (b)\alpha^{r+s-\min(r,m)} \tag{2}$$

where $m, n, r, s \geq 0$.

Proof. To prove (1), we consider the following four cases.

Case 1 : $r < m$. We see that

$$\begin{aligned} (ab)\alpha^r &= a_{r+1} \cdots a_m b = (a)\alpha^r b \\ &= (a)\alpha^r (b)\alpha^{r-\min(r,m)} = (a)\alpha^r (b)\alpha^0. \end{aligned}$$

Case 2 : $r = m$. Then

$$(ab)\alpha^r = b = (a)\alpha^r (b) = (a)\alpha^r (b)\alpha^0 = (a)\alpha^r (b)\alpha^{r-\min(r,m)}.$$

Case 3 : $m < r < m + 1$. Since

$$(a)\alpha^r = 1,\ (a)\alpha^r (b)\alpha^{r-\min(r,m)} = (b)\alpha^{r-m} = b_{r-m+1} \cdots b_n = (ab)\alpha^r.$$

Case 4 : $m + n \leq r$. In this case the sides (1) are equal to 1.
(2) follows by taking $b = (b)\alpha^s$ in (1). □

Definition 2 ([4]). The *bicyclic semigroup* is the semigroup with identity element generated by p, q subject to the relation $pq = 1$. Thus $C(p, q)$ is of the form $q^m p^n$, $m, n \geq 0$ and for any $q^m p^n, q^r p^s \in C(p, q)$, $q^m p^n q^r p^s = q^{m+r-t} p^{n+s-t}$, where $t = \min(n, r)$. The semilattice of idempotents of $C(p, q)$ is $\{q^m p^m : m \geq 0\}$ and $(q^m p^n)^{-1} = q^n p^m$.

The Green's relations on $C(p,q)$ are given by

$$q^m p^n R q^r p^s \Leftrightarrow m = r, \quad q^m p^n L q^r p^s \Leftrightarrow n = s, \quad q^m p^n H q^r p^s \Leftrightarrow m = r,\ n = s$$
$$\text{and} \quad q^m p^n D q^r p^s \quad \text{for all } q^m p^n, q^r p^s \in C(p,q).$$

2. Description of $\overline{A}(1,2)$

Let $A(1,2) = \{(a, q^m p^n) : q^m p^n \in C(p,q),\ a \in F^m\}$, where $C(p,q)$ is the bicyclic semigroup. Define a multiplication on $A(1,2)$ by

$$(a, q^m p^n)(b, q^r p^s) = (a(b)\alpha^n, q^{m+r-l} p^{n+s-l}) \tag{3}$$

where $l = \min(n,r)$.

Proposition 3. *$A(1,2)$ is an orthodox semigroup, with identity $(1,1)$ generated by (x,q) (y,q) and $(1,p)$ such that*

$$(1,p)(x,q) = (1,p)(y,q) = (1,1). \tag{4}$$

The b and $E(A(1,2))$ of idempotents of $A(1,2)$ is

$$E(A(1,2)) = \{(a, q^m p^m) : a \in F^m\} \tag{5}$$

and, for any $(a, q^m p^n) \in A(1,2)$,

$$V(a, q^m p^n) = \{(d, q^n p^m) : d \in F^n\}. \tag{6}$$

Proof. Take any $(a, q^m p^n), (b, q^r p^s), (c, q^u p^v) \in A(1,2)$. Then using Lemma 1 and the associativity of the multiplication in $C(p,q)$ we obtain

$$\begin{aligned} ((a, q^m p^n)(b, q^r p^s))\,(c, q^u p^v) &= (a(b)\alpha^n,\ q^m p^n q^r p^s)(c, q^u p^v) \\ &= (a(b)\alpha^n (c)\alpha^{n+s-\min(n,r)},\ q^m p^n q^r p^s q^u p^v) \\ &= (a(b(c)\alpha^s)\alpha^n,\ q^m p^n q^r p^s q^u p^v). \\ &= (a, q^m p^n)(b(c)\alpha^s, q^r p^s q^u p^v) \\ &= (a, q^m p^n)((b, q^r p^s)(c, q^u p^v)). \end{aligned}$$

So $A(1,2)$ is a semigroup. For any $(a, q^m p^n) \in A(1,2)$, with $a = a_1 a_2 \cdots a_m \in F^m$, we have

$$\begin{aligned} (a, q^m p^n) &= (a, q^m)(1, p^n) \\ &= (a_1, q)(a_2, q) \cdots (a_m, q)(1,p) \cdots_{n-times} (1,p), \end{aligned} \tag{7}$$

where $a_i \in \{x, y\}$. Hence $A(1,2)$ is generated by $(x,q),\ (y,q)$ and $(1,p)$. Clearly

$$(1,p)(x,q) = (1,1) = (1,p)(y,q).$$

The last two statements are easy to verify. Finally, since $E(A(1,2))$ is a band with identity $(1,1), A(1,2)$ is an orthodox semigroup with identity $(1,1)$. □

The following corollary describes the Green's relations on $A(1,2)$.

Corollary 4. *For* $(a, q^m p^n)(b, q^r p^s) \in A(1,2)$:

$$\begin{array}{ll} \text{(i)} & (a, q^m p^n)L(b, q^r p^s) \Leftrightarrow n = s, \\ \text{(ii)} & (a, q^m p^n)R(b, q^r p^s) \Leftrightarrow a = b, \\ \text{(iii)} & (a, q^m p^n)H(b, q^r p^s) \Leftrightarrow a = b \text{ and } n = s, \\ \text{(iv)} & (a, q^m p^n)D(b, q^r p^s). \end{array}$$

Proof. Follows from the corresponding descriptions of the Green's relations on $C(p,q)$. □

The following result is immediate from Corollary 4.

Corollary 5. *In* $A(1,2)$

$$\begin{array}{ll} \text{(i)} & E(A(1,2)) \cap L_{(a,q^m p^n)} = \{(d, q^n p^n) : d \in F^n\}, \\ \text{(ii)} & E(A(1,2)) \cap R_{(a,q^m p^n)} = \{(a, q^m p^m)\}, \\ \text{(iii)} & R_{(1,1)} = \{(1, p^n)\}, \ \textit{free monoid generated by } (1,p), \\ \text{(iv)} & L_{(1,1)} = \{(a, q^m) : a \in F^m\}, \ \textit{free monoid generated by } (x,p) \textit{ and } (y,q). \end{array}$$

Definition 6 ([3]). $\overline{A}(1,2)$ is the *$R-$unipotent bisimple fundamental* orthodox semigroup generated by $\overline{p}$, $\overline{q}$, $\overline{t}$ with the relations

$$\overline{p}\,\overline{q} = \overline{p}\,\overline{t} = 1.$$

Thus every element of $\overline{A}(1,2)$ is of the form kl where $k \in F^1_{\{\overline{q},\overline{t}\}}$ and $l \in F^1_{\{\overline{p}\}}$. The set of idempotents of $\overline{A}(1,2)$ is given by

$$E(\overline{A}(1,2)) = \{k\overline{p^m} : k \in F^1_{(q,t)}\} \textit{ and } m = \textit{length of } k\}. \tag{8}$$

Theorem 7. $A(1,2)$ *is isomorphic to* $\overline{A}(1,2)$.

Proof. By Proposition 3, $(x,q),(y,q)$ and $(1,p)$ are the generators of $A(1,2)$such that $(1,p)(x,q) = (1,p)(y,q) = (1,1)$. So the map $\overline{p} \to (1,p)$, $\overline{q} \to (x,q)$, $\overline{t} \to (y,q)$ extends to a homomorphism $\chi : \overline{A}(1,2) \to A(1,2)$ of $\overline{A}(1,2)$ on to $A(1,2)$. From (5) and (8), it is clear that χ is one to-one on the idempotents of $\overline{A}(1,2)$. Since $\overline{A}(1,2)$ is fundamental, χ is an isomorphism. □

Definition 8 ([6]). An *inverse transversal* of a regular semigroup T is an inverse subsemigroup S that contains a unique inverse of each element of T, that is $|V(x) \cap S| = 1$ for all $x \in T$. In this case, we denote by x^0 the unique element of $V(x) \cap S$, and write $x^{00} = (x^0)^0$.

Definition 9 ([9]). If U is a regular semigroup, then a *coextension* of U is a pair (T, φ), where T is a regular semigroup and φ is a homomorphism of T on to U. (T, φ) is called a *split coextension* if there exits a homomorphism $\chi : U \to T$ such that $\chi\varphi = id$ on U. (T, φ) is called a *coextension* of U by left zero semigroups if $e\varphi^{-1}$ is a left zero semigroup for each $e \in E(U)$, the set of idempotents of U.

Proposition 10. *The map $\theta : A(1,2) \to C(p,q)$ given by $(a, q^m p^n)\theta = q^m p^n$ defines a split coextension $(A(1,2), \theta)$ of $C(p,q)$ by left zero semigroups with a splitting $\eta : C(p,q) \to A(1,2)$ given by $(q^m p^n)\eta = (x^m, q^m, p^n)$.*
If we identify $C(p,q)$ with $C(p,q)\eta$, via η, then $C(p,q)$ is an inverse transversal of $A(1,2)$.

Proof. It is clear that θ and η are homomorphisms with $\eta\theta = id$ on $C(p,q)$. So that $(A(1,2), \theta)$ is a split coextension. For $q^m p^m \in E(C(p,q))$, $(q^m p^m)\theta^{-1} = \{(a, q^m p^m) : a \in F^m\}$ is a left zero semigroup by (i) of Corollary 5. If we identify $C(p,q)$ with $C(p,q)\eta$, via η, then for any $(a, q^m p^n) \in A(1,2)$, $V(a, q^m p^n) \cap C(p,q) = \{(x^n, q^n p^m)\}$, $C(p,q)$ is an inverse transversal of $A(1,2)$. This completes the proof of the proposition. □

3. Description of DSp_4

Definition 11 ([3]). The *double four spiral semigroup DSp_4* is the regular semigroup generated by five idempotents $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, $\widetilde{d}$ and $\widetilde{e}$ with the relations.

$$\widetilde{a} \; R \; \widetilde{b} \; L \; \widetilde{c} \; R \; \widetilde{d} \; L \; \widetilde{e}, \; \widetilde{a}\widetilde{e} = \widetilde{e} = \widetilde{e}\,\widetilde{a}.$$

Definition 12 ([5]). Let T be a regular semigroup and S a regular subsemigroup of T. A map $\theta : T \to S$ is a *split map* if the following conditions are satisfied:

$(S1)$ $x\theta = x$ for all $x \in S$,

$(S2)$ $V_s(x\theta) \subseteq V(x)$ for all $x \in T$,

$(S3)$ $(xy)\theta = (x\theta)(x^* x y y^*)\theta(y\theta)$, for all $x, y \in T$, $x^* \in V_s(x\theta)$, $y^* \in V_s(y\theta)$.

Here for $t \in T, V_s(t)$ (resp. $V(t)$) denotes the set of all inverses of t in S(resp. T).

Before proceeding further let us fix some notations. Let S be a regular semigroup $E(S)$ the set of idempotents of S. For each $x \in S$, let $r(x) = R_x \cap E(S) = \{e \in E(S) : eR \quad x\}$ and $l(x) = L_x \cap E(S) = \{e \in E(S) : eL \quad x\}$
in particular, if $e \in E(S)$ then $r(e)$ (resp. $l(e)$) is the $R-$ class (resp. $L-$ class) of e in $E(S)$. Let $E(S)/R$ be the partially ordered set of $R-$classes of $E(S)$ and $E(S)/L$

be the partially ordered set of $L-$classes of $E(S)$, where $r(e) \geq r(f)$ if and only if $ef = f$ and $l(e) \geq l(f)$ if and only if $fe = f$ for all $e,\ f \in E(S)$. In the following we shall regard $E(S)/R$ and $E(S)/L$ as small categories. Thus, for example, the objects of $E(S)/R$ are the $R-$classes of $E(S)$ and, for any two objects $r(e), r(f)$, there is exactly one morphism, denoted $(r(e), r(f))$, from $r(e)$ to $r(f)$ if $r(e) \geq r(f)$; otherwise there are no morphisms from $r(e)$ to $r(f)$.

We denote P the category of pointed sets and base point preserving maps. Given a functor $F : C \to P$ from a category C to P. We always assume that $F_e \cap F_f = \phi$ whenever e and f are distinct objects of C. We denote the base point of F_e by e itself.

Definition 13 ([5]). Let S be a regular semigroup. An *S−pair* (A, B) is a pair of functors

$$A : E(S)/R \to P,\ B : E(S)/L \to P.$$

Given an $S-$pair (A, B), a $B \times A$ *matrix over* S is a function

$$* : (b, a) \to b \times a : \cup_{l(e)\in E(S)/L} B_{l(e)} \times \cup_{r(f)\in E(S)/R} A_{r(f)} \longrightarrow S.$$

We use the following theorem to give an alternative description of DSp_4.

Theorem 14 ([5]). *Let S be a regular semigroup and let (A, B) be an $S-$pair. Let $*$ be a $B \times A$ matrix over S which satisfies the following conditions:*

(i) *If $b \in B_{l(e)}$ and $a \in A_{r(f)}$ then $b * a \in l(e)Sr(f)$.*

(ii) *For all $e,\ e',f,\ f' \in E(S)$ with $l(e) \geq l(e'), r(f) \geq r(f'),\ a \in A_{r(f)}, b \in B_{l(e)}$, $e(b * aA(r(f), r(f')))f' = e(b * a)f'$ and $e'(bB(l(e), l(e')) * a)f = e'(b * a)f$.*

(iii) *For any $b \in B_{l(e)}, a \in A_{r(f)},\ b * r(f), l(e) * a \in l(e)r(f)$.*

Then $W = W(S; A, B; *) = \{(a, x, b) : x \in S, a \in A_{r(x)}, b \in B_{l(x)}\}$ is a regular semigroup under the multiplication $(a, x, b)(c, y, d) = (aA(r(x), r(z)), z, dB(l(y), l(z)))$, where $z = x(b * c)y$. The map $\eta : S \to W,\ x\eta = (r(x), x, l(x))$ is an injective homomorphism of S to W. If we identify S with $S\eta$, via η, then $\theta : W \to S$, defined by $(a, x, b)\theta = (r(x), x, l(x))$, is a split map such that

$$ewf = e(w\theta)f \quad \text{for all } e, f \in E(S), w \in W. \tag{9}$$

Conversely, every regular semigroup T with a split map $\theta : T \to S$ satisfying (9) *can be constructed in this way.*

Let $A : E(A(1, 2))/R(= E(A(1, 2))) \to P$ and $B : E(A(1, 2))/L \to P$ be the constant functors at the two elements set $\{1, 2\}$ with base point 1. Thus, A associates with each $r(e) \in E(A(1, 2))/R$ the point set $\{1, 2\}$ and with each pair $r(e) \geq r(f)$ the identity function $id : \{1, 2\} \to \{1, 2\}$. Similarly, B associates with each $l(e) \in E(A(1,2))/L$ the pointed set $\{1,2\}$ and with each pair $l(e) \geq l(f)$ the identity function $id : \{1,2\} \to \{1,2\}$. Then (A,B) is an $A(1,2)-$ pair. The maps

$$* : B_{l(d,q^np^n)} \times A_{r(a,q^mp^m)} \to A(1,2)$$

given by

$$\begin{aligned} 1*1 = 1*2 = 2*1 &= (x^n, q^np^n)(a, q^mp^m) \\ \text{and } 2*2 &= (x^n, q^np^n)(x,q)(a, q^mp^m) \end{aligned}$$

defines a $B \times A$ matrix $* : B \times A \to A(1,2)$ over $A(1,2)$ which clearly satisfies (i), (ii) and (iii) of Theorem 14. So

$$\begin{aligned} W &= W(A(1,2); A, B; *) \\ &= \{(u,(a,q^mp^n),v) : (a,q^mp^n) \in A(1,2), u \in A_{r(a,q^mp^n)}, v \in B_{l(a,q^mp^n)}\} \\ &= \{(u,(a,q^mp^n),v) : (a,q^mp^n) \in A(1,2); u, v \in \{1,2\}\} \end{aligned}$$

with multiplication

$$(u_1,(a,q^mp^n),v_1)(u_2,(b,q^rp^s),v_2) = (u_1,(a,q^mp^n)(v_1 * u_2)(b,q^rp^s),v_2)$$

is a regular semigroup with a split map $\theta : W \to A(1,2)$ such that $ewf = e(w\theta)f$ where $e, f \in E(A(1,2)), w \in W$.

Take any $(a,q^mp^n), (b,q^rp^s) \in A(1,2)$. Then by Corollary 4, $(a,q^mp^n)L(x^n,q^np^n)$ and $(b,q^rp^s)R(b,q^rp^r)$. Therefore,

$$\begin{aligned} (10) \qquad & (u_1,(a,q^mp^n),v_1)(u_2,(b,q^rp^s),v_2) \\ = \ & (u_1,(a,q^mp^n)(v_1 * u_2)(b,q^rp^s),v_2) \\ = \ & \begin{cases} (u_1,(a,q^mp^n)(b,q^rp^s),v_2) & \text{if } v_1 = 1 \text{ or } u_2 = 1 \\ (u_1,(a,q^mp^n)(x,q)(b,q^rp^s),v_2) & \text{if } v_1 = u_2 = 2. \end{cases} \end{aligned}$$

Thus

Theorem 15. *The regular semigroup W coincides with $RM(A(1,2); \{1,2\}, \{1,2\}; p)$, the regular Rees matrix semigroup over $A(1,2)$ with sandwich matrix*

$$p = \begin{pmatrix} (1,1) & (1,1) \\ (1,1) & (x,q) \end{pmatrix}.$$

Proof. Clearly $W = RM(A(1,2); \{1,2\}, \{1,2\}; p\}$ as sets, and by (10), the multiplication in W coincide with the multiplication in $RM(A(1,2); \{1,2\}, \{1,2\}; p))$. $\square$

Theorem 16. *W is isomorphic to DSp_4.*

Proof. Take $\overline{a} = (1,(1,1),2), \overline{b} = (1,(1,1),1), \overline{c} = (2,(1,1),1), \overline{d} = (2,(1,p),2)$, $\overline{e} = (1,(y,qp),2) \in W$. Then $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}$ are idempotents and, $\overline{a}R\overline{b}L\overline{c}R\overline{d}L\overline{e}$, $\overline{ae} = \overline{e} =$

$\overline{ea}$. Therefore the relations of DSp_4 are satisfied by the generators of W, the map $\widetilde{a} \to \overline{a}, \widetilde{b} \to \overline{b}, \widetilde{c} \to \overline{c}, \widetilde{d} \to \overline{d}$ and $\widetilde{e} \to \overline{e}$ extends to a homomorphism $\chi : DSp_4 \to W$ from DSp_4 to W. Let $(u, (a, q^m p^n), v) \in W$ with $a = a_1 a_2 \cdots a_m \in F_m$. Then, we have

$$(11)\quad \begin{aligned}(u, (a, q^m p^n), v) \;=\; & (u, (1, 1), 1)(1, (a_1, qp), 2)(2, (1, 1), 1)(1, (a_2, qp), 2)\\ & (2, (1, 1), 1) \cdots (1, (a_m, qp), 2)(2, (1, 1), 1)\\ & [(1, (1, 1), 1)(2, (1, p), 2)]^n (1, (1, 1), v).\end{aligned}$$

This implies that W is an idempotent generated semigroup generated by $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}$ and hence χ is onto. It is easy to see that, the set $E(W)$ of idempotents of W is given by

$$E(W) = \begin{cases} (1, (a, q^m p^m), 2) \\ (1, (a, q^m p^m), 1) \\ (2, (a, q^m p^m), 1) \\ (2, (a, q^m p^{m+1}), 2), \qquad a \in F^m. \end{cases}$$

Further, for any $m, n \geq 0$, $(1, (a, q^m p^m), 2) \leq (1, (1, 1), 2)$, $(1, (a, q^m p^m), 1) \leq (1, (1, 1), 1)$, $(2, (a, q^m p^m), 1) \leq (2, (1, 1), 1)$ and $(2, (a, q^m p^{m+1}), 2) \leq (2, (1, p), 2)$. So χ is one-to-one on idempotents. Thus χ is an idempotent separating homomorphism from DSp_4 on to W. Since DSp_4 is fundamental, χ is an isomorphism. □

Corollary 17. *DSp_4 is isomorphic to the regular Rees matrix semigroup*

$$RM\left(A(1,2); \{1,2\}, \{1,2\}; \begin{pmatrix} (1,1) & (1,1) \\ (1,1) & (x,q) \end{pmatrix} \right) \text{ over } A(1,2).$$

Theorem 18. *Let (T, θ) be a split coextension of S by left zero semigroups. Let $\overline{S} = RM(S; I, \wedge, p)$ be a regular Rees matrix semigroup over S with sandwich matrix $p : \wedge \times I \to S$. Then the regular Rees matrix semigroup $\overline{T} = RM(T, I, \wedge, p)$ contains $\overline{S}$ and the map $\overline{\theta} : \overline{T} \to \overline{S}$, defined by $(i, x, \lambda)\overline{\theta} = (i, x\theta, \lambda)$ defines a split coextension $(\overline{T}, \overline{\theta})$ of $\overline{S}$ by left zero semigroups.*

Proof. By [8, Lemma 1.1],

$$\begin{aligned}(i, x, \lambda) \in \overline{T} \quad & \Rightarrow \quad V(x) \cap p_{\lambda j} T p_{\gamma i} \neq \phi \text{ for some } j \in I,\ \gamma \in \wedge.\\ & \Rightarrow \quad V(x\theta) \cap p_{\lambda j} S p_{\gamma i} \neq \phi \text{ since } \theta \text{ is a homomorphism}\\ & \Rightarrow \quad (i, x\theta, \lambda) \in \overline{S}.\end{aligned}$$

Therefore $\overline{\theta} : \overline{T} \to \overline{S}$ is a well defined map. Since θ is a homomorphism and p takes values in S on which θ is an identity map, $\overline{\theta}$ is a homomorphism. Now it is enough to prove that for each $(i, x, \lambda) \in E(\overline{S}), (i, x, \lambda)\theta^{-1}$ is a left zero semigroup. Again by [8, Lemma 1.1], $(i, x, \lambda)(i, x, \lambda) = (i, x, \lambda)$ implies that $xp_{\lambda i}x = x$; in particular $xp_{\lambda i}$ and $p_{\lambda i}x$ are idempotents of S. Take $(i, y, \lambda), (i, z, \lambda)$ in $(i, x, \lambda)\theta^{-1}$.

Then $y\theta = z\theta = x$, $p_{\lambda i}y, p_{\lambda i}z \in (p_{\lambda i}x)\theta^{-1}$ so that $p_{\lambda i}yp_{\lambda i} = p_{\lambda i}y$, since $(p_{\lambda i}x)\theta^{-1}$ is a left zero semigroup. Therefore

$$(i, y, \lambda)(i, z, \lambda) = (i, yp_{\lambda i}z, \lambda) = (i, yp_{\lambda i}z\lambda p_{\lambda i}z, \lambda) = (i, yp_{\lambda i}y, \lambda) = (i, y, \lambda).$$

Hence $(i, x, \lambda)\theta^{-1}$ is a left zero semigroup. □

The fundamental four-spiral semigroup

$$Sp_4 \cong RM\left(C(p,q); \{1,2\}, \{1,2\}; \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \right)$$

over the bicyclic semigroup $C(p, q)$ ([1]). Applying Theorem 18 to the Proposition 10, we get

Corollary 19. (See [3], Theorem 14) *DSp_4 is a split coextension of Sp_4 by left zero semigroups.*

Remark 20. A regular semigroup T is called an idempotent - separating extension of S if there is an idempotent - separating homomorphism θ from T on to S. In [7], Meakin described the structure of Sp_4 as a semigroup of ordered pairs and studied the idempotent separating extensions of Sp_4 analogous to Reilly's theorem ([10]). Since we described $A(1, 2)$ and DSp_4 explicitly one can study the idempotent - separating extension of $A(1, 2)$ and DSp_4 analogous to [10] and [7].

Problem 21. Determine the idempotent - separating extension of $A(1, 2)$ and DSp_4.

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