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Remarks on the Valid Equations in Lattice Implication Algebras

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Abstract. We present a set of equations that axiomatizes the class of all lattice implication algebras. The construction is different from the one given in [7], and the proof is direct: i.e., it does not rely on results from outside the realm of the lattice implication algebras, such as the theory of BCK-algebras. Then we show that the lattice H implication algebras are nothing more than the familiar Boolean algebras. Finally we obtain some negative results for the embedding of lattice implication algebras into Boolean algebras.

1. Introduction

Classical logic based on two truth values is sometimes not adequate to deal with mathematical problems encountered in some disciplines of science. The lattice implication algebra, which is introduced by Y. Xu in 1993 ([5]), is one of the various attempts to overcome this shortcomings of classical logic.

Definition 1.1 (Y. Xu [5]). By a *lattice implication algebra* we mean a bounded lattice $\langle L, \vee, \wedge, 0, 1\rangle$ with order-reversing involution "$'$" and a binary operation "$\to$" satisfying the following axioms:

$$
\begin{aligned}
&\text{(I1)} \quad x \to (y \to z) = y \to (x \to z),\\
&\text{(I2)} \quad x \to x = 1,\\
&\text{(I3)} \quad x \to y = y' \to x',\\
&\text{(I4)} \quad x \to y = y \to x = 1 \Rightarrow x = y,\\
&\text{(I5)} \quad (x \to y) \to y = (y \to x) \to x,\\
&\text{(L1)} \quad (x \vee y) \to z = (x \to z) \wedge (y \to z),\\
&\text{(L2)} \quad (x \wedge y) \to z = (x \to z) \vee (y \to z),
\end{aligned}
$$

for all $x, y, z \in L$. We denote the class of all lattice implication algebras by LIA. By abuse of notation we may abbreviate "lattice implication algebra" as LIA.

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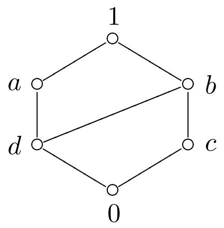
Recall that an *involution* on a set X means a bijective mapping $\sigma : X \to X$ such that $\sigma^2(x) = x$ for all $x \in X$.

It is easy to verify that a Boolean algebra can be expanded to a lattice implication algebra in the obvious way: $'$ is the complementation and $x \to y \overset{\text{def}}{=} x' \vee y$. Indeed all the axioms for LIA holds in any power set Boolean algebra, and by the Stone's representation theorem we can safely conclude our claim. Of course not all LIA's arise in this way. For instance there are LIA's in which $x \vee x' \neq 1$. One such example is given below.

Example 1.2 (Y. Jun, E. Roh, Y. Xu [2])**.** Let $L = \{0, a, b, c, d, 1\}$ with the following structure.



x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

In proving the Stone's theorem, it is vital that every Boolean algebra is isomorphically embedded in some power of the two element Boolean algebra, and this in turn relies on the fact that Boolean algebras are equationally axiomatized: i.e., there exists a set Σ of equations such that an algebra A in the signature $\langle \vee, \wedge, ', 0, 1\rangle$ is a Boolean algebra if and only if every element of Σ is valid in A.

The axiom system for LIA in definition is not equational, because of (I4) and the "order-reversing" requirement for involution: the latter may be expressed as $x \leq y \Rightarrow y' \leq x'$, which is apparently not an equation.

We know that lattices are equationally axiomatized and involution is forced by the single equation $x'' = x$, $\forall x$. So we need to find some equations that can replace (I4) and the order-reversing axiom.

The reason why we look for an equational axiom system for LIA is that equationally axiomatizable class of algebras enjoys the merits of utilizing the extensive set of tools from the theory of universal algebras. Perhaps a sort of representation theorem, such as the Stone's theorem for Boolean algebras or the Priestley's theorem for bounded distributive lattices, may be obtained for (some interesting subclasses of) LIA.

For the problem of characterizing the set of all valid equations in LIA, a necessary condition is easy to obtain, and given below.

Proposition 1.3. *If an equation $t_1 = t_2$ is valid in LIA then $(t_1 \to t_2) \wedge (t_2 \to t_1)$ must be a tautology. In particular, if an equation $t = 1$ is valid in LIA, then t must be a tautology.*

Proof. Let us abbreviate $(t_1 \to t_2) \wedge (t_2 \to t_1)$ as $t_1 \leftrightarrow t_2$. It is well-known that if t_1 and t_2 are terms in the signature of Boolean algebras (or to say the same thing, a propositional formula) and if the equation $t_1 = t_2$ is valid in the two-element Boolean algebra, denoted by $\mathbf{B}_2$, then $t_1 \leftrightarrow t_2$ is a tautology. (Actually the converse holds too.)

Since every axiom for LIA holds in $\mathbf{B}_2$, we have $\mathbf{B}_2 \in \mathsf{LIA}$. Thus, if $t_1 = t_2$ is valid in LIA, then so is it in $\mathbf{B}_2$. Hence $t_1 \leftrightarrow t_2$ must be a tautology. $\square$

2. Preliminaries

As a lattice $(L, \vee, \wedge)$ can be viewed as a poset $(L, \leq)$ by

$$x \leq y \Leftrightarrow x \vee y = y \Leftrightarrow x \wedge y = x, \tag{2.1}$$

we can define a partial order $\leq$ on an LIA by

$$x \to y = 1 \Leftrightarrow x \leq y. \tag{2.2}$$

The fact is that (2.2) indeed defines a partial order that coincides with the lattice order defined in (2.1). The proof is rather easily done, perhaps by using some of the following facts. The proofs for these facts are either easy or found in [5], [6].

Proposition 2.1. *Let L be an LIA. Then we have*

$$(x \vee y)' = x' \wedge y', \;\; (x \wedge y)' = x' \vee y', \;\; 1' = 0, \;\; 0' = 1, \quad \text{for all} \quad x, y \in L$$

.

Proposition 2.2. *Let L be an LIA and let $x, y, z \in L$. Then we have the following.*

(1) $x \to 1 = 1$, $0 \to x = 1$.
(2) $x \to (y \to x) = 1$.
(3) $1 \to x = x$, $x \to 0 = x'$.
(4) $x \leq y$ *implies* $y \to z \leq x \to z$.
(5) $x \leq y$ *implies* $z \to x \leq z \to y$.
(6) $x \to ((x \to y) \to y) = 1$.
(7) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$.
(8) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$.
(9) $x \vee y = (x \to y) \to y$.

Definition 2.3 (Y. Xu, K. Qin [6]). A *lattice H implication algebra* is a lattice implication algebra L satisfying

$$x \vee y \vee ((x \wedge y) \to z) = 1 \tag{2.3}$$

for all $x, y, z \in L$. We denote the class of all lattice H implication algebras by LHIA.

It is shown that the following two equations are valid for LHIA([6]).

$$x \to (x \to y) = x \to y,$$
$$x \to (y \to z) = (x \to y) \to (x \to z).$$

3. Equational axiomatization

Proposition 3.1. *The order-reversing property of the involution in lattice implication algebras can be specified by a single equational axiom*

$$(*) \qquad (x \wedge y)' \wedge y' = y'.$$

Proof. Suppose $x \leq y$. We will show $y' \leq x'$. From the supposition we get $x = x \wedge y$. Thus $x' \wedge y' = (x \wedge y)' \wedge y' = y'$ by $(*)$, which implies $y' \leq x'$ as desired. □

Theorem 3.2. *The class LIA of all lattice implication algebras is equationally axiomatized.*

Proof. The class of bounded lattices is equationally axiomatized. To this collection of equations, we add the 6 equations in definition (excluding the non-equation (I4)) and the following 4 more.

$$x'' = x, \tag{3.1}$$
$$(x \wedge y)' \wedge y' = y', \tag{3.2}$$
$$x \vee y = (x \to y) \to y, \tag{3.3}$$
$$1 \to y = y. \tag{3.4}$$

First we show that all these 4 axioms are valid in LIA. (3.1) simply says that "$'$" is an involution. (3.2) holds because $x \wedge y \leq y$ and "$'$" is order-reversing. (3.3) and (3.4) are chosen from fact .

Now we assume the four axioms (3.1)–(3.4) and are going to show that "$'$" is order-reversing and (I4) is valid. The fact that "$'$" is order-reversing has been shown earlier in proposition . It remains to show $x \to y = 1 \Rightarrow x \leq y$ — this will suffice to prove (I4).

If $x \to y = 1$, then $(x \to y) \to y = y$ by (3.4). But then $x \vee y = y$ by (3.3). Therefore we get $x \leq y$ as desired. □

In fact, an excellent equational axiomatization of LIA is found by Yiquan and Wenbiao in 2001 ([7]). But their proof utilizes the notion of BCK-*algebras*, which is axiomatized by the following

(1) $((x * y) * (x * z)) * (z * y) = 0,$

(2) $(x * (x * y)) * y = 0,$

(3) $x * x = 0$,

(4) $x * y = 0 = y * x \Rightarrow x = y$,

(5) $0 * x = 0$.

$x * y$ is something like the set difference $x - y$ or $(x \to y)'$. One can define a partial order on a BCK-algebra by letting $x \leq y \Leftrightarrow x * y = 0$. However one cannot recover $*$ from $\leq$. The proof in [7] relies on some theorems in the theory of BCK-algebras ([1]). Our axiomatization is self-contained and easier to understand.

Next we turn our attention to LHIA, which is defined in 2.3. We find that a lattice H implication algebra is nothing but an expansion of a Boolean algebra.

Proposition 3.3. *Every lattice H implication algebra is obtained by taking an expansion of a Boolean algebra in the standard way: i.e.,*

$$x \to y = x' \vee y. \tag{3.5}$$

Thus the familiar equational axioms for Boolean algebras plus (3.5) *form an equational axiom system for LHIA.*

Proof. The axiom (2.3) for LHIA is $x \vee y \vee ((x \wedge y) \to z) = 1,\ \forall x, y, z \in L$. So, if we take $x = y$ and $z = 0$, we get the familiar $x \vee x' = 1$. By taking the involution of both sides as guaranteed by proposition , we get $x \wedge x' = 0$. So we get all the axioms for Boolean algebras except the distributivity. But every LIA is distributive as shown in [7]. $\square$

Now we may say that LHIA is relatively uninteresting because it is nothing new.

4. Embedding lattice implication algebras into Boolean algebras

As we mentioned earlier, it would be nice if we could embed (some subclass of) lattice implication algebras into Boolean algebras. The standard way of doing this job is to associate, to each element a in a lattice implication algebra L, a set of maximal ideals in L, or equivalently a set of (0,1)-valued homomorphisms from L.

Henceforth L denotes a lattice implication algebra unless stated otherwise.

Definition 4.1 (Y. Jun, E. Roh, Y. Xu [2])**.** An *LI-ideal* of L is a subset $\emptyset \neq A \subseteq L$ such that

$$0 \in A,$$
$$(x \to y)' \in A \text{ and } y \in A \Rightarrow x \in A,$$

for all $x, y \in A$.

Note that $\{0\}$ and L are (improper) ideals of L. Recall that an ordinary lattice ideal (compared to our LI-ideals) is defined to be a nonempty subset A of the lattice satisfying $x \in A,\ y \leq x \Rightarrow y \in A$ and $x, y \in A \Rightarrow x \vee y \in A$. In [2], it is shown that

LI-ideals are necessarily lattice ideals but not conversely. In example , $\{0, c\}$ is an LI-ideal, and $\{0, d\}$ is a lattice ideal which is not an LI-ideal.

The reason why we would need LI-ideals instead of ordinary lattice ideals is that an LI-ideal induces a congruence relation on a the lattice implication algebra in the following way.

$$x \sim y \text{ if and only if } \{(x \to y)', (y \to x)'\} \subseteq A.$$

This enables us to form the quotient structure L/A and start the whole business of homomorphisms, kernels etc.

Proposition 4.2. *Let $\mathcal{A}$ be a set of proper LI-ideals of L which is linearly ordered: i.e. $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. Then $\bigcup \mathcal{A}$ is again a proper LI-ideal of L.*

Proof. Let $A = \bigcup \mathcal{A}$, and let $(x \to y)' \in A$, $y \in A$. We want to show $x \in A$. Take $A_1, A_2 \in \mathcal{A}$ so that $(x \to y)' \in A_1$ and $y \in A_2$. Then, without loss of generality, both $(x \to y)'$ and y belong to A_1. Thus $x \in A_1 \subseteq A$ as desired. Obviously $0 \in A_1 \subseteq A$. To show that A is proper, just note that an ideal B is proper iff $1 \notin B$ and A is a union of proper ideals. □

It is known that given any lattice element $x \in L$ there always exists a least LI-ideal that contains x ([2]). This fact, combined with proposition and the Zorn's lemma, guarantees that each element x is contained in a maximal LI-ideal of L, *provided that the least LI-ideal containing x is proper.* For ordinary lattice ideals this problem does not occur, as $\{y \leq x \mid y \in L\}$ is the least ideal containing x, which is proper unless $x = 1$.

The standard trick of embedding L into a Boolean algebra of sets goes as follows. Let $\mathcal{A}$ be the set of all maximal LI-ideals of L. Define a map ϕ from L into the power set Boolean algebra $\mathcal{P}(\mathcal{A})$ by $\phi(a) = \{A \in \mathcal{A} \mid a \in A\}$. We show ϕ is one-to-one and preserves some operations among $\to$, $\vee$, $\wedge$ and $'$.

Let us compute the least LI-ideal $\langle b \rangle$ containing b in example . We know $d \in \langle b \rangle$ as $d \leq b$. From $(1 \to b)' = b' = d \in \langle b \rangle$ we get $1 \in \langle b \rangle$. Thus $\langle b \rangle = L$, the improper ideal.

So this approach is hopeless for LIA. Once we tried to embed LHIA, which seemed to be a sensible subclass of LIA, into Boolean algebras in this manner and made success, which is of course pointless as LHIA is essentially the class of all Boolean algebras.

Perhaps we could add an axiom weaker than (2.3) to the axioms of LIA and find an adequate subclass of LIA that yields an interesting result.

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