KYUNGPOOK Math. J. 43(2003), 587-592

Strong Reducedness and Strong Regularity for Near-rings

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Abstract. A near-ring N is called strongly reduced if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$, where N_c denotes the constant part of N. We investigate some properties of strongly reduced near-rings and apply those to the study of left strongly regular near-rings. Finally we classify all reduced, and strongly reduced near-rings of order ≤ 7.

1. Introduction

A right near-ring N is called left strongly regular if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$. Mason ([3]) introduced this notion and characterized left strongly regular zero-symmetric near-rings. Several authors ([2], [4], [5], [7] etc.) studied them. In particular, Reddy and Murty ([7]) extended some results in [3] to the non-zero symmetric case. They observed that every left strongly regular near-ring has some interesting property. In this paper we consider this property. Let N be a right near-ring and let N_c denote the constant part of N. We define N to be strongly reduced if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$. Obviously a strongly reduced near-ring N is reduced. We show that strong reducedness is a more general concept than the property (*) in Reddy and Murty ([7]). Left or right strongly regular near-rings form one of the important class of strongly reduced near-rings. Using strong reducibility, we characterize left strongly regular near-rings and (P_0)-near-rings. Finally we classify all reduced, and strongly reduced near-rings of order ≤ 7 using the description given by Clay ([1]).

2. Results

Throughout this paper we work with right near-rings. For notation and basic results, we shall refer to Pilz ([6]). Recall that a near-ring N is reduced if, for $a \in N$, $a^2 = 0$ implies $a = 0$. For a near-ring N, N_c denotes the constant part of

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Received November 8, 2002.
2000 Mathematics Subject Classification: 16Y30.
Key words and phrases: reduced, strongly reduced, left strongly regular, strongly regular.

N, that is, $N_c = \{x \in N \mid x = x0\}$. A near-ring N is said to be *strongly reduced* if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$. Obviously N is strongly reduced if and only if, for $a \in N$ and any positive integer n, $a^n \in N_c$ implies $a \in N_c$. We will show that a strongly reduced near-ring is reduced, that is, for $a \in N$, $a^2 = 0$ implies $a = 0$. A near-ring N is said to be *left strongly regular* if, for each $a \in N$, there exists $x \in N$ such that $a = xa^2$. Right strong regularity is defined in a symmetric way.

A subnear-ring H of a near-ring N is called *invariant* if $NH \subseteq H$ and $HN \subseteq H$. For a subset S of N, $\langle S \rangle$ stands for the invariant subnear-ring of N generated by S. We give some sufficient conditions for a near-ring to be strongly reduced.

Proposition 1.

(1) *Let N be a near-ring. If $a \in \langle a^2 \rangle$ for each $a \in N$, then N is strongly reduced. In particular, right or left strongly regular near-rings are strongly reduced.*

(2) *Every integral near-ring N is strongly reduced. Hence a subdirect sum of integral near-rings is strongly reduced.*

Proof. (1) Note that the constant part N_c is an invariant subnear-ring of N. Suppose $a \in \langle a^2 \rangle$ for each $a \in N$. If $a^2 \in N_c$ then $a \in \langle a^2 \rangle \subseteq N_c$.

(2) Let $a \in N$ with $a^2 \in N_c$. Then $(a - a^2)a = 0$, and hence $a = a^2 \in N_c$. □

We state some basic properties of a strongly reduced near-ring.

Proposition 2. *Let N be a strongly reduced near-ring and let $a, b, x \in N$. Then we have the following.*

(1) *N is reduced.*

(2) *If $ab^n \in N_c$ for some positive integer n, then $\{ab, ba\} \cup aNb \cup bNa \subseteq N_c$.*

(3) *If $ab^n = 0$ for some positive integer n, then $ab = 0$ and $ba = b0$.*

Proof. (1) Assume that $a^2 = 0$. Then $a^2 \in N_c$, and hence $a \in N_c$. Then we see $a = a0 = a0a = aa = 0$.

(2) First suppose $ab \in N_c$. Then $(ba)^2 = baba = bab0a = bab0 \in N_c$. Since N is strongly reduced, we have $ba \in N_c$. Then we obtain $xba \in N_c$ for each $x \in N$, whence $(axb)^2 \in N_c$. By the strong reducibility of N, we obtain $axb \in N_c$ for each $x \in N$. Since $ba \in N_c$, we also obtain $bNa \subseteq N_c$. Now suppose $ab^n \in N_c$. Then $(ab)^n \in N_c$ by the above argument. Since N is strongly reduced, this implies $ab \in N_c$. Hence by the first paragraph, the claim is proved.

(3) If $ab^n = 0$ for some $n \geq 1$, then $ab \in N_c$ by (2). Hence $ab = abb^{n-1} = ab^n = 0$. Then $(ba)^2 = baba = b0 \in N_c$. Hence $ba \in N_c$. Therefore $(ba)^2 - ba \in N_c$. Then $(ba)^2 - ba = \{(ba)^2 - ba\}b = babab - bab = b0 - b0 = 0$. Hence we obtain $ba = (ba)^2 = b0$. □

Clearly, if N is a zero-symmetric near-ring, then N is strongly reduced if and only if N is reduced. The following example shows that a reduced near-ring is not necessarily strongly reduced.

Example 1. Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with addition modulo 6 and define multiplication as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	3	1	3	1	1
2	0	0	2	0	2	2
3	3	3	3	3	3	3
4	0	0	4	0	4	4
5	3	3	5	3	5	5

Obviously this is a reduced near-ring. The constant part of $\mathbb{Z}_6$ is $\{0, 3\}$. Since $1^2 = 3$ is a constant element but 1 is not, this near-ring is not strongly reduced. Also note that $1^n \neq 1$ for any integer $n > 1$.

Following Reddy and Murty ([7]) we say that a near-ring N has the property (*) if it satisfies

(i) for any $a, b \in N$, $ab = 0$ implies $ba = b0$.

(ii) for $a \in N$, $a^3 = a^2$ implies $a^2 = a$.

We give equivalent conditions for a near-ring N to be strongly reduced.

Theorem 1. *The following statements are equivalent for a near-ring N:*

(1) *N is strongly reduced.*

(2) *For $a \in N$, $a^3 = a^2$ implies $a^2 = a$.*

(3) *N has the property (*).*

(4) *If $a^{n+1} = xa^{n+1}$ for $a, x \in N$ and some nonnegative integer n, then $a = xa = ax$.*

Proof. (1) $\Longrightarrow$ (2) Assume that $a^3 = a^2$. Then $(a^2 - a)a = 0$, whence $a(a^2 - a) = a0 \in N_c$ by Proposition 2 (3). Then $(a^2 - a)a^2 = (a^3 - a^2)a = 0a = 0$. Again by Proposition 2 (3) $a^2(a^2 - a) = a^2 0 \in N_c$. Hence $(a^2 - a)^2 = a^2(a^2 - a) - a(a^2 - a) = a^2 0 - a0 = (a^2 - a)0 \in N_c$. This implies $a^2 - a \in N_c$. Hence $a^2 - a = (a^2 - a)0 = (a^2 - a)a = 0$.

(2) $\Longrightarrow$ (3) This follows from Proposition 2 (3).

(3) $\Longrightarrow$ (1) Assume $a^2 \in N_c$. Then $a^3 = a^2 a = a^2$. By condition (2), this implies $a = a^2 \in N_c$.

(1) $\Longrightarrow$ (4) Suppose $a^{n+1} = xa^{n+1}$ for some $n \geq 0$. Then $(a - xa)a^n = 0$. Hence $(a - xa)a = 0$ by Proposition 2 (3), and so $(a - xa)^2 \in N_c$ by Proposition 2 (2). Since N is strongly reduced, we have $a - xa \in N_c$. Then $a - xa = (a - xa)a = 0$,

that is $a = xa$. Now $(a - ax)a = a^2 - axa = a^2 - a^2 = 0 \in N_c$. Hence $(a - ax)^2 = a(a - ax) - ax(a - ax) \in N_c$ by Proposition 2 (2), and so $a - ax \in N_c$. Therefore $a - ax = (a - ax)a = 0$.

(4) $\Longrightarrow$ (2) This is obvious. $\square$

Left strongly regular near-rings are studied by several authors ([2]-[6], [9] etc.) Since all left strongly regular near-rings are strongly reduced, we can use it to study left strongly regular near-rings.

The following is a generalization of [7], Theorem 3.

Lemma 1. *Let N be a strongly reduced near-ring and let $a, x \in N$. If $a^n = xa^{n+1}$ for some positive integer n, then $a = xa^2 = axa$ and $ax = xa$.*

Proof. Assume that $a^n = xa^{n+1}$ for some $n \geq 1$. By Theorem 1, $a = xa^2 = axa$. Then $(ax - xa)a = 0$. Hence, by Proposition 1 (2), $(ax - xa)^2 = ax(ax - xa) - xa(ax - xa) \in N_c$. Since N is strongly reduced, $ax - xa \in N_c$. Hence $ax - xa = (ax - xa)a = 0$. $\square$

A near-ring N is said to be *left π-regular* if, for each $a \in N$, there exists a positive integer n and an element $x \in N$ such that $a^n = xa^{n+1}$. Here we give some characterizations of left strongly regular near-rings.

Theorem 2. *Let N be a near-ring. Then the following statements are equivalent:*

1) *N is left strongly regular.*

2) *N is strongly reduced and left π-regular.*

3) *For each $a \in N$, there exists $x, y \in N$ such that $a = xa^2ya$.*

4) *For each $a \in N$, $a \in \langle a^2 \rangle \cap aNa$.*

Proof. 1) $\Longrightarrow$ 2) - 4) By Proposition 1 (1), a right strongly regular near-ring is strongly reduced. Hence this follows from Lemma 1.

2) $\Longrightarrow$ 1) This also follows from Lemma 1.

3) $\Longrightarrow$ 1) By hypothesis, N is strongly reduced. If $a = xa^2ya$, then $ya = yxa^2(ya)$. By Theorem 1, $ya = yayxa^2$. Thus $a = xa^2yayxa^2$. This implies that N is left strongly regular.

4) $\Longrightarrow$ 1) Since $a \in \langle a^2 \rangle$ for each $a \in N$, N is strongly reduced by Proposition 1 (1). Hence N satisfies (4) in Theorem 1. Since $a \in aNa$, there exists $x \in N$ such that $a = axa$. Hence $a = (ax)a = a(ax) = a^2x$. Then we have $a = axa = (a^2x)xa = a^2x^2a$. Then, by the same way as in 3) $\Longrightarrow$ 1), we conclude that N is left strongly regular. $\square$

A near-ring is said to be *periodic* if, for each $a \in N$, there exist distinct positive integers m, n such that $a^m = a^n$. A near-ring N is called a (P_0)-near-ring if, for each $a \in N$, there exists an integer $n > 1$ such that $a = a^n$ (See [6], 9.4, p.289).

Obviously a (P_0)-near-ring is strongly reduced (cf. Proposition 1(1)). Hence the proof of the following corollary follows directly from Lemma 1.

Corollary 2. *Let N be a near-ring. Then the following statements are equivalent:*

1) *N is periodic and strongly reduced.*

2) *N is a (P_0)-near-ring.*

As a special case of this corollary, we have

Corollary 3. *Let N be a finite near-ring. Then the following statements are equivalent:*

1) *N is strongly reduced.*

2) *N is left strongly regular.*

3) *N is a (P_0)-near-ring.*

Now we classify all reduced, and strongly reduced near-rings of order ≤ 7. To do it, we use Clay's tables ([1]). For example, according to [1], 2.1, p. 367, on the cyclic group $\mathbb{Z}_4$ of order 4, there are 12 equivalence classes of near-rings: 1)-12).

Near-Rings of Order ≤ 7

Groups	zero-symmetric and reduced	non-zero-symmetric, reduced and non-strongly reduced	non-zero-symmetric and strongly reduced
$\mathbb{Z}_4$	8), 10), 11)		9)
V	1), 6)	21)	18), 20), 23)
$\mathbb{Z}_5$	7), 8), 10)		9)
$\mathbb{Z}_6$	27), 47)	21), 32), 38), 56), 59)	24), 35) 48), 49), 52), 53)
S_3	34)	12), 16), 31)	11), 14), 30), 39)
$\mathbb{Z}_7$	18), 20), 21), 23), 24)		19)

Acknowledgements. The first author gratefully acknowledges the kind hospitality he enjoyed at Okayama University.

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