Structure of the Double Four-spiral Semigroup

V. M. Chandrasekaran

Department of Mathematics, Vellore Institute of Technology, Deemed University, Vellore - 632014, India
e-mail: vmcsn@yahoo.com

M. Loganathan

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai - 600 005, India

Abstract. In this paper, we first give an alternative description of the fundamental orthodox semigroup $A(1, 2)$. We then use this to represent the double four-spiral semigroup $DSp_4$ as a regular Rees matrix semigroup over $A(1, 2)$.

1. Introduction

In [2], [3] Byleen, Meakin and Pastijn introduced the four-spiral semigroup $Sp_4$ and the double four-spiral semigroup $DSp_4$ and studied their properties in detail. These regular semigroups play an important role in the theory of idempotent generated bismple but not completely simple semigroups. The semigroup $\overline{A}(1, 2)$ was introduced in [3] as a tool to analyse the structure of $DSp_4$. In this paper we give an alternative description of $\overline{A}(1, 2)$ and we show that the bicyclic semigroup $C(p, q)$ is an inverse transversal of $\overline{A}(1, 2)$. We then represent $DSp_4$ as a regular Rees matrix semigroup over $\overline{A}(1, 2)$ which is analogous to the Byleen’s representation of $Sp_4$ as a regular Rees matrix semigroup over the bicyclic semigroup $C(p, q)$ ([1]).

First we will introduce the terminologies which are used in this paper. We use whenever possible the notation of Clifford and Preston ([4]).

Given a nonempty set $A$ we denote by $F_A$ the free semigroup on $A$. The elements of $F_A$ are the nonempty finite words $a_1a_2\cdots a_m, \ a_i \in A, \ 1 \leq i \leq m$. The multiplication on $F_A$ is given by

$$(a_1a_2\cdots a_m)(b_1b_2\cdots b_n) = a_1a_2\cdots a_mb_1b_2\cdots b_n.$$ 

If 1 denotes the empty word, then $F_A^1 = F_A \cup \{1\}$ is called the free monoid on $A$. For any word $a = a_1a_2\cdots a_m$ in $F_A$, the integer $m$ is called the length of $a$ and the length of 1 is by definition 0.

Received February 20, 2002, and in revised form, July 15, 2003.
2000 Mathematics Subject Classification: 20M17.
Key words and phrases: four-spiral semigroup, double four-spiral semigroup, regular Rees matrix semigroup.
Let \( F = F^1_{\{x,y\}} = F_{\{x,y\}} \cup \{1\} \) be the free monoid on \( \{x, y\} \). For each \( m \geq 0 \), let \( F^m \) be the subset \( F \) consisting of words of length \( m \). Thus, for example, \( F^0 = \{1\} \), \( F^1 = \{x, y\} \) and \( F^2 = \{x^2, xy, yx, y^2\} \). Define \( \alpha : F \to F \) by \( \alpha(1) = 1 \), \( \alpha(x) = 1 \), \( \alpha(y) = 1 \) and \( \alpha(a) = a_2a_3 \cdots a_m \) for any word \( a = a_1a_2 \cdots a_m \) of length \( m \geq 2 \). For each \( r > 0 \) let \( \alpha^r \) denote the composite of \( \alpha \) \( r \) times and let \( \alpha^0 = \text{id} \) on \( F \). Thus for any \( a = a_1a_2 \cdots a_m \in F \),

\[
(\alpha)a^r = \begin{cases} 1 & \text{if } r \geq m \\
 a_{r+1}a_{r+2} \cdots a_m & \text{if } r < m.
\end{cases}
\]

**Lemma 1.** If \( a = a_1a_2 \cdots a_m \) is a word of length \( m \) and \( b = b_1b_2 \cdots b_n \) is a word of length \( n \), then

\[
(1) \quad (ab)a^r = (a)a^r(b)b = (a)a^r = (a)a^{r - \min(r, m)}
\]

and

\[
(2) \quad (a(b)a^s)a^r = (a)a^r(b)a^{r + s - \min(r, m)}
\]

where \( m, n, r, s \geq 0 \).

**Proof.** To prove (1), we consider the following four cases.

**Case 1:** \( r < m \). We see that

\[
(ab)a^r = a_{r+1} \cdots a_mb = (a)a^rb = (a)a^{r - \min(r, m)} = (a)a^rb.
\]

**Case 2:** \( r = m \). Then

\[
(ab)a^r = b = (a)a^rb = (a)a^rb = (a)a^{r - \min(r, m)}.
\]

**Case 3:** \( m < r < m + 1 \). Since

\[
(a)a^r = 1, \quad (a)a^rb = (a)a^rb = (a)a^{r - \min(r, m)} = (a)a^{r - \min(r, m)} = (b)r_m = b_{r_m} \cdots b_n = (ab)a^r.
\]

**Case 4:** \( m + n \leq r \). In this case the sides (1) are equal to 1.

(2) follows by taking \( b = (b)a^s \) in (1). \( \Box \)

**Definition 2 ([4]).** The **bicyclic semigroup** is the semigroup with identity element generated by \( p, q \) subject to the relation \( pq = 1 \). Thus \( C(p, q) \) is of the form \( q^mp^n, m, n \geq 0 \) and for any \( q^mp^n, q^rp^s \in C(p, q) \), \( q^mp^nq^rp^s = q^{m+n-t}p^{n+s-t} \), where \( t = \min(n, r) \). The semilattice of idempotents of \( C(p, q) \) is \( \{q^mp^m : m \geq 0\} \) and \( (q^mp^n)^{-1} = q^mp^n \).
The Green’s relations on \( C(p, q) \) are given by

\[ q^m p^n R_q p^s \Leftrightarrow m = r, \quad q^m p^n L_q p^s \Leftrightarrow n = s, \quad q^m p^n H_q p^s \Leftrightarrow m = r, \quad n = s \]

and \( q^m p^n D_q p^s \) for all \( q^m p^n, q^r p^s \in C(p, q) \).

2. Description of \( \overline{A}(1, 2) \)

Let \( A(1, 2) = \{(a, q^m p^n) : q^m p^n \in C(p, q), \ a \in F^m\} \), where \( C(p, q) \) is the bicyclic semigroup. Define a multiplication on \( A(1, 2) \) by

\[ (a, q^m p^n) \cdot (b, q^r p^s) = (a(b)\alpha^n, q^{m+r-\min(n, r)}p^{n+s-1}) \]

where \( l = \min(n, r) \).

Proposition 3. \( A(1, 2) \) is an orthodox semigroup, with identity \((1, 1)\) generated by \((x, q)\) \((y, q)\) and \((1, p)\) such that

\[ (1, p)(x, q) = (1, p)(y, q) = (1, 1). \]

The \( b \) and \( E(A(1, 2)) \) of idempotents of \( A(1, 2) \) is

\[ E(A(1, 2)) = \{(a, q^m p^n) : a \in F^m\} \]

and, for any \((a, q^m p^n) \in A(1, 2), \)

\[ V(a, q^m p^n) = \{(d, q^n p^m) : d \in F^n\}. \]

Proof. Take any \((a, q^m p^n), (b, q^r p^s), (c, q^t p^u) \in A(1, 2)\). Then using Lemma 1 and the associativity of the multiplication in \( C(p, q) \) we obtain

\[ ((a, q^m p^n) \cdot (b, q^r p^s)) \cdot (c, q^t p^u) = (a(b)\alpha^n, q^{m+p^n}q^r p^s)(c, q^t p^u) \]

\[ = (a(b)\alpha^n(c)\alpha^t, q^{m+p^n}q^r p^s q^t) \]

\[ = (a(b(c)\alpha^t)\alpha^n, q^{m+p^n}q^r p^s q^t). \]

\[ = (a, q^m p^n) \cdot (b(c)\alpha^t q^r p^s). \]

\[ = (a, q^m p^n) \cdot (b, q^r p^s) \cdot (c, q^t p^u). \]

So \( A(1, 2) \) is a semigroup. For any \((a, q^m p^n) \in A(1, 2), \) with \( a = a_1 a_2 \cdots a_m \in F^m, \) we have

\[ (a, q^m p^n) = (a, q^m)(1, p^n) \]

\[ = (a_1, q)(a_2, q) \cdots (a_m, q)(1, p) \cdots \text{n-times} \ (1, p), \]

where \( a_i \in \{x, y\}. \) Hence \( A(1, 2) \) is generated by \((x, q), \ (y, q)\) and \((1, p)\). Clearly

\[ (1, p)(x, q) = (1, 1) = (1, p)(y, q). \]
The last two statements are easy to verify. Finally, since \( E(A(1,2)) \) is a band with identity \((1,1)\), \( A(1,2) \) is an orthodox semigroup with identity \((1,1)\).

The following corollary describes the Green’s relations on \( A(1,2) \).

**Corollary 4.** For \((a, q^m p^n)(b, q^r p^s) \in A(1,2) :\)

1. \((a, q^m p^n)L(b, q^r p^s) \iff n = s,\)
2. \((a, q^m p^n)R(b, q^r p^s) \iff a = b,\)
3. \((a, q^m p^n)H(b, q^r p^s) \iff a = b \text{ and } n = s,\)
4. \((a, q^m p^n)D(b, q^r p^s).\)

**Proof.** Follows from the corresponding descriptions of the Green’s relations on \( C(p, q) \).

The following result is immediate from Corollary 4.

**Corollary 5.** In \( A(1,2) \)

(i) \( E(A(1,2)) \cap L_{(a, q^m p^n)} = \{ (d, q^n p^a) : d \in F^n \}, \)
(ii) \( E(A(1,2)) \cap R_{(a, q^m p^n)} = \{ (a, q^n p^a) \}, \)
(iii) \( R_{(1,1)} = \{ (1, p^n) \}, \) free monoid generated by \((1, p)\),
(iv) \( L_{(1,1)} = \{ (a, q^m) : a \in F^m \}, \) free monoid generated by \((x, p)\) and \((y, q)\).

**Definition 6 ([3]).** \( \overline{A}(1,2) \) is the \( R\)-unipotent bisimple fundamental orthodox semigroup generated by \( p, q, t \) with the relations

\[
\overline{p} \overline{q} = \overline{p} \overline{t} = 1.
\]

Thus every element of \( \overline{A}(1,2) \) is of the form \( kl \) where \( k \in F^1_{(q,t)} \) and \( l \in F^1_{(p)} \).

The set of idempotents of \( \overline{A}(1,2) \) is given by

\[
E(\overline{A}(1,2)) = \{ k \overline{p} \overline{m} : k \in F^1_{(q,t)} \text{ and } m = \text{length of } k \}.
\]

**Theorem 7.** \( A(1,2) \) is isomorphic to \( \overline{A}(1,2) \).

**Proof.** By Proposition 3, \((x, q),(y, q)\) and \((1, p)\) are the generators of \( A(1,2) \) such that \((1, p)(x, q) = (1, p)(y, q) = (1, 1)\). So the map \( \overline{p} \rightarrow (1, p), \overline{q} \rightarrow (x, q), \overline{t} \rightarrow (y, q) \) extends to a homomorphism \( \chi : \overline{A}(1,2) \rightarrow A(1,2) \) of \( \overline{A}(1,2) \) on to \( A(1,2) \). From (5) and (8), it is clear that \( \chi \) is one to one on the idempotents of \( \overline{A}(1,2) \). Since \( \overline{A}(1,2) \) is fundamental, \( \chi \) is an isomorphism.
Definition 8 ([6]). An inverse transversal of a regular semigroup $T$ is an inverse subsemigroup $S$ that contains a unique inverse of each element of $T$, that is $|V(x) \cap S| = 1$ for all $x \in T$. In this case, we denote by $x^0$ the unique element of $V(x) \cap S$, and write $x^{00} = (x^0)^0$.

Definition 9 ([9]). If $U$ is a regular semigroup, then a coextension of $U$ is a pair $(T, \varphi)$, where $T$ is a regular semigroup and $\varphi$ is a homomorphism of $T$ onto $U$. $(T, \varphi)$ is called a split coextension if there exits a homomorphism $\chi : U \to T$ such that $\chi \varphi = id$ on $U$. $(T, \varphi)$ is called a coextension of $U$ by left zero semigroups if $e\varphi^{-1}$ is a left zero semigroup for each $e \in E(U)$, the set of idempotents of $U$.

Proposition 10. The map $\theta : A(1, 2) \to C(p, q)$ given by $(a, q^n p^m) \theta = q^m p^n$ defines a split coextension $(A(1, 2), \theta)$ of $C(p, q)$ by left zero semigroups with a splitting $\eta : C(p, q) \to A(1, 2)$ given by $(q^m p^n) \eta = (x^m, q^n, p^n)$. If we identify $C(p, q)$ with $C(p, q) \eta$, via $\eta$, then $C(p, q)$ is an inverse transversal of $A(1, 2)$.

Proof. It is clear that $\theta$ and $\eta$ are homomorphisms with $\eta \theta = id$ on $C(p, q)$. So that $(A(1, 2), \theta)$ is a split coextension. For $q^m p^n \in E(C(p, q))$, $(q^m p^n) \theta^{-1} = \{(a, q^n p^m) : a \in F^m\}$ is a left zero semigroup by (i) of Corollary 5. If we identify $C(p, q)$ with $C(p, q) \eta$, via $\eta$, then for any $(a, q^n p^m) \in A(1, 2)$, $V(a, q^n p^m) \cap C(p, q) = \{(x^m, q^n p^m)\}$, $C(p, q)$ is an inverse transversal of $A(1, 2)$. This completes the proof of the proposition. 

3. Description of $DS_{p4}$

Definition 11 ([3]). The double four spiral semigroup $DS_{p4}$ is the regular semigroup generated by five idempotents $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ and $\tilde{e}$ with the relations:

$$\tilde{a} R \tilde{b} L \tilde{c} R \tilde{d} L \tilde{e}, \tilde{a}\tilde{e} = \tilde{e} \tilde{a}.$$

Definition 12 ([5]). Let $T$ be a regular semigroup and $S$ a regular subsemigroup of $T$. A map $\theta : T \to S$ is a split map if the following conditions are satisfied:

(S1) $x \theta = x$ for all $x \in S$,
(S2) $V_s(x \theta) \subseteq V(x)$ for all $x \in T$,
(S3) $(x y) \theta = (x \theta)(x^* x y y^*) \theta(y \theta)$, for all $x, y \in T$, $x^* \in V_s(x \theta)$, $y^* \in V_s(y \theta)$.

Here for $t \in T, V_s(t)$ (resp. $V(t)$) denotes the set of all inverses of $t$ in $S$ (resp. $T$).

Before proceeding further let us fix some notations. Let $S$ be a regular semigroup $E(S)$ the set of idempotents of $S$. For each $x \in S$, let $r(x) = R_x \cap E(S) = \{ e \in E(S) : e R \ x \}$ and $l(x) = L_x \cap E(S) = \{ e \in E(S) : e L \ x \}$ in particular, if $e \in E(S)$ then $r(e)$ (resp. $l(e)$) is the $R-$ class (resp. $L-$ class) of $e$ in $E(S)$. Let $E(S)/R$ be the partially ordered set of $R-$classes of $E(S)$ and $E(S)/L$
be the partially ordered set of \(L\)-classes of \(E(S)\), where \(r(e) \geq r(f)\) if and only if \(ef = f\) and \(l(e) \geq l(f)\) if and only if \(fe = f\) for all \(e, f \in E(S)\). In the following we shall regard \(E(S)/R\) and \(E(S)/L\) as small categories. Thus, for example, the objects of \(E(S)/R\) are the \(R\)-classes of \(E(S)\) and, for any two objects \(r(e), r(f)\), there is exactly one morphism, denoted \((r(e), r(f))\), from \(r(e)\) to \(r(f)\) if \(r(e) \geq r(f)\); otherwise there are no morphisms from \(r(e)\) to \(r(f)\).

We denote \(P\) the category of pointed sets and base point preserving maps. Given a functor \(F : C \to P\) from a category \(C\) to \(P\). We always assume that \(F_e \cap F_f = \emptyset\) whenever \(e\) and \(f\) are distinct objects of \(C\). We denote the base point of \(F_e\) by \(e\) itself.

**Definition 13** ([5]). Let \(S\) be a regular semigroup. An \(S\)-pair \((A, B)\) is a pair of functors

\[
A : E(S)/R \to P, \quad B : E(S)/L \to P.
\]

Given an \(S\)-pair \((A, B)\), a \(B \times A\) matrix over \(S\) is a function

\[
* : (b, a) \to b \times a : \cup_{l(e) \in E(S)/L} B_{l(e)} \times \cup_{r(f) \in E(S)/R} A_{r(f)} \to S.
\]

We use the following theorem to give an alternative description of \(DSp_4\).

**Theorem 14** ([5]). Let \(S\) be a regular semigroup and let \((A, B)\) be an \(S\)-pair. Let \(*\) be a \(B \times A\) matrix over \(S\) which satisfies the following conditions:

(i) If \(b \in B_{l(e)}\) and \(a \in A_{r(f)}\) then \(b * a \in l(e)Sr(f)\).

(ii) For all \(e, e', f, f' \in E(S)\) with \(l(e) \geq l(e'), r(f) \geq r(f')\), \(a \in A_{r(f)}, b \in B_{l(e)}\),

\[
e(b * aA(r(f), r(f')))f' = e(b * a)f'\text{ and } e'(bB(l(e), l(e')))a f = e'(b * a)f.
\]

(iii) For any \(b \in B_{l(e)}, a \in A_{r(f)}, b * r(f), l(e) * a \in l(e)r(f)\).

Then \(W = W(S; A, B; \ast) = \{(a, x, b) : x \in S, a \in A_{r(x)}, b \in B_{l(x)}\}\) is a regular semigroup under the multiplication \((a, x, b)(c, y, d) = (aA(r(x), r(z)), z, dB(l(y), l(z)))\), where \(z = x(b * c)y\). The map \(\eta : S \to W, \eta(x) = (r(x), x, l(x))\) is an injective homomorphism of \(S\) to \(W\). If we identify \(S\) with \(S\eta\) via \(\eta\), then \(\theta : W \to S\), defined by \((a, x, b)\theta = (r(x), x, l(x))\), is a split map such that

\[
e\theta f = e(w\theta)f\text{ for all } e, f \in E(S), w \in W.
\]  

Conversely, every regular semigroup \(T\) with a split map \(\theta : T \to S\) satisfying (9) can be constructed in this way.

Let \(A : E(A(1, 2))/R(= E(A(1, 2))) \to P\) and \(B : E(A(1, 2))/L \to P\) be the constant functors at the two elements set \(\{1, 2\}\) with base point 1. Thus, \(A\) associates with each \(r(e) \in E(A(1, 2))/R\) the point set \(\{1, 2\}\) and with each pair \(r(e) \geq r(f)\) the identity function \(id : \{1, 2\} \to \{1, 2\}\). Similarly, \(B\) associates with
each \( l(e) \in E(A(1,2)) \) is the pointed set \( \{1, 2\} \) and with each pair \( l(e) \geq l(f) \) the identity function \( \text{id} : \{1, 2\} \to \{1, 2\} \). Then \((A, B)\) is an \( A(1, 2)\) - pair. The maps
\[
* : B_l(d, q^mp^n) \times A_r(a, q^mp^n) \to A(1, 2)
\]
given by
\[
1 * 1 = 1 * 2 = 2 * 1 = (x^n, q^mp^n)(a, q^mp^n)
\]
and
\[
2 * 2 = (x^n, q^mp^n)(x, q)(a, q^mp^n)
\]
defines a \( B \times A \) matrix \( * : B \times A \to A(1, 2) \) over \( A(1, 2) \) which clearly satisfies (i), (ii) and (iii) of Theorem 14. So
\[
W = W(A(1, 2); A, B; *)
\]
\[
= \{(u, (a, q^mp^n), v) : (a, q^mp^n) \in A(1, 2), u \in A_r(a, q^mp^n), v \in B_l(a, q^mp^n)\}
\]
with multiplication
\[
(u_1, (a, q^mp^n), v_1)(u_2, (b, q^p^n), v_2) = (u_1, (a, q^mp^n)(v_1 * u_2)(b, q^p^n), v_2)
\]
is a regular semigroup with a split map \( \theta : W \to A(1, 2) \) such that \( ewf = e(w\theta)f \) where \( e, f \in E(A(1, 2)), w \in W \).

Take any \( (a, q^mp^n), (b, q^p^n) \in A(1, 2) \). Then by Corollary 4, \( (a, q^mp^n)L(x^n, q^n p^n) \) and \( (b, q^p^n)R(b, q^p^n) \). Therefore,
\[
(u_1, (a, q^mp^n), v_1)(u_2, (b, q^p^n), v_2)
\]
\[
= \{(u_1, (a, q^mp^n)(v_1 * u_2)(b, q^p^n), v_2)
\]
\[
= \{(u_1, (a, q^mp^n)(b, q^p^n), v_2) \quad \text{if} \quad v_1 = 1 \quad \text{or} \quad u_2 = 1
\]
\[
= \{(u_1, (a, q^mp^n)(x, q)(b, q^p^n), v_2) \quad \text{if} \quad v_1 = u_2 = 2
\]

Thus

**Theorem 15.** The regular semigroup \( W \) coincides with \( RM(A(1, 2); \{1, 2\}, \{1, 2\}; p) \), the regular Rees matrix semigroup over \( A(1, 2) \) with sandwich matrix
\[
p = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & x & q
\end{pmatrix}
\]

**Proof.** Clearly \( W = RM(A(1, 2); \{1, 2\}, \{1, 2\}; p) \) as sets, and by (10), the multiplication in \( W \) coincide with the multiplication in \( RM(A(1, 2); \{1, 2\}, \{1, 2\}; p) \). □

**Theorem 16.** \( W \) is isomorphic to \( DSp_4 \).

**Proof.** Take \( \pi = (1, (1, 1), 2), \delta = (1, (1, 1), 1), \tau = (2, (1, 1), 1), \alpha = (2, (1, p), 2), \beta = (1, (y, qp), 2) \in W \). Then \( \pi, \delta, \tau, \alpha, \beta \) are idempotents and, \( \pi \delta \alpha R \alpha \delta L = \tau \delta L \).

\( \pi \delta \alpha R \alpha \delta L = \tau \delta L \)
Therefore the relations of \( DS_{p4} \) are satisfied by the generators of \( W \), the map \( \tilde{a} \rightarrow \pi, \tilde{b} \rightarrow \tilde{b}, \tilde{c} \rightarrow \tau, \tilde{d} \rightarrow \tilde{d} \) and \( \tilde{e} \rightarrow \tau \) extends to a homomorphism \( \chi : DS_{p4} \rightarrow W \) from \( DS_{p4} \) to \( W \). Let \((u, (a, q^m p^n), v) \in W \) with \( a = a_1 a_2 \cdots a_m \in F_m \). Then, we have

\[
(11) \quad (u, (a, q^m p^n), v) = (u, (1, 1), 1)(1, (a_1, q), 2)(2, (1, 1), 1)(1, (a_2, q), 2) \\
(2, (1, 1), 1) \cdots (1, (a_m, q), 2)(2, (1, 1), 1) \\
[1, (1, 1), 1][2, (1, p), 2]^m[1, (1, 1), v].
\]

This implies that \( W \) is an idempotent generated semigroup generated by \( \pi, \tilde{b}, \tau, \tilde{d}, \tau \) and hence \( \chi \) is onto. It is easy to see that, the set \( E(W) \) of idempotents of \( W \) is given by

\[
E(W) = \left\{ \begin{array}{ll}
(1, (a, q^m p^n), 2) \\
(1, (a, q^m p^n), 1) \\
(2, (a, q^m p^n), 1) \\
(2, (a, q^m p^{n+1}), 2), \quad a \in F^m.
\end{array} \right.
\]

Further, for any \( m, n \geq 0 \), \( (1, (a, q^m p^n), 2) \leq (1, (1, 1), 2), \ (1, (a, q^m p^n), 1) \leq (1, (1, 1), 1), \ (2, (a, q^m p^n), 1) \leq (2, (1, 1), 1) \) and \( (2, (a, q^m p^{n+1}), 2) \leq (2, (1, p), 2) \). So \( \chi \) is one-to-one on idempotents. Thus \( \chi \) is an idempotent separating homomorphism from \( DS_{p4} \) on to \( W \). Since \( DS_{p4} \) is fundamental, \( \chi \) is an isomorphism.

**Corollary 17.** \( DS_{p4} \) is isomorphic to the regular Rees matrix semigroup

\[
RM \left( A(1, 2); \{1, 2\}, \{1, 2\}; \begin{pmatrix} (1, 1) & (1, 1) \\
(1, 1) & (x, q) \end{pmatrix} \right) \text{ over } A(1, 2).
\]

**Theorem 18.** Let \((T, \theta)\) be a split coextension of \( S \) by left zero semigroups. Let \( \overline{S} = RM(S; I, \wedge, p) \) be a regular Rees matrix semigroup over \( S \) with sandwich matrix \( p : \wedge \times I \rightarrow S \). Then the regular Rees matrix semigroup \( \overline{T} = RM(T, I, \wedge, p) \) contains \( \overline{S} \) and the map \( \overline{\theta} : \overline{T} \rightarrow \overline{S} \), defined by \((i, x, \lambda)\overline{\theta} = (i, x \theta, \lambda) \) defines a split coextension \( \overline{(T, \theta)} \) of \( \overline{S} \) by left zero semigroups.

**Proof.** By [8, Lemma 1.1],

\[
(i, x, \lambda) \in \overline{T} \quad \Rightarrow \quad V(x) \cap p_{\lambda j} T p_{\gamma i} \neq \emptyset \quad \text{for some } j \in I, \gamma \in \wedge.
\]

\[
\Rightarrow \quad V(x \theta) \cap p_{\lambda j} S p_{\gamma i} \neq \emptyset \quad \text{since } \theta \text{ is a homomorphism}
\]

\[
\Rightarrow \quad (i, x \theta, \lambda) \in \overline{S}.
\]

Therefore \( \overline{\theta} : \overline{T} \rightarrow \overline{S} \) is a well defined map. Since \( \theta \) is a homomorphism and \( p \) takes values in \( S \) on which \( \theta \) is an identity map, \( \overline{\theta} \) is a homomorphism. Now it is enough to prove that for each \((i, x, \lambda) \in E(\overline{S}), (i, x, \lambda)\theta^{-1} \) is a left zero semigroup. Again by [8, Lemma 1.1], \((i, x, \lambda)(i, x, \lambda) = (i, x, \lambda) \) implies that \( x p_{\lambda i} x = x \); in particular \( x p_{\lambda i} \) and \( p_{\lambda i} x \) are idempotents of \( S \). Take \((i, y, \lambda), (i, z, \lambda) \) in \((i, x, \lambda)\theta^{-1} \).
Then \( y^\theta = z^\theta = x \), so that \( p_{\lambda_i} y p_{\lambda_i} = p_{\lambda_i} y \), since \( (p_{\lambda_i} x)^{\theta^{-1}} \) is a left zero semigroup. Therefore

\[
(i, y, \lambda)(i, z, \lambda) = (i, y p_{\lambda_i} z \lambda p_{\lambda_i} z, \lambda) = (i, y p_{\lambda_i} y, \lambda) = (i, y, \lambda).
\]

Hence \( (i, x, \lambda)^{\theta^{-1}} \) is a left zero semigroup. \( \square \)

The fundamental four-spiral semigroup

\[
Sp_4 \cong RM \left( C(p, q); \{1, 2\}, \{1, 2\}; \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \right)
\]

over the bicyclic semigroup \( C(p, q) \) ([1]). Applying Theorem 18 to the Proposition 10, we get

**Corollary 19.** (See [3], Theorem 14) \( DSp_4 \) is a split coextension of \( Sp_4 \) by left zero semigroups.

**Remark 20.** A regular semigroup \( T \) is called an idempotent - separating extension of \( S \) if there is an idempotent - separating homomorphism \( \theta \) from \( T \) on to \( S \). In [7], Meakin described the structure of \( Sp_4 \) as a semigroup of ordered pairs and studied the idempotent separating extensions of \( Sp_4 \) analogous to Reilly’s theorem ([10]). Since we described \( A(1, 2) \) and \( DSp_4 \) explicitly one can study the idempotent - separating extension of \( A(1, 2) \) and \( DSp_4 \) analogous to [10] and [7].

**Problem 21.** Determine the idempotent - separating extension of \( A(1, 2) \) and \( DSp_4 \).

**References**


