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A Note on Maass-Jacobi Forms

Jae-Hyun Yang
Department of Mathematics, Inha University, Incheon 402-751, Korea
e-mail : jhyang@inha.ac.kr

Abstract. In this paper, we introduce the notion of Maass-Jacobi forms and investigate some properties of these new automorphic forms. We also characterize these automorphic forms in several ways.

1. Introduction

We let $SL_{2,1}(\mathbb{R}) = SL(2,\mathbb{R}) \ltimes \mathbb{R}^{(1,2)}$ be the semi-direct product of the special linear group $SL(2,\mathbb{R})$ of degree 2 and the commutative group $\mathbb{R}^{(1,2)}$ equipped with the following multiplication law

$$(g,\alpha) * (h,\beta) = (gh,\, \alpha\, {}^t h^{-1} + \beta), \quad g,h \in SL(2,\mathbb{R}), \quad \alpha,\beta \in \mathbb{R}^{(1,2)}, \tag{1.1}$$

where $\mathbb{R}^{(1,2)}$ denotes the set of all 1×2 real matrices. We let

$$SL_{2,1}(\mathbb{Z}) = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^{(1,2)}$$

be the discrete subgroup of $SL_{2,1}(\mathbb{R})$ and $K = SO(2)$ the special orthogonal group of degree 2.

Throughout this paper, for brevity we put

$$G = SL_{2,1}(\mathbb{R}), \quad \Gamma_1 = SL(2,\mathbb{Z}) \quad \text{and} \quad \Gamma = SL_{2,1}(\mathbb{Z}).$$

Let $\mathbb{H}$ be the Poincaré upper half plane. Then G acts on $\mathbb{H}\times\mathbb{C}$ transitively by

$$(g,\alpha)\circ(\tau,z) = ((d\tau - c)(-b\tau + a)^{-1}, (z + \alpha_1\tau + \alpha_2)(-b\tau + a)^{-1}), \tag{1.2}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^{(1,2)}$ and $(\tau,z) \in \mathbb{H}\times\mathbb{C}$. We observe that K is the stabilizer of this action (1.2) at the origin $(i,0)$. $\mathbb{H}\times\mathbb{C}$ may be identified with the homogeneous space G/K in a natural way.

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The aim of this paper is to define the notion of Maass-Jacobi forms generalizing that of Maass wave forms and study some properties of these new automorphic forms. For the convenience of the reader, we review Maass wave forms. For $s \in \mathbb{C}$, we denote by $W_s(\Gamma_1)$ the vector space of all smooth bounded functions $f : SL(2,\mathbb{R}) \longrightarrow \mathbb{C}$ satisfying the following conditions (a) and (b):

(a) $f(\gamma g k) = f(g)$ for all $\gamma \in \Gamma_1,\ g \in SL(2,\mathbb{R})$ and $k \in K$.

(b) $\Delta_0 f = \frac{1-s^2}{4} f$,

where $\Delta_0 = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x \partial \theta} + \frac{5}{4}\frac{\partial^2}{\partial \theta^2}$ is the Laplace-Beltrami operator associated to the $SL(2,\mathbb{R})$-invariant Riemannian metric

$$ds_0^2 = \frac{1}{y^2}(dx^2 + dy^2) + \left(d\theta + \frac{dx}{2y}\right)^2$$

on $SL(2,\mathbb{R})$ whose coordinates $x, y, \theta\ (\,x \in \mathbb{R},\ y > 0,\ 0 \le \theta < 2\pi\,)$ are given by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad g \in SL(2,\mathbb{R})$$

by means of the Iwasawa decomposition of $SL(2,\mathbb{R})$. The elements in $W_s(\Gamma_1)$ are called *Maass wave forms.* It is well known that $W_s(\Gamma_1)$ is nontrivial for infinitely many values of s. For more detail, we refer to [6], [9], [13], [17] and [20].

The paper is organized as follows. In Section 2, we calculate the algebra of all invariant differential operators under the action (1.2) of G on $\mathbb{H} \times \mathbb{C}$ completely. In addition, we provide a G-invariant Riemannian metric on $\mathbb{H} \times \mathbb{C}$ and compute its Laplace-Beltrami operator. In Section 3, using the above Laplace-Beltrami operator, we introduce a concept of Maass-Jacobi forms generalizing that of Maass wave forms. We characterize Maass-Jacobi forms as smooth functions on G or $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ satisfying a certain invariance property, where $S\mathcal{P}_2$ denotes the symmetric space consisting of all 2×2 positive symmetric real matrices Y with $\det Y = 1$. In Section 4, we find the unitary dual of G and present some properties of G. In Section 5, we describe the decomposition of the Hilbert space $L^2(\Gamma\backslash G)$. In the final section, we make some comments on the Fourier expansion of Maass-Jacobi forms.

Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{Z}^+$ denotes the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix A, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. We denote the identity matrix of degree n by E_n. $\mathbb{H}$ denotes the Poincaré upper-half plane.

2. Invariant Differential Operators on $\mathbb{H} \times \mathbb{C}$

We recall that $S\mathcal{P}_2$ is the symmetric space consisting of all 2×2 positive symmetric real matrices Y with $\det Y = 1$. Then G acts on $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ transitively by

$$(g, \alpha) \cdot (Y, V) = (\, gY\, {}^t\!g,\; (V + \alpha)\, {}^t\!g \,), \tag{2.1}$$

where $g \in SL(2, \mathbb{R}),\ \alpha \in \mathbb{R}^{(1,2)},\ Y \in S\mathcal{P}_2$ and $V \in \mathbb{R}^{(1,2)}$. It is easy to see that K is a maximal compact subgroup of G stabilizing the origin $(E_2, 0)$. Thus $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ may be identified with the homogeneous space G/K as follows:

$$G/K \ni (g, \alpha)K \longmapsto (g, \alpha) \cdot (E_2, 0) \in S\mathcal{P}_n \times \mathbb{R}^{(m,n)}, \tag{2.2}$$

where $g \in SL(2, \mathbb{R})$ and $\alpha \in \mathbb{R}^{(1,2)}$.

We know that $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ transitively by

$$g < \tau >= (a\tau + b)(c\tau + d)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad \tau \in \mathbb{H}.$$

Now we observe that the action (1.2) of G on $\mathbb{H} \times \mathbb{C}$ may be rewritten as

$$(g, \alpha) \circ (\tau, z) = (\, {}^t\!g^{-1} < \tau >,\, (z + \alpha_1\tau + \alpha_2)(-b\tau + a)^{-1} \,),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),\ \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(1,2)}$, and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$. Since the action (1.2) is transitive and K is the stabilizer of this action at the origin $(i, 0)$, $\mathbb{H} \times \mathbb{C}$ can be identified with the homogeneous space G/K as follows:

$$G/K \ni (g, \alpha)K \longmapsto (g, \alpha) \circ (i, 0). \tag{2.3}$$

We see that we can express an element Y of $S\mathcal{P}_2$ uniquely as

$$Y = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & x^2y^{-1} + y \end{pmatrix} \tag{2.4}$$

with $x, y \in \mathbb{R}$ and $y > 0$.

Lemma 2.1. *We define the mapping* $T : S\mathcal{P}_n \times \mathbb{R}^{(m,n)} \longrightarrow \mathbb{H} \times \mathbb{C}$ *by*

$$T(Y, V) = (\, x + iy,\; v_1(x + iy) + v_2 \,), \tag{2.5}$$

where Y *is of the form* (2.4) *and* $V = (v_1, v_2) \in \mathbb{R}^{(1,2)}$. *Then the mapping* T *is a bijection which is compatible with the above two actions* (1.2) *and* (2.1).

For any $Y \in S\mathcal{P}_2$ of the form (2.4), we put

$$g_Y = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{pmatrix} = \begin{pmatrix} y^{-1/2} & 0 \\ -xy^{-1/2} & y^{1/2} \end{pmatrix}. \tag{2.6}$$

and

$$\alpha_{Y,V} = V\,{}^t g_Y^{-1}. \tag{2.7}$$

Then we have

$$T(Y,V) = (\,g_Y\,,\,\alpha_{Y,V}\,)\circ(i,0). \tag{2.8}$$

Proof. It is easy to prove the lemma. So we leave the proof to the reader. □

Now we give a complete description of the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ of all differential operators on $\mathbb{H}\times\mathbb{C}$ invariant under the action (1.2) of G. First we note that the Lie algebra $\mathfrak{g}$ of G is given by $\mathfrak{g} = \left\{\,(X,Z)\,|\,X\in\mathbb{R}^{(2,2)},\ \sigma(X)=0,\ Z\in\mathbb{R}^{(1,2)}\,\right\}$ equipped with the following Lie bracket

$$[(X_1,Z_1),\,(X_2,Z_2)] = (\,[X_1,X_2]_0,\,Z_2\,{}^tX_1 - Z_1\,{}^tX_2\,),$$

where $[X_1,X_2]_0 = X_1X_2 - X_2X_1$ denotes the usual matrix bracket and (X_1,Z_1), $(X_2,Z_2)\in\mathfrak{g}$. And $\mathfrak{g}$ has the following decomposition

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p} \qquad (\text{ direct sum }),$$

where $\mathfrak{k} = \left\{(X,0)\in\mathfrak{g}\ |\ X = \begin{pmatrix} 0 & x \\ -x & 0\end{pmatrix},\quad x\in\mathbb{R}\right\}$ and $\mathfrak{p} = \left\{(X,Z)\in\mathfrak{g}|\ X = {}^tX\in\mathbb{R}^{(2,2)},\ \sigma(X)=0,\ Z\in\mathbb{R}^{(1,2)}\right\}$.

We observe that $\mathfrak{k}$ is the Lie algebra of K and that we have the following relations

$$[\mathfrak{k},\,\mathfrak{k}]\subset\mathfrak{k} \qquad \text{and} \qquad [\mathfrak{k},\,\mathfrak{p}]\subset\mathfrak{p}.$$

Thus the coset space $G/K\cong\mathbb{H}\times\mathbb{C}$ is a *reductive* homogeneous space in the sense of [12], p. 284. It is easy to see that the adjoint action Ad of K on $\mathfrak{p}$ is given by

$$\mathrm{Ad}\,(k)((X,Z)) = (\,kX\,{}^tk,\ Z\,{}^tk\,), \tag{2.9}$$

where $k\in K$ and $(X,Z)\in\mathfrak{p}$ with $X={}^tX$, $\sigma(X)=0$. The action (2.9) extends uniquely to the action ρ of K on the polynomial algebra $\mathrm{Pol}\,(\mathfrak{p})$ of $\mathfrak{p}$ given by

$$\rho\,:\,K\longrightarrow \mathrm{Aut}\,(\,\mathrm{Pol}(\mathfrak{p}\,)). \tag{2.10}$$

Let $\mathrm{Pol}\,(\mathfrak{p})^K$ be the subalgebra of $\mathrm{Pol}\,(\mathfrak{p})$ consisting of all invariants of the action ρ of K. Then according to [12], Theorem 4.9, p. 287, there exists a canonical linear bijection $\lambda\,(\,P\longmapsto D_{\lambda(P)})$ of $\mathrm{Pol}\,(\mathfrak{p})^K$ onto $\mathbb{D}(\mathbb{H}\times\mathbb{C})$. Indeed, if $(\xi_k)\,(\,1\le k\le 4\,)$ is any basis of $\mathfrak{p}$ and $P\in\mathrm{Pol}\,(\mathfrak{p})^K$, then

$$\big(D_{\lambda(P)}f\big)\,(\tilde g\circ(i,0)) = \left[P\left(\frac{\partial}{\partial t_k}\right)f((\tilde g * \exp\,(\sum_{k=1}^4 t_k\xi_k))\circ(i,0))\right]_{(t_k)=0}, \tag{2.11}$$

where $\tilde{g} \in G$ and $f \in C^\infty(\mathbb{H} \times \mathbb{C})$.

We put

$$e_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (0,0) \right), \quad e_2 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (0,0) \right)$$

and

$$f_1 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right), \quad f_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,1) \right).$$

Then e_1, e_2, f_1, f_2 form a basis of $\mathfrak{p}$. We write for coordinates (X,Z) by

$$X = \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2)$$

with real variables x, y, z_1 and z_2.

Lemma 2.2. *The following polynomials*

$$\begin{aligned} P(X,Z) &= \frac{1}{8}\,\sigma(X^2) = \frac{1}{4}\,(x^2+y^2), \\ \xi(X,Z) &= Z\,{}^tZ = z_1^2 + z_2^2, \\ P_1(X,Z) &= -\frac{1}{2}\,ZX\,{}^tZ = \frac{1}{2}\,(z_2^2 - z_1^2)x - z_1 z_2 y \quad \textit{and} \\ P_2(X,Z) &= \frac{1}{2}\,(z_2^2 - z_1^2)\,y + z_1 z_2\, x \end{aligned}$$

are algebraically independent generators of $Pol(\mathfrak{p})^K$.

Proof. We leave the proof of the above lemma to the reader. □

Now we are ready to compute the G-invariant differential operators D, Ψ, D_1 and D_2 corresponding to the K-invariants P, ξ, P_1 and P_2 respectively under the canonical linear bijection (2.11). For real variables $t = (t_1, t_2)$ and $s = (s_1, s_2)$, we have

$$\exp(t_1e_1 + t_2e_2 + s_1f_1 + s_2f_2) = \left(\begin{pmatrix} a_1(t,s) & a_3(t,s) \\ a_3(t,s) & a_2(t,s) \end{pmatrix}, \, (b_1(t,s), \, b_2(t,s)) \right),$$

where

$$a_1(t,s) = 1 + t_1 + \frac{1}{2!}\,(t_1^2 + t_2^2) + \frac{1}{3!}\,t_1(t_1^2 + t_2^2) + \frac{1}{4!}\,(t_1^2 + t_2^2)^2 + \cdots$$

$$a_2(t,s) = 1 - t_1 + \frac{1}{2!}\,(t_1^2 + t_2^2) - \frac{1}{3!}\,t_1(t_1^2 + t_2^2) + \frac{1}{4!}\,(t_1^2 + t_2^2)^2 - \cdots,$$

$$a_3(t,s) = t_2 + \frac{1}{3!}\,t_2(t_1^2 + t_2^2) + \frac{1}{5!}\,t_2(t_1^2 + t_2^2)^2 + \cdots,$$

$$b_1(t,s) = s_1 - \frac{1}{2!}\,(s_1t_1 + s_2t_2) + \frac{1}{3!}\,s_1(t_1^2 + t_2^2) - \frac{1}{4!}\,(s_1t_1 + s_2t_2)(t_1^2 + t_2^2) + \cdots,$$

$$b_2(t,s) = s_2 - \frac{1}{2!}\,(s_1t_2 - s_2t_1) + \frac{1}{3!}\,s_2(t_1^2 + t_2^2) - \frac{1}{4!}\,(s_1t_2 - s_2t_1)(t_1^2 + t_2^2) + \cdots.$$

For brevity, we write $a_j,\, b_k$ for $a_j(t,s),\, b_k(t,s)$ $(\, j=1,2,3,\ k=1,2\,)$ respectively. We now fix an element $(g,\alpha)\in G$ and write

$$g\,=\,\begin{pmatrix} g_1 & g_{12}\\ g_{21} & g_2\end{pmatrix}\in SL(2,\mathbb{R})\quad \text{and}\quad \alpha\,=\,(\alpha_1,\alpha_2)\in\mathbb{R}^{(1,2)}.$$

We put $(\,\tau(t,s),\,z(t,s)\,)=(\,(g,\alpha)*\exp\,(\,t_1e_1\,+\,t_2e_2\,+\,s_1f_1\,+\,s_2f_2\,))\circ(i,0)$ with $\tau(t,s)=x(t,s)\,+\,i\,y(t,s)$ and $z(t,s)=u(t,s)\,+\,i\,v(t,s)$.
Here $x(t,s)$, $y(t,s)$, $u(t,s)$ and $v(t,s)$ are real. By an easy calculation, we obtain

$$\begin{aligned} x(t,s)\,&=\,-(\,\tilde a\tilde c+\tilde b\tilde d\,)\,(\,\tilde a^2+\tilde b^2\,)^{-1},\\ y(t,s)\,&=\,(\,\tilde a^2+\tilde b^2\,)^{-1},\\ u(t,s)\,&=\,(\,\tilde a\,\tilde\alpha_2-\tilde b\tilde\alpha_1\,)\,(\,\tilde a^2+\tilde b^2\,)^{-1},\\ v(t,s)\,&=\,(\,\tilde a\,\tilde\alpha_1+\tilde b\,\tilde\alpha_2\,)\,(\,\tilde a^2+\tilde b^2\,)^{-1},\end{aligned}$$

where $\tilde a=\,g_1a_1\,+\,g_{12}a_3,\ \ \tilde b=\,g_1a_3+\,g_{12}a_2,\ \tilde c=\,g_{21}a_1\,+\,g_2a_3,\ \ \tilde d=\,g_{21}a_3\,+\,g_2a_2,$ $\tilde\alpha_1=\,\alpha_1a_2\,-\,\alpha_2a_3\,+\,b_1,\ \tilde\alpha_2=\,-\alpha_1a_3\,+\,\alpha_2a_1\,+\,b_2.$
By an easy calculation, at $t=s=0$, we have

$$\begin{aligned}\frac{\partial x}{\partial t_1}\,&=\,4\,g_1\,g_{12}\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial y}{\partial t_1}\,&=\,-2\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial u}{\partial t_1}\,&=\,4\,g_1\,g_{12}\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial v}{\partial t_1}\,&=\,-\,2\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{2},\\ \frac{\partial^2 x}{\partial t_1^2}\,&=\,-\,16\,g_1\,g_{12}\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3},\\ \frac{\partial^2 y}{\partial t_1^2}\,&=\,8\,(\,g_1^2\,-\,g_{12}^2\,)^2\,(\,g_1^2\,+\,g_{12}^2\,)^{-3}\,-\,4\,(\,g_1^2\,+\,g_{12}^2\,)^{-1},\\ \frac{\partial^2 u}{\partial t_1^2}\,&=\,-\,16\,g_1\,g_{12}\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3},\\ \frac{\partial^2 v}{\partial t_1^2}\,&=\,4\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^4\,+\,g_{12}^4\,-\,6\,g_1^2\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial x}{\partial t_2}\,&=\,-\,2\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial y}{\partial t_2}\,&=\,-\,4\,g_1\,g_{12}\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial u}{\partial t_2}\,&=\,-\,2\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\end{aligned}$$

$$\begin{aligned}
\frac{\partial v}{\partial t_2} &= -4\,g_1\,g_{12}\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\
\frac{\partial^2 x}{\partial t_2^2} &= 16\,g_1\,g_{12}\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3},\\
\frac{\partial^2 y}{\partial t_2^2} &= 32\,g_1^2\,g_{12}^2\,(\,g_1^2\,+\,g_{12}^2\,)^{-3}\,-\,4\,(\,g_1^2\,+\,g_{12}^2\,)^{-1},\\
\frac{\partial^2 u}{\partial t_2^2} &= 16\,g_1\,g_{12}\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3},\\
\frac{\partial^2 v}{\partial t_2^2} &= -4\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^4\,+\,g_2^4\,-\,6\,g_1\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3}.
\end{aligned}$$

We note that $\tilde{a}\tilde{d}\,-\,\tilde{b}\tilde{c}\,=\,1,\ \ a_1a_2 - a_3^2 = 1$ and $g_1g_2 - g_{12}g_{21} = 1$.

Using the above facts and applying the chain rule, we can easily compute the differential operators D, Ψ, D_1 and D_2. It is known that the images of generators P, ξ, P_1 and P_2 under λ are generators of $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ (cf. [11]).

Summarizing, we have the following.

Theorem 2.3. *The algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is generated by the following differential operators*

(2.12)
$$D = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right),$$

$$\Psi = y\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right), \tag{2.13}$$

$$D_1 = 2\,y^2\,\frac{\partial^3}{\partial x\partial u\partial v}\,-\,y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right)+\left(v\,\frac{\partial}{\partial v}+1\right)\Psi \tag{2.14}$$

and

$$D_2 = y^2\,\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial v^2}-\frac{\partial^2}{\partial u^2}\right)-2\,y^2\,\frac{\partial^3}{\partial y\partial u\partial v}\,-\,v\,\frac{\partial}{\partial u}\Psi, \tag{2.15}$$

where $\tau = x+iy$ and $z = u+iv$ with real variables x,y,u,v. Moreover, we have

$$\begin{aligned}
[D,\,\Psi] = D\Psi-\Psi D\, &= 2\,y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right)-\,4\,y^2\,\frac{\partial^3}{\partial x\partial u\partial v}\\
&\quad -\,2\left(v\,\frac{\partial}{\partial v}\Psi\,+\,\Psi\right).
\end{aligned}$$

In particular, the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ *is not commutative. Thus the homogeneous space* $\mathbb{H}\times\mathbb{C}$ *is not weakly symmetric in the sense of A. Selberg* ([19]).

Now we provide a natural G-invariant Kähler metric on $\mathbb{H}\times\mathbb{C}$.

Proposition 2.4. *The Riemannian metric* ds^2 *on* $\mathbb{H}\times\mathbb{C}$ *defined by*

$$ds^2 \,=\, \frac{y\,+\,v^2}{y^3}\,(\,dx^2\,+\,dy^2\,)\,+\,\frac{1}{y}\,(\,du^2\,+\,dv^2\,)\;-\;\frac{2v}{y^2}\,(\,dx\,du\,+\,dy\,dv\,)$$

is invariant under the action (1.2) *of* G *and is a Kähler metric on* $\mathbb{H}\times\mathbb{C}$. *The Laplace-Beltrami operator* Δ *of the Riemannian space* $(\,\mathbb{H}\times\mathbb{C},\,ds^2\,)$ *is given by*

$$\Delta\,=\,y^2\left(\frac{\partial^2}{\partial x^2}\,+\,\frac{\partial^2}{\partial y^2}\right)+(\,y\,+\,v^2\,)\left(\frac{\partial^2}{\partial u^2}\,+\,\frac{\partial^2}{\partial v^2}\right)+2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}\,+\,\frac{\partial^2}{\partial y\partial v}\right).$$

That is, $\Delta\,=\,D\,+\,\Psi$.

Proof. For $Y\in S\mathcal{P}_2$ of the form (2.4) and $(v_1,v_2)\in\mathbb{R}^{(1,2)}$, it is easy to see that

$$dY\,=\,\begin{pmatrix} -\,y^{-2}\,dy & -\,y^{-1}\,dx\,+\,x\,y^{-2}\,dy \\ -\,y^{-1}\,dx\,+\,x\,y^{-2}\,dy & 2\,x\,y^{-1}\,dx\,+\,(\,1\,-\,x^2\,y^{-2}\,)\,dy \end{pmatrix}$$

and $dV\,=\,(dv_1,dv_2)$. Then we can show that the following metric $d\tilde{s}^2$ on $S\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ defined by

$$d\tilde{s}^2\,=\,\frac{dx^2\,+\,dy^2}{y^2}\,+\,\frac{1}{y}\,\left\{\,(\,x^2\,+\,y^2\,)\,dv_1^2\,+\,2\,x\,dv_1\,dv_2\,+\,dv_2^2\,\right\}$$

is invariant under the action (2.1) of G. Indeed, since

$$Y^{-1}\,=\,\begin{pmatrix} y\,+\,x^2\,y^{-1} & x\,y^{-1} \\ x\,y^{-1} & y^{-1}\end{pmatrix},$$

we can easily show that $d\tilde{s}^2\,=\,\frac{1}{2}\,\sigma(Y^{-1}dYY^{-1}dY)\,+\,dV\,Y^{-1}\,{}^t(dV)$.
For an element $(\,g,\,\alpha\,)\in G$ with $g\in SL(2,\mathbb{R})$ and $\alpha\in\mathbb{R}^{(1,2)}$, we put

$$(\,Y^*,\,V^*\,)=(\,g,\,\alpha\,)\cdot(Y,V)\,=\,(\,gY\,{}^tg,\,(V+\alpha)\,{}^tg\,).$$

Since $Y^*\,=\,gY\,{}^tg$ and $V^*\,=\,(\,V\,+\,\alpha\,)\,{}^tg$, we get $dY^*\,=\,g\,dY\,{}^tg$ and $V^*\,=\,(\,V\,+\,\alpha\,)\,{}^tg$.
Therefore by a simple calculation, we can show that

$$\begin{aligned}&\sigma\left(\,Y^{*-1}\,dY^*\,Y^{*-1}\,dY^*\,\right)\,+\,dV^*\,Y^{*-1}\,{}^t(dV^*)\\ &=\,\sigma(\,Y^{-1}dY\,Y^{-1}\,dY\,)\,+\,dV\,Y^{-1}\,{}^t(dV).\end{aligned}$$

Hence the metric $d\tilde{s}^2$ is invariant under the action (2.1) of G.

Using this fact and Lemma 2.1, we can prove that the metric ds^2 in the above theorem is invariant under the action (1.2). Since the matrix form (g_{ij}) of the metric ds^2 is given by

$$(g_{ij}) = \begin{pmatrix} (y+v^2)\,y^{-3} & 0 & -v\,y^{-2} & 0 \\ 0 & (y+v^2)\,y^{-3} & 0 & -v\,y^{-2} \\ -v\,y^{-2} & 0 & y^{-1} & 0 \\ 0 & -v\,y^{-2} & 0 & y^{-1} \end{pmatrix}$$

and $\det(g_{ij}) = y^{-6}$, the inverse matrix (g^{ij}) of (g_{ij}) is easily obtained by

$$(g^{ij}) = \begin{pmatrix} y^2 & 0 & y\,v & 0 \\ 0 & y^2 & 0 & y\,v \\ y\,v & 0 & y+v^2 & 0 \\ 0 & y\,v & 0 & y+v^2 \end{pmatrix}.$$

Now it is easily shown that $D+\Psi$ is the Laplace-Beltrami operator of $(\mathbb{H}\times\mathbb{C},\, ds^2)$. □

Remark 2.5. We can show that for any two positive real numbers α and β, the following metric

$$ds^2_{\alpha,\beta} = \alpha\,\frac{dx^2+dy^2}{y^2} + \beta\,\frac{v^2(dx^2+dy^2)+y^2(du^2+dv^2)-2\,yv\,(dx\,du+dy\,dv)}{y^3}$$

is also a Riemannian metric on $\mathbb{H}\times\mathbb{C}$ which is invariant under the action (1.2) of G. In fact, we can see that the two-parameter family of $ds^2_{\alpha,\beta}\ (\alpha>0,\ \beta>0)$ provides a complete family of Riemannian metrics on $\mathbb{H}\times\mathbb{C}$ invariant under the action of (1.2) of G. It can be easily seen that the Laplace-Beltrami operator $\Delta_{\alpha,\beta}$ of $ds^2_{\alpha,\beta}$ is given by

$$\begin{aligned} \Delta_{\alpha,\beta} &= \frac{1}{\alpha}\,y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \left(\frac{y}{\beta}+\frac{v^2}{\alpha}\right)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \\ &\quad + \frac{2\,yv}{\alpha}\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right) \\ &= \frac{1}{\alpha}\,D + \frac{1}{\beta}\,\Psi. \end{aligned}$$

Remark 2.6. By a tedious computation, we see that the scalar curvature of $(\mathbb{H}\times\mathbb{C},\, ds^2)$ is -3.

We want to propose the following problem to be studied in the future.

Problem 2.7. Find all the eigenfunctions of Δ.

We will give some examples of eigenfunctions of Δ.

(1) $h(x,y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|a|y)\, e^{2\pi iax} \quad (s \in \mathbb{C},\ a \neq 0\,)$ with eigenvalue $s(s-1)$, where

$$K_s(z) := \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\} t^{s-1}\, dt, \qquad \mathrm{Re}\, z > 0. \tag{2.16}$$

(2) $y^s,\ y^s x,\ y^s u\ (s \in \mathbb{C})$ with eigenvalue $s(s-1)$.

(3) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(4) $x,\ y,\ u,\ v,\ xv,\ uv$ with eigenvalue 0.

(5) All Maass wave forms.

3. Maass-Jacobi forms

Let Δ be the Laplace-Beltrami operator of the Riemannian metric ds^2 on $\mathbb{H}\times\mathbb{C}$ defined in Proposition 2.4. Using this operator, we define the notion of Maass-Jacobi forms.

Definition 3.1. A smooth bounded function $f : \mathbb{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ is called a *Maass-Jacobi form* if it satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) $f(\tilde{\gamma}\circ(\tau,z)) = f(\tau,z) \quad$ for all $\tilde{\gamma} \in \Gamma$ and $(\tau,z) \in \mathbb{H}\times\mathbb{C}$.

(MJ2) f is an eigenfunction of the Laplace-Beltrami operator Δ.

(MJ3) f has a polynomial growth, that is, f fulfills a boundedness condition.

For a complex number $\lambda \in \mathbb{C}$, we denote by $MJ(\Gamma,\lambda)$ the vector space of all Maass-Jacobi forms f such that $\Delta f = \lambda f$. We note that, since $\Delta f = \lambda f$ is an elliptic partial differential equation, Maass-Jacobi forms are real analytic (see [8]). Professor Berndt kindly informed me that he also considered such automorphic forms in ([1]) (also see [4], p.82).

Let $f \in MJ(\Gamma,\lambda)$ be a Maass-Jacobi form with eigenvalue λ. Then it is easy to see that the function $\phi_f : G \longrightarrow \mathbb{C}$ defined by

$$\phi_f(g,\alpha) = f((g,\alpha)\circ(i,0)), \quad (g,\alpha) \in G \tag{3.1}$$

satisfies the following conditions $(\mathrm{MJ1})^0$-$(\mathrm{MJ3})^0$:

$(\mathrm{MJ1})^0$ $\phi_f(\gamma x k) = \phi_f(x)$ for all $\gamma \in \Gamma,\ x \in G$ and $k \in K$.

$(\mathrm{MJ2})^0$ phi_f is an eigenfunction of the Laplace-Beltrami operator Δ_0 of $(G,\, ds_0^2)$, where ds_0^2 is a G-invariant Riemannian metric on G induced by $(\mathbb{H}\times\mathbb{C},\, ds^2)$.

$(\mathrm{MJ3})^0$ ϕ_f has a suitable polynomial growth (cf. [5]).

For any right K-invariant function $\phi : G \longrightarrow \mathbb{C}$ on G, we define the function $f_\phi : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ by

$$f_\phi(\tau, z) = \phi(g, \alpha), \quad (\tau, z) \in \mathbb{H} \times \mathbb{C}, \tag{3.2}$$

where (g, α) is an element of G such that $(g, \alpha) \circ (i, 0) = (\tau, z)$. Obviously it is well defined because (3.2) is independent of the choice of $(g, \alpha) \in G$ such that $(g, \alpha) \circ (i, 0) = (\tau, z)$. It is easy to see that if ϕ is a smooth bounded function on G satisfying the conditions $(\mathrm{MJ1})^0$-$(\mathrm{MJ3})^0$, then the function f_ϕ defined by (3.2) is a Maass-Jacobi form.

Now we characterize Maass-Jacobi forms as smooth eigenfunctions on $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ satisfying a certain invariance property.

Proposition 3.2. *Let $f : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ be a nonzero Maass-Jacobi form in $MJ(\Gamma, \lambda)$. Then the function $h_f : S\mathcal{P}_n \times \mathbb{R}^{(m,n)} \longrightarrow \mathbb{C}$ defined by*

$$h_f(Y, V) = f((g,\, V\, {}^t g^{-1}) \circ (i, 0)) \quad \text{for some } g \in SL(2, \mathbb{R}) \text{ with } Y = g\, {}^t g \tag{3.3}$$

satisfies the following conditions:

$(\mathrm{MJ1})^*$ $h_f(\gamma Y\, {}^t\gamma,\, (V{+}\delta)\, {}^t\gamma) = h_f(Y, V)$ *for all* $(\gamma, \delta) \in \Gamma$ *with* $\gamma \in SL(2, \mathbb{Z})$ *and* $\delta \in \mathbb{Z}^{(1,2)}$.

$(\mathrm{MJ2})^*$ *h_f is an eigenfunction of the Laplace-Beltrami operator $\tilde{\Delta}$ on the homogeneous space $(S\mathcal{P}_n \times \mathbb{R}^{(m,n)}, d\tilde{s}^2)$, where $d\tilde{s}^2$ is the G-invariant Riemannian metric on $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ induced from $d\tilde{s}^2$.*

$(\mathrm{MJ3})^*$ *h_f has a suitable polynomial growth.*

Here if (Y, V) is a coordinate of $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ given in Lemma 2.1, then the G-invariant Riemannian metric $d\tilde{s}^2$ and its Laplace-Beltrami operator $\tilde{\Delta}$ on $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ are given by

$$d\tilde{s}^2 = \frac{1}{y^2}(dx^2 + dy^2) \,+\, \frac{1}{y}\left\{(x^2 + y^2)dv_1^2 \,+\, 2x dv_1 dv_2 \,+\, dv_2^2\right\}$$

and

$$\tilde{\Delta} = y^2 \left(\frac{\partial^2}{\partial x^2} \,+\, \frac{\partial^2}{\partial y^2}\right) \,+\, \frac{1}{y}\left\{\frac{\partial^2}{\partial v_1^2} - 2x\frac{\partial^2}{\partial v_1 \partial v_2} + (x^2 + y^2)\frac{\partial^2}{\partial v_2^2}\right\}.$$

Conversely, if h is a smooth bounded function on $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ satisfying the above conditions $(\mathrm{MJ1})^*$-$(\mathrm{MJ3})^*$, then the function $f_h : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f_h(\tau, z) = \, h(g\, {}^t g,\, \alpha\, {}^t g) \tag{3.4}$$

for some $(g, \alpha) \in G$ with $(g, \alpha) \circ (i, 0) = (\tau, z)$ is a Maass-Jacobi form on $\mathbb{H} \times \mathbb{C}$.

Proof. First of all, we note that h_f is well defined because (3.3) is independent

of the choice of g with $Y = g\,{}^tg$. If $(\gamma,\delta) \in \Gamma$ with $\gamma \in \Gamma_1,\ \delta \in \mathbb{Z}^{(1,2)}$ and $(Y,V) \in S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ with $Y = g\,{}^tg$ for some $g \in SL(2,\mathbb{R})$, then the element $g_\gamma := \gamma g$ satisfies $\gamma Y\,{}^t\gamma = \gamma g\,{}^t(\gamma g)$.
Thus according to the definition of h_f, for all $(\gamma,\delta) \in \Gamma$ and $(Y,V) \in S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$, we have

$$\begin{aligned} h_f(\gamma Y\,{}^t\gamma,\, (V+\delta)\,{}^t\gamma) &= f((\gamma g,\, (V+\delta)\,{}^t\gamma\,{}^t(\gamma g)^{-1}) \circ (i,0)) \\ &= f((\gamma g,\, (V+\delta)\,{}^tg^{-1}) \circ (i,0)) \\ &= f(((\gamma,\delta) * (g,\, V\,{}^tg^{-1})) \circ (i,0)) \\ &= f((g,\, V\,{}^tg^{-1}) \circ (i,0)) \quad \text{(because } f \text{ is } \Gamma-\text{invariant)} \\ &= h_f(Y,V). \end{aligned}$$

Therefore this proves the condition (MJ1)*. $d\tilde{s}^2$ and $\tilde{\Delta}$ are obtained from Lemma 2.1 and Proposition 2.3. Hence h_f is an eigenfunction of $\tilde{\Delta}$. Clearly h_f satisfies the condition (MJ3)*.

Conversely we note that f_h is well defined because (3.4) is independent of the choice of $(g,\alpha) \in G$ with $(g,\alpha) \circ (i,0) = (\tau,z)$. If $\tilde{\gamma} = (\gamma,\delta) \in \Gamma$ and $(\tau,z) \in \mathbb{H} \times \mathbb{C}$ with $(g,\alpha) \circ (i,0) = (\tau,z)$, then we have

$$\begin{aligned} f_h(\tilde{\gamma} \circ (\tau,z)) &= f_h(\tilde{\gamma} \circ ((g,\alpha) \circ (i,0))) \\ &= f_h((\tilde{\gamma} * (g,\alpha)) \circ (i,0)) \\ &= f_h((\gamma g,\, \delta\,{}^tg^{-1} + \alpha) \circ (i,0)) \\ &= h((\gamma g)\,{}^t(\gamma g),\, (\delta\,{}^tg^{-1} + \alpha)\,{}^t(\gamma g)) \\ &= h((\gamma(g\,{}^tg)\,{}^t\gamma,\, (\delta + \alpha\,{}^tg)\,{}^t\gamma) \\ &= h(g\,{}^tg,\, \alpha\,{}^tg) \\ &= f_h((g,\alpha) \circ (i,0)) = f_h(\tau,z). \end{aligned}$$

Thus f_h satisfies the condition (MJ1). It is easy to see that f_h satisfies the conditions (MJ2) and (MJ3). □

Definition 3.3. A smooth bounded function on G or $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ is also called a *Maass-Jacobi form* if it satisfies the conditions (MJ1)^0-(MJ3)^0 or (MJ1)*-(MJ3)*.

Remark 3.4. We note that Maass wave forms are special ones of Maass-Jacobi forms. Thus the number of λ's with $MJ(\Gamma,\lambda) \neq 0$ is infinite.

Theorem 3.5. *For any complex number $\lambda \in \mathbb{C}$, the vector space $MJ(\Gamma,\lambda)$ is finite dimensional.*

Proof. The proof follows from [10], Theorem 1, p. 8 and [5], p. 191. □

4. On the group $SL_{2,1}(\mathbb{R})$

For brevity, we set $H = \mathbb{R}^{(1,2)}$. Then we have the split exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow SL(2,\mathbb{R}) \longrightarrow 1.$$

We see that the unitary dual $\hat{H}$ of H is isomorphic to $\mathbb{R}^2$. The unitary character $\chi_{(\lambda,\mu)}$ of H corresponding to $(\lambda,\mu) \in \mathbb{R}^2$ is given by

$$\chi_{(\lambda,\mu)}(x,y) = e^{2\pi i(\lambda x+\mu y)}, \qquad (x,y) \in H.$$

G acts on H by conjugation and hence this action induces the action of G on $\hat{H}$ as follows.

$$G \times \hat{H} \longrightarrow \hat{H}, \quad (g,\chi) \mapsto \chi^g, \quad g \in G,\ \chi \in \hat{H}, \tag{4.1}$$

where the character χ^g is defined by $\chi^g(a) = \chi(gag^{-1}),\ a \in H.$
If $g = (g_0, \alpha) \in G$ with $g_0 \in SL(2,\mathbb{R})$ and $\alpha\ inH$, it is easy to check that for each $(\lambda,\mu) \in \mathbb{R}^2$,

$$\chi^g_{(\lambda,\mu)} = \chi_{(\lambda,\mu)g_0}. \tag{4.2}$$

We see easily from (4.2) that the G-orbits in $\hat{H} \cong \mathbb{R}^2$ consist of two orbits Ω_0, Ω_1 given by

$$\Omega_0 = \{(0,0)\}, \quad \Omega_1 = \mathbb{R}^2 - \{(0,0)\}.$$

We observe that Ω_0 is the G-orbit of $(0,0)$ and Ω_1 is the G-orbit of any element $(\lambda,\mu) \neq 0$.

Now we choose the element $\delta = \chi_{(1,0)}$ of $\hat{H}$. That is, $\delta(x,y) = e^{2\pi i x}$ for all $(x,y) \in \mathbb{R}^2$. It is easy to check that the stabilizer of $\chi_{(0,0)}$ is G and the stabilizer G_δ of δ is given by

$$G_\delta = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \mid c \in \mathbb{R},\ \alpha \in \mathbb{R}^{(1,2)} \right\}.$$

We see that H is regularly embedded. This means that for every G-orbit Ω in $\hat{H}$ and for every $\sigma \in \Omega$ with stabilizer G_σ of σ, the canonical bijection $G_\sigma \backslash G \longrightarrow \Omega$ is a homeomorphism.

According to G. Mackey ([18]), we obtain

Theorem 4.1. *The irreducible unitary representations of G are the following:*

(a) *The irreducible unitary representations π, where the restriction of π to H is trivial and the restriction of π to $SL(2,\mathbb{R})$ is an irreducible unitary representation of $SL(2,\mathbb{R})$. For the unitary dual of $SL(2,\mathbb{R})$, we refer to* [7] *or* [15], p. 123.

(b) *The representations* $\pi_{(r)} = Ind_{G_\delta}^G \sigma_r\ (r \in \mathbb{R})$ *induced from the unitary character* σ_r *of* G_δ *defined by*

$$\sigma_r\left(\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, (\lambda, \mu)\right)\right) = \delta(rc+\lambda) = e^{2\pi i(rc+\lambda)}, \quad c, \lambda, \mu \in \mathbb{R}.$$

Proof. The proof of the above theorem can be found in [22], p. 850. □

We put

$$W_1 = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, (0,0)\right), \quad W_2 = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, (0,0)\right), \quad W_3 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (0,0)\right)$$

and

$$W_4 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1,0)\right), \qquad W_5 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,1)\right).$$

Clearly $W_1, \cdots, W_5$ form a basis of $\mathfrak{g}$.

Lemma 4.2. *We have the following relations.*

$$[W_1, W_2] = W_3, \quad [W_3, W_1] = 2W_1, \quad [W_3, W_2] = -2W_2,$$
$$[W_1, W_4] = 0, \quad [W_1, W_5] = -W_4, \quad [W_2, W_4] = W_5, \quad [W_2, W_5] = 0,$$
$$[W_3, W_4] = W_4, \quad [W_3, W_5] = -W_5, \quad [W_4, W_5] = 0.$$

Proof. The proof follows from an easy computation. □

Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexfication of $\mathfrak{g}$. We put

$$\mathfrak{k}_{\mathbb{C}} = \mathbb{C}\,(W_1 - W_2), \qquad \mathfrak{p}_{\pm} = \mathbb{C}\,(W_3 \pm i(W_1 + W_2)).$$

Then we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_-, \quad [\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\pm}] \subset \mathfrak{p}_{\pm}, \quad \mathfrak{p}_- = \overline{\mathfrak{p}_+}.$$

We note that $\mathfrak{k}_{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}$ of K.

We set $\mathfrak{a} = \mathbb{R}\,W_3$. By Lemma 4.2, the roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ are given by $\pm e$, $\pm 2e$, where e is the linear functional $e : \mathfrak{a} \longrightarrow \mathbb{C}$ defined by $e(W_3) = 1$. The set $\Sigma^+ = \{e,\, 2e\}$ is the set of positive roots of $\mathfrak{g}$ relative to $\mathfrak{a}$. We recall that for a root α, the root space $\mathfrak{g}_\alpha$ is defined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Then we see easily that

$$\mathfrak{g}_e = \mathbb{R}\,W_4, \quad \mathfrak{g}_{-e} = \mathbb{R}\,W_5, \quad \mathfrak{g}_{2e} = \mathbb{R}\,W_1, \quad \mathfrak{g}_{-2e} = \mathbb{R}\,W_2$$

and

$$\mathfrak{g} = \mathfrak{g}_{-2e} \oplus \mathfrak{g}_{-e} \oplus \mathfrak{a} \oplus \mathfrak{g}_{e} \oplus \mathfrak{g}_{2e}.$$

Proposition 4.3. *The Killing form B of $\mathfrak{g}$ is given by*

$$B((X_1, Z_1), (X_2, Z_2)) = 5\,\sigma(X_1 X_2), \tag{4.3}$$

where $(X_1, Z_1), (X_2, Z_2) \in \mathfrak{g}$ with $X_1, X_2 \in \mathfrak{sl}(2, \mathbb{R})$ and $Z_1, Z_2 \in \mathbb{R}^{(1,2)}$. Hence the Killing form is highly nondegenerate. The adjoint representation Ad of G is given by

$$Ad((g, \alpha))(X, Z) = (gXg^{-1},\, (Z - \alpha\, {}^tX)\, {}^tg), \tag{4.4}$$

where $(g, \alpha) \in G$ with $g \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^{(1,2)}$ and $(X, Z) \in \mathfrak{g}$ with $X \in \mathfrak{sl}(2, \mathbb{R})$, $Z \in \mathbb{R}^{(1,2)}$.

Proof. The proof follows immediately from a direct computation. □

An Iwasawa decomposition of the group G is given by

$$G = NAK, \tag{4.5}$$

where

$$N = \left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, a \right) \in G \,\middle|\, x \in \mathbb{R},\ a \in \mathbb{R}^{(1,2)} \right\}$$

and

$$A = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 0 \right) \in G \,\middle|\, a > 0 \right\}.$$

An Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k},$$

where

$$\mathfrak{n} = \left\{ \left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, Z \right) \in \mathfrak{g} \,\middle|\, x \in \mathbb{R},\ Z \in \mathbb{R}^{(1,2)} \right\}$$

and

$$\mathfrak{a} = \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, 0 \right) \in \mathfrak{g} \,\middle|\, x \in \mathbb{R} \right\}.$$

In fact, $\mathfrak{a}$ is the Lie algebra of A and $\mathfrak{n}$ is the Lie algebra of N.

Now we compute the Lie derivatives for functions on G explicitly. We define the differential operators $L_k,\, R_k\ (1 \leq k \leq 5)$ on G by

$$L_k f(\tilde{g}) = \frac{d}{dt}\bigg|_{t=0} f(\tilde{g} * \exp tW_k)$$

and

$$R_k f(\tilde{g}) = \frac{d}{dt}\bigg|_{t=0} f(\exp tW_k * \tilde{g}),$$

where $f \in C^\infty(G)$ and $\tilde{g} \in G$.

By an easy calculation, we get

$$\exp tW_1 = \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0, 0) \right), \quad \exp tW_2 = \left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, (0, 0) \right)$$

$$\exp tW_3 = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, (0, 0) \right), \quad \exp tW_4 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (t, 0) \right)$$

and

$$\exp tW_5 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0, t) \right).$$

Now we use the following coordinates (g, α) in G given by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4.6}$$

and

$$\alpha = (\alpha_1, \alpha_2), \tag{4.7}$$

where $x, \alpha_1, \alpha_2 \in \mathbb{R},\ y > 0$ and $0 \le \theta < 2\pi$. By an easy computation, we have

$$\begin{aligned}
L_1 &= y\cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} + \sin^2\theta \frac{\partial}{\partial \theta} - \alpha_2 \frac{\partial}{\partial \alpha_1}, \\
L_2 &= y\cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} - \cos^2\theta \frac{\partial}{\partial \theta} - \alpha_1 \frac{\partial}{\partial \alpha_2}, \\
L_3 &= -2y\sin 2\theta \frac{\partial}{\partial x} + 2y \cos 2\theta \frac{\partial}{\partial y} + \sin 2\theta \frac{\partial}{\partial \theta} - \alpha_1 \frac{\partial}{\partial \alpha_1} + \alpha_2 \frac{\partial}{\partial \alpha_2}, \\
L_4 &= \frac{\partial}{\partial \alpha_1}, \\
L_5 &= \frac{\partial}{\partial \alpha_2}, \\
R_1 &= \frac{\partial}{\partial x}, \\
R_2 &= (y^2 - x^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} - y \frac{\partial}{\partial \theta}, \\
R_3 &= 2x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, \\
R_4 &= y^{-1/2} \cos\theta \frac{\partial}{\partial \alpha_1} + y^{-1/2} \sin\theta \frac{\partial}{\partial \alpha_2}, \\
R_5 &= -y^{-1/2} (x\cos\theta + y \sin\theta) \frac{\partial}{\partial \alpha_1} + y^{-1/2} (y\cos\theta - x\sin\theta) \frac{\partial}{\partial \alpha_2}.
\end{aligned}$$

In fact, the calculation for L_3 and R_5 can be found in [22], p. 837-839.

We define the differential operators $\mathbb{L}_j\ (\,1 \leq j \leq 5\,)$ on $\mathbb{H} \times \mathbb{C}$ by

$$\mathbb{L}_j f(\tau,z) = \frac{d}{dt}\bigg|_{t=0} f(\exp tW_j \circ (\tau,z)), \quad 1 \leq j \leq 5,$$

where $f \in C^\infty(\mathbb{H} \times \mathbb{C})$. Using the coordinates $\tau = x + iy$ and $z = u + iv$ with x, y, u, v real and $y > 0$, we can easily compute the explicit formulas for $\mathbb{L}_j$'s. They are given by

$$\begin{aligned}
\mathbb{L}_1 &= (x^2 - y^2)\,\frac{\partial}{\partial x} + 2xy\,\frac{\partial}{\partial y} + (xu - yv)\,\frac{\partial}{\partial u} + (yu + xv)\,\frac{\partial}{\partial v},\\
\mathbb{L}_2 &= -\,\frac{\partial}{\partial x},\\
\mathbb{L}_3 &= -2x\,\frac{\partial}{\partial x} - 2y\,\frac{\partial}{\partial y} - u\,\frac{\partial}{\partial u} - v\,\frac{\partial}{\partial v},\\
\mathbb{L}_4 &= x\,\frac{\partial}{\partial u} + y\frac{\partial}{\partial v},\\
\mathbb{L}_5 &= \frac{\partial}{\partial u}.
\end{aligned}$$

5. The decomposition of $L^2(\Gamma\backslash G)$

Let R be the right regular representation of G on the Hilbert space $L^2(\Gamma\backslash G)$. We set $G_1 = SL(2,\mathbb{R})$. Then the decomposition of R is given by

$$L^2(\Gamma\backslash G) = L^2_{\rm disc}(\Gamma_1\backslash G_1) \bigoplus L^2_{\rm cont}(\Gamma_1\backslash G_1) \bigoplus \int_{-\infty}^{\infty} \mathcal{H}_{(r)} dr, \tag{5.1}$$

where $L^2_{\rm disc}(\Gamma_1\backslash G_1)$ (resp. $L^2_{\rm cont}(\Gamma_1\backslash G_1)$) is the discrete (resp. continuous) part of $L^2(\Gamma_1\backslash G_1)$ (cf. [14], [15]) and $\mathcal{H}_{(r)}$ is the representation space of $\pi_{(r)}$ (cf. Theorem 4.1. (b)).

We recall the result of Rolf Berndt (cf. [2], [3], [4]). Let $H^{(1,1)}_{\mathbb{R}}$ denote the Heisenberg group which is $\mathbb{R}^3$ as a set and is equipped with the following multiplication

$$(\lambda,\mu,\kappa)\,(\lambda',\mu',\kappa') = (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\mu'-\mu\lambda').$$

We let $G^J = SL(2,\mathbb{R}) \ltimes H^{(1,1)}_{\mathbb{R}}$ be the semidirect product of $SL(2,\mathbb{R})$ and $H^{(1,1)}_{\mathbb{R}}$, called the Jacobi group whose multiplication law is given by

$$(M,(\lambda,\mu,\kappa))\cdot(M',(\lambda',\mu',\kappa')) = (MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu',\kappa+\kappa'+\tilde{\lambda}\mu'-\tilde{\mu}\lambda'))$$

with $M, M' \in SL(2,\mathbb{R}), (\lambda,\mu,\kappa),\, (\lambda',\mu',\kappa') \in H_{\mathbb{R}}^{(1,1)}$ and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)M'$. Obviously the center $Z(G^J)$ of G^J is given by $\{(0,0,\kappa)\,|\,\kappa\in\mathbb{R}\,\}$. We denote

$$H_{\mathbb{Z}}^{(1,1)} = \{\,(\lambda,\mu,\kappa)\in H_{\mathbb{R}}^{(1,1)}\ |\ \lambda,\mu,\kappa \text{ integral}\,\}.$$

We set

$$\Gamma^J = SL(2,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(1,1)}, \quad K^J = K\, \times Z(G^J).$$

R. Berndt proved that the decomposition of the right regular representation R^J of G^J in $L^2(\Gamma^J\backslash G^J)$ is given by

$$L^2(\Gamma^J\backslash G^J) = \left(\bigoplus_{m,n\in\mathbb{Z}} \mathcal{H}_{m,n} \right) \bigoplus \left(\bigoplus_{\nu=\pm\frac{1}{2}} \int_{\substack{\mathrm{Re}\, s=0\\ \mathrm{Im}\, s>0}} \mathcal{H}_{m,s,\nu}\, ds \right), \tag{5.2}$$

where the $\mathcal{H}_{m,n}$ is the irreducible unitary representation isomorphic to the discrete series $\pi_{m,k}^{\pm}$ or the principal series $\pi_{m,s,\nu}$, and the $\mathcal{H}_{m,s,\nu}$ is the representation space of $\pi_{m,s,\nu}$ (cf. [4], p. 47-48). For more detail on the decomposition of $L^2(\Gamma^J\backslash G^J)$, we refer to [4], p. 75-103.

Since $\mathbb{H}\times\mathbb{C} = K^J\backslash G^J = K\backslash G$, the space of the Hilbert space $L^2(\Gamma\backslash(\mathbb{H}\times\mathbb{C}))$ consists of K^J-fixed elements in $L^2(\Gamma^J\backslash G^J)$ or K-fixed elements in $L^2(\Gamma\backslash G)$. Hence we obtain the spectral decomposition of $L^2(\Gamma\backslash(\mathbb{H}\times\mathbb{C}))$ for the Laplacian Δ or $\Delta_{\alpha,\beta}$ (cf. Proposition 2.4 or Remark 2.5).

6. Remarks on Fourier expansions of Maass-Jacobi forms

We let $f:\mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$ be a Maass-Jacobi form with $\Delta f = \lambda f$. Then f satisfies the following invariance relations

$$f(\tau+n,\,z) \,=\, f(\tau,z) \quad \text{for all } n\in\mathbb{Z} \tag{6.1}$$

and

$$f(\tau,\,z\,+\,n_1\tau\,+\,n_2) \,=\, f(\tau,z) \quad \text{for all } n_1,\,n_2\in\mathbb{Z}. \tag{6.2}$$

Therefore f is a smooth function on $\mathbb{H}\times\mathbb{C}$ which is periodic in x and u with period 1. So f has the following Fourier series

$$f(\tau,z) \,=\, \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} c_{n,r}(y,v)\, e^{2\pi i(nx+ru)}. \tag{6.3}$$

For two fixed integers n and r, we have to calculate the function $c_{n,r}(y,v)$. For brevity, we put $F(y,v) = c_{n,r}(y,v)$. Then F satisfies the following differential equation

$$\left[y^2\,\frac{\partial^2}{\partial y^2} \,+\, (y+v^2)\,\frac{\partial^2}{\partial v^2} \,+\, 2yv\,\frac{\partial^2}{\partial y\partial v} \,-\, \left\{\,(ay+bv)^2 \,+\, b^2 y \,+\, \lambda\,\right\}\right] F \,=\, 0. \tag{6.4}$$

Here $a = 2\pi n$ and $b = 2\pi r$ are constant. We note that the function $u(y) = y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|y)$ satisfies the differential equation (6.4) with $\lambda = s(s-1)$. Here $K_s(z)$ is the K-Bessel function defined by (2.16) (see Lebedev [16] or Watson [21]). The problem is that if there exist solutions of the differential equation (6.4), we have to find their solutions explicitly.

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