Strong Reducedness and Strong Regularity for Near-rings

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Abstract. A near-ring $N$ is called strongly reduced if, for $a \in N$, $a^2 \in Nc$ implies $a \in Nc$, where $Nc$ denotes the constant part of $N$. We investigate some properties of strongly reduced near-rings and apply those to the study of left strongly regular near-rings. Finally we classify all reduced, and strongly reduced near-rings of order $\leq 7$.

1. Introduction

A right near-ring $N$ is called left strongly regular if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$. Mason ([3]) introduced this notion and characterized left strongly regular zero-symmetric near-rings. Several authors ([2], [4], [5], [7] etc.) studied them. In particular, Reddy and Murty ([7]) extended some results in [3] to the non-zero symmetric case. They observed that every left strongly regular near-ring has some interesting property. In this paper we consider this property. Let $N$ be a right near-ring and let $Nc$ denote the constant part of $N$. We define $N$ to be strongly reduced if, for $a \in N$, $a^2 \in Nc$ implies $a \in Nc$. Obviously a strongly reduced near-ring $N$ is reduced. We show that strong reducedness is a more general concept than the property (*) in Reddy and Murty ([7]). Left or right strongly regular near-rings form one of the important class of strongly reduced near-rings. Using strong reducibility, we characterize left strongly regular near-rings and $(P_3)$-near-rings. Finally we classify all reduced, and strongly reduced near-rings of order $\leq 7$ using the description given by Clay ([1]).

2. Results

Throughout this paper we work with right near-rings. For notation and basic results, we shall refer to Pilz ([6]). Recall that a near-ring $N$ is reduced if, for $a \in N$, $a^2 = 0$ implies $a = 0$. For a near-ring $N$, $Nc$ denotes the constant part of

Received November 8, 2002.
2000 Mathematics Subject Classification: 16Y30.
Key words and phrases: reduced, strongly reduced, left strongly regular, strongly regular.
\[N\], that is, \(N_c = \{x \in N \mid x = xa\}\). A near-ring \(N\) is said to be strongly reduced if, for \(a \in N\), \(a^2 \in N_c\) implies \(a \in N_c\). Obviously \(N\) is strongly reduced if and only if, for \(a \in N\) and any positive integer \(n\), \(a^n \in N_c\) implies \(a \in N_c\). We will show that a strongly reduced near-ring is reduced, that is, for \(a \in N\), \(a^2 = 0\) implies \(a = 0\). A near-ring \(N\) is said to be left strongly regular if, for each \(a \in N\), there exists \(x \in N\) such that \(a = xax^2\). Right strong regularity is defined in a symmetric way.

A subnear-ring \(H\) of a near-ring \(N\) is called invariant if \(NH \subseteq H\) and \(HN \subseteq H\). For a subset \(S\) of \(N\), \(\langle S\rangle\) stands for the invariant subnear-ring of \(N\) generated by \(S\). We give some sufficient conditions for a near-ring to be strongly reduced.

**Proposition 1.**

(1) Let \(N\) be a near-ring. If \(a \in \langle a^2 \rangle\) for each \(a \in N\), then \(N\) is strongly reduced. In particular, right or left strongly regular near-rings are strongly reduced.

(2) Every integral near-ring \(N\) is strongly reduced. Hence a subdirect sum of integral near-rings is strongly reduced.

**Proof.** (1) Note that the constant part \(N_c\) is an invariant subnear-ring of \(N\). Suppose \(a \in \langle a^2 \rangle\) for each \(a \in N\). If \(a^2 \in N_c\) then \(a \in \langle a^2 \rangle \subseteq N_c\).

(2) Let \(a \in N\) with \(a^2 \in N_c\). Then \((a - a^2)a = 0\), and hence \(a = a^2 \in N_c\). \(\Box\)

We state some basic properties of a strongly reduced near-ring.

**Proposition 2.** Let \(N\) be a strongly reduced near-ring and let \(a, b, x \in N\). Then we have the following.

(1) \(N\) is reduced.

(2) If \(ab^n \in N_c\) for some positive integer \(n\), then \(\{ab, ba\} \cup aNb \cup bNa \subseteq N_c\).

(3) If \(ab^n = 0\) for some positive integer \(n\), then \(ab = 0\) and \(ba = 0\).

**Proof.** (1) Assume that \(a^2 = 0\). Then \(a^2 \in N_c\), and hence \(a \in N_c\). Then we see \(a = a0 = a0a = aa = 0\).

(2) First suppose \(ab \in N_c\). Then \((ba)^2 = bababab0a = bab0a \in N_c\). Since \(N\) is strongly reduced, we have \(ba \in N_c\). Then we obtain \(xba \in N_c\) for each \(x \in N\), whence \((xab)^2 \in N_c\). By the strong reducibility of \(N\), we obtain \(xab \in N_c\) for each \(x \in N\). Since \(ba \in N_c\), we also obtain \(bNa \subseteq N_c\). Now suppose \(ab^n \in N_c\). Then \((ab)^n \in N_c\) by the above argument. Since \(N\) is strongly reduced, this implies \(ab \in N_c\). Hence by the first paragraph, the claim is proved.

(3) If \(ab^n = 0\) for some \(n \geq 1\), then \(ab \in N_c\) by (2). Hence \(ab = abab^{n-1} = ab^n = 0\). Then \((ba)^2 = babab = b0 \in N_c\). Hence \(ba \in N_c\). Therefore \((ba)^2 - ba \in N_c\). Then \((ba)^2 - ba = ((ba)^2 - ba)b = babab - bab = b0 - b0 = 0\). Hence we obtain \(ba = (ba)^2 = b0\). \(\Box\)

Clearly, if \(N\) is a zero-symmetric near-ring, then \(N\) is strongly reduced if and only if \(N\) is reduced. The following example shows that a reduced near-ring is not
necessarily strongly reduced.

**Example 1.** Let \( \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\} \) with addition modulo 6 and define multiplication as follows:

\[
\begin{array}{c|cccccc}
\cdot & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 3 & 1 & 1 \\
2 & 0 & 0 & 2 & 0 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 0 & 0 & 4 & 0 & 4 & 4 \\
5 & 3 & 3 & 5 & 3 & 5 & 5 \\
\end{array}
\]

Obviously this is a reduced near-ring. The constant part of \( \mathbb{Z}_6 \) is \( \{0, 3\} \). Since \( 1^2 = 3 \) is a constant element but 1 is not, this near-ring is not strongly reduced. Also note that \( 1^n \neq 1 \) for any integer \( n > 1 \).

Following Reddy and Murty ([7]) we say that a near-ring \( N \) has the property (\( * \)) if it satisfies

(i) for any \( a, b \in N \), \( ab = 0 \) implies \( ba = b0 \).

(ii) for \( a \in N \), \( a^3 = a^2 \) implies \( a^2 = a \).

We give equivalent conditions for a near-ring \( N \) to be strongly reduced.

**Theorem 1.** The following statements are equivalent for a near-ring \( N \):

1. \( N \) is strongly reduced.
2. For \( a \in N \), \( a^3 = a^2 \) implies \( a^2 = a \).
3. \( N \) has the property (\( * \)).
4. If \( a^{n+1} = xa^{n+1} \) for \( a, x \in N \) and some nonnegative integer \( n \), then \( a = xa = ax \).

**Proof.**

1. \( \implies \) (2) Assume that \( a^3 = a^2 \). Then \( (a^2 - a)a = 0 \), whence \( a(a^2 - a) = a0 \in N_c \) by Proposition 2 (3). Then \( (a^2 - a)a = (a^3 - a^2)a = 0a = 0 \). Again by Proposition 2 (3) \( a^2(a^2 - a) = a^20 \in N_c \). Hence \( (a^2 - a)^2 = a^2(a^2 - a) - a(a^2 - a) = a^20 - a0 = (a^2 - a)0 \in N_c \). This implies \( a^2 - a \in N_c \). Hence \( a^2 - a = (a^2 - a)0 = (a^2 - a)a = 0 \).

2. \( \implies \) (3) This follows from Proposition 2 (3).

3. \( \implies \) (1) Assume \( a^2 \in N_c \). Then \( a^3 = a^2a = a^2 \). By condition (2), this implies \( a = a^2 \in N_c \).

1. \( \implies \) (4) Suppose \( a^{n+1} = xa^{n+1} \) for some \( n \geq 0 \). Then \( (a - xa)a^{n} = 0 \). Hence \( (a - xa)a = 0 \) by Proposition 2 (3), and so \( (a - xa)^2 \in N_c \) by Proposition 2 (2). Since \( N \) is strongly reduced, we have \( a - xa \in N_c \). Then \( a - xa = (a - xa)a = 0 \),
that is \(a = xa\). Now \((a - ax)a = a^2 - axa = a^2 - a^2 = 0 \in N_c\). Hence \((a - ax)^2 = a(a - ax) - ax(a - ax) \in N_c\) by Proposition 2 (2), and so \(a - ax \in N_c\). Therefore \(a - ax = (a - ax)a = 0\).

(4) \(\implies\) (2) This is obvious. \(\square\)

Left strongly regular near-rings are studied by several authors ([2]-[6], [9] etc.) Since all left strongly regular near-rings are strongly reduced, we can use it to study left strongly regular near-rings.

The following is a generalization of [7], Theorem 3.

**Lemma 1.** Let \(N\) be a strongly reduced near-ring and let \(a, x \in N\). If \(a^n = xa^{n+1}\) for some positive integer \(n\), then \(a = xa^n = axa\) and \(ax = xa\).

**Proof.** Assume that \(a^n = xa^{n+1}\) for some \(n \geq 1\). By Theorem 1, \(a = xa^2 = axa\). Then \((ax - xa)a = 0\). Hence, by Proposition 1 (2), \((ax - xa)^2 = ax(ax - xa) - xa(ax - xa) \in N_c\). Since \(N\) is strongly reduced, \(ax - xa \in N_c\). Hence \(ax - xa = (ax - xa)a = 0\). \(\square\)

A near-ring \(N\) is said to be left \(\pi\)-regular if, for each \(a \in N\), there exists a positive integer \(n\) and an element \(x \in N\) such that \(a^n = xa^{n+1}\). Here we give some characterizations of left strongly regular near-rings.

**Theorem 2.** Let \(N\) be a near-ring. Then the following statements are equivalent:

1) \(N\) is left strongly regular.

2) \(N\) is strongly reduced and left \(\pi\)-regular.

3) For each \(a \in N\), there exists \(x, y \in N\) such that \(a = xa^2ya\).

4) For each \(a \in N\), \(a \in \langle a^2 \rangle \cap aNa\).

**Proof.** 1) \(\implies\) 2) - 4) By Proposition 1 (1), a right strongly regular near-ring is strongly reduced. Hence this follows from Lemma 1.

2) \(\implies\) 1) This also follows from Lemma 1.

3) \(\implies\) 1) By hypothesis, \(N\) is strongly reduced. If \(a = xa^2ya\), then \(ya = yxa^2(ya)\). By Theorem 1, \(ya = yaxa^2\). Thus \(a = xa^2yaxa^2\). This implies that \(N\) is left strongly regular.

4) \(\implies\) 1) Since \(a \in \langle a^2 \rangle\) for each \(a \in N\), \(N\) is strongly reduced by Proposition 1 (1). Hence \(N\) satisfies (4) in Theorem 1. Since \(a \in aNa\), there exists \(x \in N\) such that \(a = axa\). Hence \(a = (ax)a = a(ax) = a^2x\). Then we have \(a = axa = (a^2x)xa = a^2x^2a\). Then, by the same way as in 3) \(\implies\) 1), we conclude that \(N\) is left strongly regular. \(\square\)

A near-ring is said to be periodic if, for each \(a \in N\), there exist distinct positive integers \(m, n\) such that \(a^n = a^m\). A near-ring \(N\) is called a \((P_0)\)-near-ring if, for each \(a \in N\), there exists an integer \(n > 1\) such that \(a = a^n\) (See [6], 9.4, p.289).
Obviously a \((P_0)\)-near-ring is strongly reduced (cf. Proposition 1(1)). Hence the proof of the following corollary follows directly from Lemma 1.

**Corollary 2.** Let \(N\) be a near-ring. Then the following statements are equivalent:

1) \(N\) is periodic and strongly reduced.

2) \(N\) is a \((P_0)\)-near-ring.

As a special case of this corollary, we have

**Corollary 3.** Let \(N\) be a finite near-ring. Then the following statements are equivalent:

1) \(N\) is strongly reduced.

2) \(N\) is left strongly regular.

3) \(N\) is a \((P_0)\)-near-ring.

Now we classify all reduced, and strongly reduced near-rings of order \(\leq 7\). To do it, we use Clay’s tables ([1]). For example, according to [1], 2.1, p. 367, on the cyclic group \(Z_4\) of order 4, there are 12 equivalence classes of near-rings: 1)-12).

### Near-Rings of Order \(\leq 7\)

<table>
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<tr>
<th>Groups</th>
<th>zero-symmetric and reduced</th>
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<td>(V)</td>
<td>1, 6)</td>
<td>21)</td>
<td>18, 20, 23)</td>
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<td>(Z_5)</td>
<td>7, 8, 10)</td>
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<td>34)</td>
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<td>(Z_7)</td>
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**Acknowledgements.** The first author gratefully acknowledges the kind hospitality he enjoyed at Okayama University.
References