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Steinhaus Graphs with Minimum Degree Two

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Abstract. In this paper, we classify the Steinhaus graphs with minimum degree two.

1. Introduction

Let $T = a_{12}a_{13}\cdots a_{1n}$ be an $(n-1)$-long string of zeros and ones. The *Steinhaus graph* G, generated by T has as its adjacency matrix, the *Steinhaus matrix*, $A(G) = [a_{ij}]$ which is obtained from the following, called the *Steinhaus property*:

$$a_{ij} = \begin{cases} 0 & \text{if } 1 \le i = j \le n; \\ a_{i-1,j-1} + a_{i-1,j} \pmod 2 & \text{if } 1 < i < j \le n; \\ a_{ji} & \text{if } 1 \le j < i \le n. \end{cases}$$

In this case, T is called the *generating string* of G. A *Steinhaus triangle* is the upper-triangular part of a Steinhaus matrix (excluding the diagonal) and hence, is generated by the first row (which is the generating string) in the triangle. It is obvious that there are exactly 2^{n-1} Steinhaus graphs of order n. The vertices of a Steinhaus graph are usually labelled by their corresponding row numbers. In Figure 1, the Steinhaus graph generated by 0110110 is pictured.

Let G be a Steinhaus graph of order n generated by $T = a_{12}a_{13}\cdots a_{1n}$. The *partner* of G, $P(G)$, is the Steinhaus graph generated by the reverse of the last column of the adjacency matrix of G, i.e., $a_{n-1,n}a_{n-2,n}\cdots a_{1n}$ is the generating string of $P(G)$. Note that a Steinhaus graph G is isomorphic to its partner $P(G)$. For further results for Steinhaus graphs (See [2], [3], [4] and [6]).

We often prefer to think the sequence of zeros and ones that generates a Steinhaus graph as a number. Since the sequence 0110110 generates the graph in Figure 1, we say that this graph is generated by $k = 54 = (110110)_2$. Hence the graph with n vertices generated by k will be denoted $H_{n,k}$. In Figure 1, the Steinhaus graph is denoted by $H_{8,54}$.

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	1	2	3	4	5	6	7	8
1	0	0	1	1	0	1	1	0
2	0	0	1	0	1	1	0	1
3	1	1	0	1	1	0	1	1
4	1	0	1	0	0	1	1	0
5	0	1	1	0	0	1	0	1
6	1	1	0	1	1	0	1	1
7	1	0	1	1	0	1	0	0
8	0	1	1	0	1	1	0	0

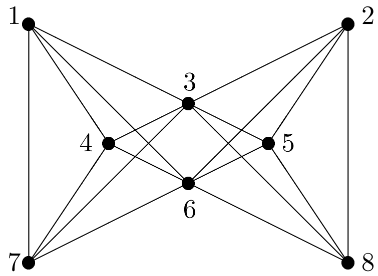

Figure 1. Steinhaus graph with the generating string 0110110

We now give some basic graph theoretical definitions. Let G be a graph. A *cut vertex* in G is a vertex whose deletion increases the number of components. Similarly, an edge in G is called a *bridge* if its deletion increases the number of components. A *pendent vertex* in G is a vertex of degree one. Let G be a connected graph. Let W be a set of vertices or a set of edges. If $G - W$ is disconnected, then we say that W *separates* G. We say that G is *k-connected* ($k \geq 2$) if no set of $k-1$ or fewer vertices separates it. Similarly, G is *k-edge-connected* ($k \geq 2$) if no set of at most $k-1$ edges separates it. As is usual, $|X|$ is the cardinality of a set X. We denote $log_2(x)$ by $lg(x)$. Any definitions not given here can be found in [1].

Now, we give some results relating to connectivity of Steinhaus graphs.

Theorem 1.1 ([5], [6]). *Let $n > 5$ and let G be a nonempty Steinhaus graph of order n. Then the following statements are equivalent.*

(1) *G is 2-connected.*

(2) *G is 2-edge-connected.*

(3) *G has no pendent vertices.*

Theorem 1.2 ([6]). *A Steinhaus graph G is 3-edge-connected if and only if its minimum degree $\delta(G)$ is larger than two.*

Theorem 1.3 ([6]). *G is a Steinhaus graph with $\delta(G) \geq 3$ if and only if G is 3-connected unless G is one of the followings:*
D_{2m}, for $m \geq 5$;
E_{2m}, for $m \geq 3$;
$H_{6,27} = P(H_{6,13}); H_{7,27} = P(H_{7,45}); H_{8,19} = P(H_{8,37})$; $H_{8,55} = P(H_{8,73})$; $H_{8,54}$ and $H_{9,37} = P(H_{9,147})$,

where D_n *is the Steinhaus graph generated by the* $(n-1)$*-long sequence*

$$0\overbrace{\underline{10}\ \ \underline{10}\cdots\underline{10}}^{(m-2)\ \ times}00$$

when n *is even and* E_n *is the Steinhaus graph generated by the* $(n-1)$*-long sequence* $1\overbrace{\underline{01}\ \ \underline{01}\cdots\underline{01}}^{(m-2)\ \ times}10$ *when* n *is even.*

Note that the all graphs in Theorem 1.3 are 3-edge-connected.

Theorem 1.4 ([5]). *Let* $n > 5$ *and let* $p(n)$ *be the number of Steinhaus graphs of order* n *having a pendent vertex. Then*

$$p(n) = 2\sum_{i=1}^{n-1}\delta_i - \sum_{j=2}^{\lfloor\frac{n+2}{2}\rfloor}\epsilon_j,$$

where $\delta_i = min\{2^m, n-i\}$ *for the nonnegative integer* m *such that* $2^{m-1} < i \leq 2^m$ *and where*

$$\epsilon_j = \begin{cases} 1 & \text{if } 2^{\lceil lg(j-1)\rceil} \text{ divides } n-j+1; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the number of 2-connected and 2-edge-connected Steinhaus graphs is equal to $2^{n-1} - p(n) - 1$.

2. Minimum degree two in Steinhaus graphs

In this section, we give an equivalent expressions for Steinhaus graphs of minimum degree two. It will be useful to denote by $G_n(k; i, j)$ the Steinhaus graph of order n generated by the string $a_{ki} = 1 = a_{kj}$ and $a_{kl} = 0$ for all l except for $i, j (i < j)$. Thus the degree of vertex k is two.

We denote S_n to be the collection of all Steinhaus graphs of order n. We set

$$A \equiv \{G_n(k; i, j) | i < j < k,\ \ i < k < j\ \ \text{or}\ \ k < i < j\}.$$

Then

$$\{G \in S_n | \delta(G) = 2\} = \{G \in A | \delta(G) = 2\}.$$

R_1^8	→	1	0	0	0	0	0	0	0
R_2^8	→	1	1	0	0	0	0	0	0
R_3^8	→	1	0	1	0	0	0	0	0
R_4^8	→	1	1	1	1	0	0	0	0
R_5^8	→	1	0	0	0	1	0	0	0
R_6^8	→	1	1	0	0	1	1	0	0
R_7^8	→	1	0	1	0	1	0	1	0
R_8^8	→	1	1	1	1	1	1	1	1

Figure 2. Pascal's square of length 8

Set $A_1 = \{G_n(k;i,j) | i < j < k\}$, $A_2 = \{G_n(k;i,j) | i < k < j\}$ and $A_3 = \{G_n(k;i,j) | k < i < j\}$. Then $A = A_1 \cup A_2 \cup A_3$. So,

$$\{G \in S_n | \delta(G) = 2\} = \bigcup_{i=1}^{3} \{G \in A_i | \delta(G) = 2\}.$$

We now present some facts concerning Pascal's triangle modulo two. The rows of the triangle are labelled $R_1, R_2, \cdots$, and so the rth element of R_p is $\binom{p-1}{r-1}$ (mod 2). If Q is a string of zeros and ones, then Q^s is the string Q concatenated with itself $s-1$ times. For example, if $Q = 01$, then $Q^4 = 01010101$. Similarly, if Q is a matrix, then Q^s is the string Q concatenated with itself $s-1$ times. Observe that $R_{2^m} = 1^{2^m}$ because $\binom{2^m-1}{r}$ is odd for $0 \le r \le 2^m - 1$. Let

$$R_t^p = R_t 0^{p-t}.$$

Then *Pascal's square* of length p consists of p rows $R_1^p, R_2^p, \cdots, R_p^p$ (see Figure 2).

From now, we give expressions relating to the parameters n, k, i, j which is equivalent to minimum degree two in Steinhaus graphs.

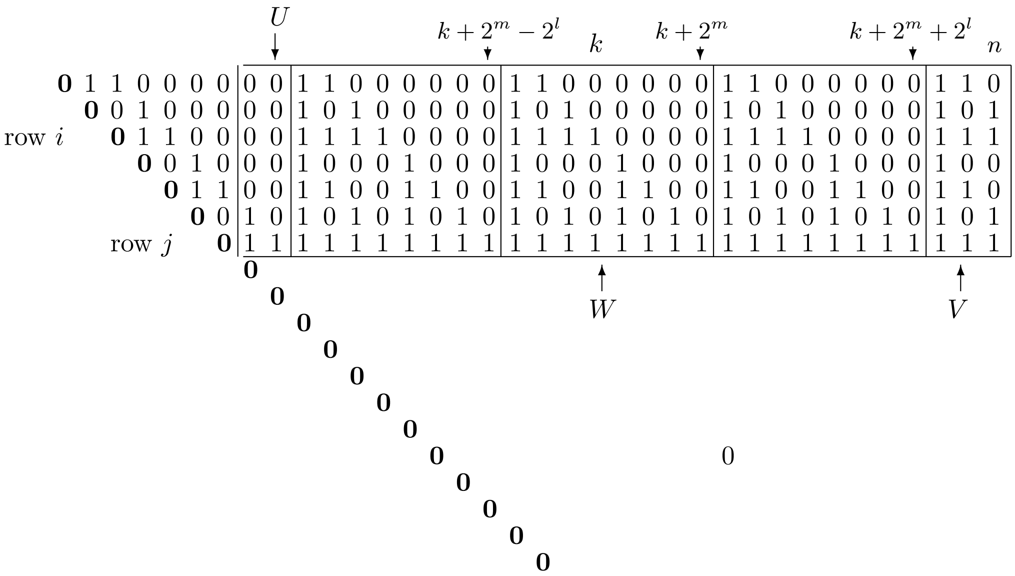

Figure 3.

Lemma 2.1. *Let $G = G_n(k; i, j) \in A_1$ with $j - i = 2^m$. Then G has a pendent vertex if and only if G satisfies one of the followings:*

(a) $n - k \geq 2^m$.

(b) $n - k \leq 2^m$ *and* $j = d2^q$ *for some* d, *where* $q = \lceil lg(k - i) \rceil$.

(c) $k = n$, $i \leq 2^m$ *and* $n - j + 1 \leq 2^m$.

(d) $j < k + 2^m - 2^l$, *where* $l = \lceil lg(j) \rceil$.

Proof. Let v be a pendent vertex in G. Let $l = \lceil lg(j) \rceil$ and so $2^{l-1} < j \leq 2^l$. Since k is of degree two, v is not equal to k. We put $B = (a_{pq})$ for $p = 1, 2, \cdots, j$ and $q = j + 1, j + 2, \cdots, n$. Then from figure 3, it is not difficult to see that B is the form UW^sV, where W is Pascal's square of length 2^l from row $2^l - j + 1$ to row 2^l, V is a prefix of W and U is a suffix of W. In Figure 3, it is illustrated for $s = 3$, $l = 3$, the rectangle W consists of 7 rows $R_2^8, R_3^8, \cdots, R_8^8$, U is identical to the last 2 columns of W and V is identical to the first 3 columns of W. By the Steinhaus property,

$$a_{r,k-i+r} = 1 \quad \text{for } r = 1, 2, \cdots, i. \tag{1}$$

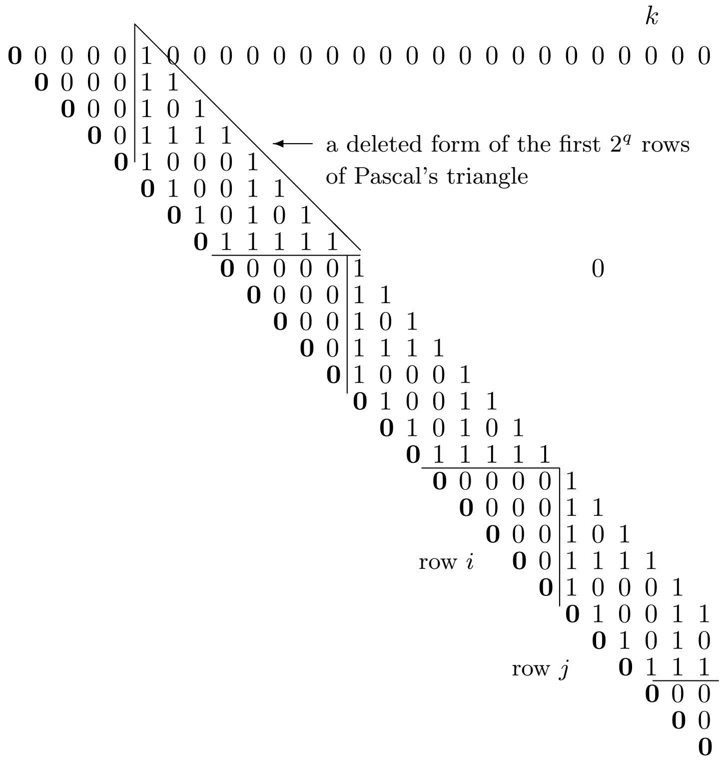

Figure 4.

Case (i): $v = 1$.
By (1), vertex v is adjacent to the vertex $k - i + 1$. Also, $a_{1,k+2^m+1} = 1$. Since v is a pendent vertex, $k + 2^m + 1 > n$. So $n - k \leq 2^m$. Let $q = \lceil lg(k - i) \rceil$ and so $2^{q-1} < k - i \leq 2^q$. As depicted in Figure 4, the Steinhaus matrix $A(G)$ mainly consists of copies of a deleted form of the first 2^q rows of Pascal's triangle. So, it is easy to see that $j = d2^q$ for some d.

Case (ii): $1 < v \leq i$.
By (1), v is adjacent to $k - i + v$. Since v is a pendent vertex, $a_{v-1,k-i+v} = 1$. This gives a contradiction because $a_{v-1,k-i+v} = 0$.

Case (iii): $v = i + 1$.
Since $a_{i+1,k-2^m+1} = 1$, v is adjacent to the vertex $k - 2^m + 1$. If $k \neq n$, then $a_{i+1,k+1} = 1$. This gives a contradiction because $i + 1$ is a pendent vertex in G. So $k = n$. If $i > 2^m$ or $n - j + 1 > 2^m$, then it is easy to see that the degree of k is at least three. This gives a contradiction because the degree of k is two. So, $k = n$, $i \leq 2^m$, and $n - j + 1 \leq 2^m$. This is illustrated in Figure 5.

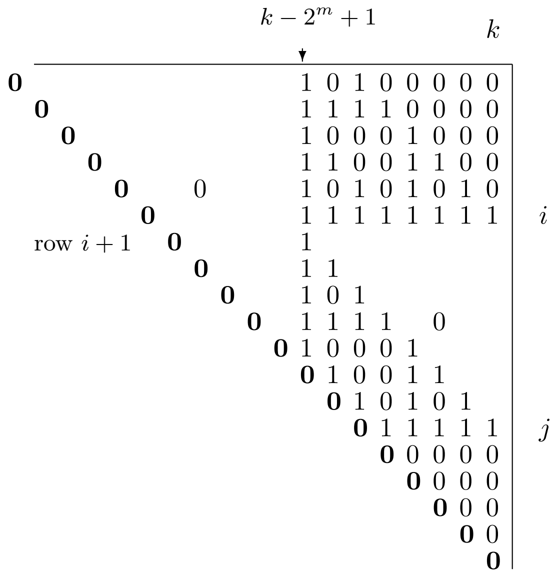

Figure 5.

Case (iv): $i+1 < v \le j$.
An analogous argument of Case (ii) leads us to a contradiction.

Case (v): $j < v < k$.
In this case, $v = k + 2^m - d2^l$ for some positive integer d. So, $j < k + 2^m - 2^l$.

Case (vi): $v > k$.
In this case, $v = k+2^m+d2^l$ for some integer d, and so $k+2^m \le n$. Hence $n-k \ge 2^m$

Conversely, if $n - k \ge 2^m$, G has a pendent vertex $k + 2^m$ (see Figure 3). Assume that $j = d2^q$ for some d. Then G has a pendent vertex 1 (see Figure 4). Assume that $k = n$, $i \le 2^m$ and $n - j + 1 \le 2^m$. Then G has a pendent vertex $i+1$ (see Figure 5). Assume that $j < k + 2^m - 2^l$, where $l = \lceil lg(j) \rceil$. Then by the Steinhaus property, G has a pendent vertex $k + 2^m - 2^l$. Hence the proof of lemma is completed. □

Lemma 2.2. *Let $G = G_n(k; i, j) \in A_1$ with $j - i \ne 2^m$. Then G has a pendent vertex if and only if G satisfies the followings:*

$$k = n \quad \text{and} \quad j - i = c2^{l_1} = d2^{l_2}$$

for some c and d, where $l_1 = \lceil lg(i) \rceil$ and $l_2 = \lceil lg(n-j) \rceil$.

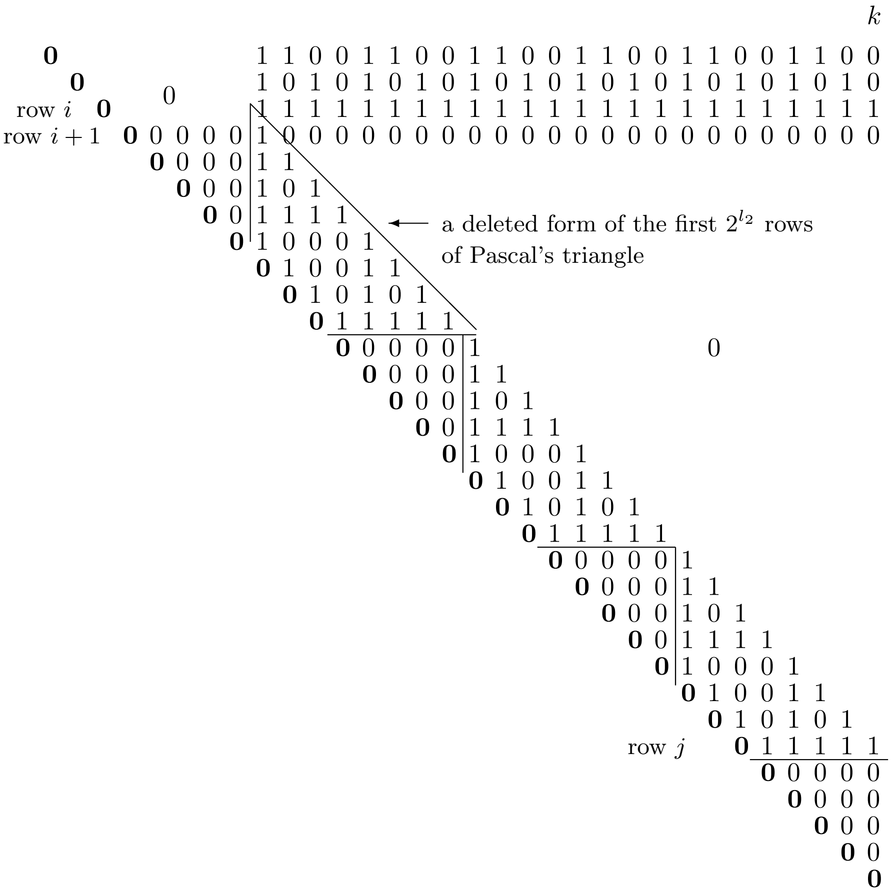

Figure 6.

Proof. Let v be a pendent vertex in G. If $2 \leq v \leq i$ or $i+2 \leq v \leq j$, then an analogous argument to Lemma 2.1 leads us to a contradiction. Also, it is not difficult to see that if $v > j$, then the degree of v is at least two. So $v = 1$ or $v = i+1$. Assume that v is the vertex 1. Then its adjacency matrix consists of a big Pascal's triangle. But in Pascal's triangle, it is impossible that $j - i \neq 2^m$. Assume that v is the vertex $i+1$. Since $a_{i+1,k-(j-i)+1} = 1$, v is adjacent to the vertex $k-(j-i)+1$. If $k \neq n$, then $a_{i+1,k+1} = 1$. This gives a contradiction because $i+1$ is a pendent vertex in G. So $k = n$. Also, it is easy to see that $j - i = c2^{l_1} = d2^{l_2}$ for some c and d, where $l_1 = \lceil lg(i) \rceil$ and $l_2 = \lceil lg(n-j) \rceil$. This is illustrated in Figure 6.

Conversely, if G has the case $k = n$ and $j - i = c2^{l_1} = d2^{l_2}$, then $i+1$ is a pendent vertex in G. Hence the proof of lemma is completed. □

By combining the above Lemmas, we prove the following theorem.

Lemma 2.3. *Let $G = G_n(k; i, j) \in A_1$. Then $\delta(G) = 2$ if and only if G satisfies one of the followings:*

(1) $j - i = 2^m$.
In this case, $n - k < 2^m$ and it satisfies the following conditions:
(a) $j \neq d2^q$ for any d, where $q = \lceil lg(k - i) \rceil$.
(b) $k \neq n$, $i > 2^m$ or $n - j + 1 > 2^m$.
(c) $j \geq k + 2^m - 2^l$, where $l = \lceil lg(j) \rceil$.

(2) $j - i \neq 2^m$.
$k \neq n$, $j - i \neq c2^{l_1}$ or $j - i \neq d2^{l_2}$ for any c and d, where $l_1 = \lceil lg(i) \rceil$ and $l_2 = \lceil lg(n - j) \rceil$.

Note that if $G \in A_3$ and $\delta(G) = 2$, then $P(G) \in A_1$ and $\delta(P(G)) = 2$. So by Theorem 2.3, we get to the following theorem.

Lemma 2.4. *Let $G \in A_3$. Then $\delta(G) = 2$ if and only if $P(G) = G_n(k; i, j)$ satisfies at least one of the following conditions:*

Case 1. $j - i = 2^m$.
In this case, $n - k < 2^m$ and it satisfies the following conditions:
(a) $j \neq d2^q$ for any d, where $q = \lceil lg(k - i) \rceil$.
(b) $k \neq n$, $i > 2^m$ or $n - j + 1 > 2^m$.
(c) $j \geq k + 2^m - 2^l$, where $l = \lceil lg(j) \rceil$.

Case 2. $j - i \neq 2^m$.
$k \neq n$ or $j - i \neq c2^{l_1}$ or $j - i \neq d2^{l_2}$ for any c and d, where $l_1 = \lceil lg(i) \rceil$ and $l_2 = \lceil lg(n - j) \rceil$.

Lemma 2.5. *Let $G \in A_2 = G_n(k; i, j)$. Then G has a pendent vertex if and only if G satisfies one of the followings:*

(a) *$k - i = 1$ and $j - k = 1$.*

(b) *$k - i = 1$ and $j - k > 1$.*
$n - j + 1 = d2^{l_1}$ and $j - k - 1 = s2^{l_2}$ for some d and s, where $l_1 = \lceil lg(j - k) \rceil$ and $l_2 = \lceil lg(i) \rceil$.

(c) *$k - i > 1$ and $j - k = 1$.*
$i = d2^{l_1}$ and $k - i - 1 = s2^{l_2}$ for some d and s, where $l_1 = \lceil lg(k - i) \rceil$ and $l_2 = \lceil lg(n - j + 1) \rceil$.

Proof. Let v be a pendent vertex in G.

Suppose that $k - i > 1$ and $j - k > 1$. Then $a_{i-r,k-r} = a_{i-r,k+1} = 1$ for $r = 0, \cdots, i - 1$ and $a_{k+r,j+r} = a_{k-1,j+r} = 1$ for all $r = 0, \cdots, n - j$. If $v \leq i$ or $v \geq j$, then v is not a pendent vertex. Now, since $a_{v,i} = a_{v,j} = 1$ for $i < v < j$, v

is at least degree two. Thus in this case, any vertex of G is not a pendent vertex. This gives a contradiction. So G satisfies one of the following three cases.

Case (i): $k-i=1, j-k=1$.
In this case, it is easily seen to imply that G is the $1-n$ path.

Case (ii): $k-i=1, j-k>1$.
Let $l_1=\lceil lg(j-k)\rceil$ and $l_2=\lceil lg(i)\rceil$.

If $v\neq n$, then by Steinhaus property, v is not a pendent vertex. So, the vertex v is equal to n. In this case, G satisfies the condition $n-j+1=d2^{l_1}$, and $j-k-1=s2^{l_2}$ for some d and s. This is illustrated in Figure 7.

Case (iii) $k-i>1, j-k=1$.
We consider the partner of G. Then $P(G)$ has the same case to Case (ii). So we obtain the desired result.

Conversely, if $k-i=1, j-k=1$, the verties 1 and n are pendent verties. Assumer that $k-i=1, j-k>1$. If $n-j+1=d2^{l_1}$ and $j-k-1=s2^{l_2}$ for some d and s, then n is pendent vertex (see Figure 3). By considering the partner of G, we prove the case (c). In this case, 1 is pendent vertex in G. Hence the proof of lemma is completed. □

From Lemma 2.5, we get to the following theorem.

Lemma 2.6. *Let $G\in A_2=G_n(k;i,j)$. Then $\delta(G)=2$ if and only if G satisfies one of the following cases:*

(1) $k-i>1$ *and* $j-k>1$.

(2) $k-i=1$ *and* $j-k>1$.
$n-j+1\neq d2^{l_1}$, *or* $j-k-1\neq s2^{l_2}$ *for any* d *and* s, *where* $l_1=\lceil lg(j-k)\rceil$ *and* $l_2=\lceil lg(i)\rceil$.

(3) $k-i>1$ *and* $j-k=1$.
$i\neq d2^{l_1}$ *or* $k-i-1\neq s2^{l_2}$ *for any* d *and* s, *where* $l_1=\lceil lg(k-i)\rceil$ *and* $l_2=\lceil lg(n-j+1)\rceil$.

From previous facts, we see that the number of 3-edge-connected Steinhaus graphs is

$$2^{n-1}-(p(n)+b(n)+1),$$

where $b(n)=|\bigcup_{i=1}^{3}\{G\in A_i|\delta(G)=2\}|$. So, in order to see the number 3-edge-connected Steinhaus graphs, we need to count the number $b(n)$ of all Steinhaus graphs with $\delta(G)=2$.

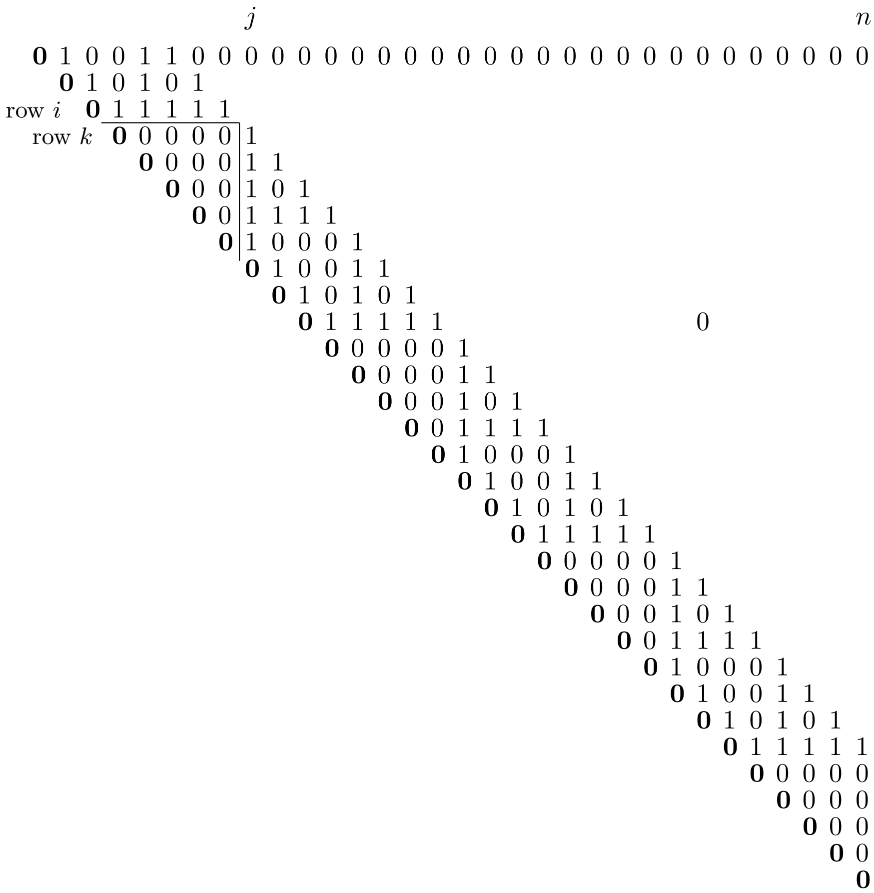

Figure 7.

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