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ALGEBRAIC OPERATIONS ON FUZZY NUMBERS USING OF LINEAR FUNCTIONS

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Abstract. In this paper, we introduce two types of algebraic operations on fuzzy numbers using piecewise linear functions and then show that the Zadeh implication is smaller than the Diense-Rescher implication, which is smaller than the Lukasiewicz implication. If $(f, *)$ is an available pair, then $A{*_m}B \leq A{*_p}B \leq A{*_j}B$.

1. Introduction

D. Dubois and H.Prade employed the extension principle to extend algebraic operations from crisp to fuzzy numbers([4], [5]). It is well-known that for two continuous fuzzy numbers, the extension principle method and the $\alpha - cut$ method are equivalent. In [2], Chung introduced the Lmap-Min method to the extend algebraic operations from crisp to fuzzy numbers using piecewise linear function and minimum($\wedge$). And it was proven that for two equipotent fuzzy numbers, the Lmap-Min method and the extension principle method are equivalent.

A $fuzzy\ set$ is a function A on a set X to the unit interval. For any $\alpha \in [0, 1]$, the $\alpha - cut$ of a fuzzy set A on a set X, A^α, is the crisp set $A^\alpha = \{x \in X | A(x) \geq \alpha\}$.

For any fuzzy set A on a set X, the *support* of A, A^{+0}, is the set $\{x \in X | A(x) > 0\}$.

A fuzzy set A on the set R of real numbers is said to be a $fuzzy$ *number* if it satisfies the following:

1) A^α is a non-empty closed interval for each $\alpha \in [0, 1]$

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2) A^{+0} is a bounded interval.

3) A is continuous on $S(A)$, where $S(A)$ denotes the closure of A^{+0} in the real line R.

We use $F(R)$ to denote the set of fuzzy numbers.

For a fuzzy number A, we write $S(A)=[s_A, S_A]$ and $A^1=[m_A, M_A]$.

For a fuzzy number A, the *left spread* of A, $L(A)$ is the interval $[s_A, m_A]$ and the *right spread* of $A, R(A)$, is the interval $[M_A, S_A]$.

A continuous binary operation $*$:$R \times R \to R$ is said to be:

1) *increasing(decreasing, resp.)* if $x < y, u < v$ imply $x * u < y * v$ $(x * u > y * v$, resp).

2) *hybrid* if $v < y, x < u$ imply $x * y < u * v$.

Throughout this paper, we use $*$ to denote the continuous binary operation on R.

Addition(+), meet $(\wedge)$ and join$(\vee)$ are continuous increasing binary operations on R and subtraction(-) is a continuous hybrid binary operation on R. Multiplication$(\times)$ is a continuous increasing binary operation on $[0, \infty)$ and a continuous decreasing binary operation on $(-\infty, 0]$.

$[ExtensionPrinciple]$ [7]. For $A, B \in F(R)$, we define a fuzzy set on R, $A{*_e}B$, by the equation $(A{*_e}B)(z) = \vee_{z=x*y} A(x) \wedge B(y)$.

REMARK 1. 1. *If* $(A{*_e}B)(z) > 0$, *then there is a pair* $(x, y) \in S(A) \times S(B)$ *such that* $(A{*_e}B)(z) = A(x) \wedge B(y)$. *We call such a pair the critical point of* $(A{*_e}B)$ *with respect to* z.

For two intervals $[a, b]$ and $[c, d]$, a linear function f:$[a, b] \to [c, d]$ is said to be:

1) *increasing* if for each $x \in [a, b]$, $f(x) = \frac{d-c}{b-a}(x-a)+c$,

2) *decreasing* if for each $x \in [a, b]$, $f(x) = \frac{c-d}{b-a}(x-a)+d$.

For $A, B \in F(R)$, a linear function f:$S(A) \to S(B)$ is said to be:

1) *piecewise increasing* if $f|_{R(A)}^{R(B)}$: $R(A) \to R(B)$, $f|_{A^1}^{B^1}$: $A^1 \to B^1$ and $f|_{L(A)}^{L(B)}$: $L(A) \to L(B)$ are increasing linear functions.

2) *piecewise decreasing* if $f|_{R(A)}^{L(B)}$: $R(A) \to L(B)$, $f|_{A^1}^{B^1}$: $A^1 \to B^1$ and $f|_{L(A)}^{R(B)}$: $L(A) \to R(B)$ are decreasing linear functions.

3) *piecewise* if it is piecewise increasing or piecewise decreasing.

For $A, B \in F(R)$ and a linear function $f{:}S(A) \to S(B)$, we use $i(A * B)$ to denote the set $\{x * f(x) | x \in S(A)\}$.

Let $A, B \in F(R)$ and $f{:}S(A) \to S(B)$ a piecewise linear function. Then a pair $(f, *)$ is called an *available pair* if it satisfies one of the following:

1) f and $*$ are both increasing.

2) f is decreasing and $*$ is hybrid.

PROPOSITION 1. 2. *Let $A, B \in F(R)$ and $f{:}S(A) \to S(B)$ be a piecewise linear function. Then one has the following: 1) $i(A * B)$ is a closed interval in R. 2) If $(f, *)$ is an available pair, then $i(A * B) = \{x * y | x \in S(A), y \in S(B)\}$.*

Proof. See[2].

[Lmap-Min Method] [2]. Let $A, B \in F(R)$ and let $f{:}S(A) \to S(B)$ be a piecewise linear function. Then, we define a fuzzy set on R, $A{*_m}B$, as follows: $(A{*_m}B)(z) = A(x) \wedge B(f(x))$ if $z = x * f(x)$ for some $x \in S(A)$ and $(A{*_m}B)(z) = 0$ if $z \neq x * f(x)$ for any $x \in S(A)$. Then if $(f, *)$ is an available pair, then $A{*_m}B$ is a fuzzy number. □

DEFINITION 1. 3. *[2] Let $A, B \in F(R)$. A piecewise linear function $f{:}S(A) \to S(B)$ is said to be a shift (from A to B) if $A(x) = B(f(x))$ for each $x \in S(A)$.*

DEFINITION 1. 4. *[2] 1) Two fuzzy numbers A and B are said to be i-equipotent(d-equipotent, resp.), symbolized as $A \sim B$ ($A \simeq B$, resp), provided that there is an increasing(decreasing, resp.) shift from A to B.*

2) Two fuzzy numbers A and B are said to be equipotent if they are i-equipotent or d-equipotent.

THEOREM 1. 5. *Let $A, B \in F(R)$. Then one has the following:*
1) If $A \sim B$ and $*$ are increasing, then $A{*_m}B = A{*_e}B$.
2) If $A \simeq B$ is decreasing and $*$ are hybrid, then $A{*_m}B = A{*_e}B$.

Proof. See[2]. □

DEFINITION 1. 6. *A fuzzy number A is said to be positive(negative, resp.) if $S(A) \subseteq [0, \infty)$ ($S(A) \subseteq (-\infty, 0]$, resp.).*

2. ALGEBRAIC OPERATIONS ON FUZZY NUMBERS

In the following theorem, we introduce the algebraic operation on fuzzy numbers using piecewise linear functions and addition(+).

THEOREM 2. 1. *[Lmap-Addition Method] Let $A, B \in F(R)$ and $f{:}S(A) \to S(B)$ a piecewise linear function. Then, we define a fuzzy set on R, $A{*_p}B$, as follows: $(A{*_p}B)(z) = (A(x)+B(f(x)))/2$ if $z = x*f(x)$ for some $x \in S(A)$ and $(A{*_p}B)(z) = 0$ if $z \neq x*f(x)$ for any $x \in S(A)$. If $(f,*)$ is an available pair, then $A{*_p}B$ is a fuzzy number.*

Proof. 1) Suppose f and $*$ are both increasing. Since $(A{*_p}B)^{+0} \subseteq i(A*B)$ and $i(A*B)$ is bounded, $(A{*_p}B)^{+0}$ is bounded. Since $(A{*_p}B)^1 \neq 0$ and for each $\alpha \in (0,1]$, $(A{*_p}B)^1 \subseteq (A{*_p}B)^\alpha$, $(A{*_p}B)^\alpha \neq 0$ for each $\alpha \in (0,1]$. Suppose $z_1, z_2 \in (A{*_p}B)^\alpha$ and $z_1 \leq z \leq z_2$. Since $(A{*_p}B)^\alpha \subseteq i(A*B)$ and $i(A*B)$ is an interval, $z \in i(A*B)$ and so there is $x \in S(A)$ such that $z = x*f(x)$. Since $z_1, z_2 \in (A{*_p}B)^\alpha$, there are $x_1, x_2 \in S(A)$ such that $z_1 = x_1 * f(x_1)$ and $z_2 = x_2 * f(x_2)$. Since f and $*$ are increasing, $x_1 \leq x \leq x_2$.

Case 1. $x_1 \leq x \leq m_A \leq x_2$: Then $A(x_1) \leq A(x)$ and $B(f(x_1)) \leq B(f(x))$. Hence $A(x_1)+B(f(x_1)) \leq A(x)+B(f(x))$ and so $(A{*_p}B)(z) \geq \alpha$.

Case 2. $x_1 \leq m_A \leq x \leq M_A \leq x_2$: Then $A(x_1) = 1$ and $B(f(x_1)) = 1$ and so $(A{*_p}B)(z) \geq \alpha$.

Case 3. $x_1 \leq M_A \leq x \leq x_2$: Then $A(x_2) \leq A(x)$ and $B(f(x_2)) \leq B(f(x))$. Hence $A(x_2)+B(f(x_2)) \leq A(x)+B(f(x))$ and so $(A{*_p}B)(z) \geq \alpha$.

Thus $(A{*_p}B)(z) = (A(x)+B(f(x)))/2 \geq \alpha$ and hence $z \in (A{*_p}B)^\alpha$. Therefore $(A{*_p}B)^\alpha$ is an interval. Let $z_0 = inf(A{*_p}B)^\alpha$. Then there is a decreasing sequence $< z_n >$ in $(A{*_p}B)^\alpha$ such that $z_n \to z_0$. Then for each $n \in N$ there is $x_n \in S(A)$ such that $z_n = x_n * f(x_n)$ and $(A(x) + B(f(x)))/2 \geq \alpha$ Since f and $*$ are increasing, $< x_n >$ is a decreasing sequence in S(A). Since $S(A)$ is bounded, there is $x_0 \in S(A)$ such that $x_n \to x_0$. Since A, B, f, $+$ and $*$ are continuous, $(A(x_n) + B(f(x_n)))/2 \to (A(x_0)+B(f(x_0)))/2$ and $z_n \to x_0 * f(x_0) = z_0$. Since $(A(x_n) + B(f(x_n)))/2 \geq \alpha$ for each $n \in N$, $(A(x_0) + B(f(x_0)))/2 \geq \alpha$ and so $z_0 \in (A{*_p}B)^\alpha$. Similarly we have $sup(A{*_p}B)^\alpha \in (A{*_p}B)^\alpha$. In all, $(A*_pB)^\alpha$ is a non-empty closed interval. Since f, $*$, A, B and $+$ are continuous, $(A*_pB)$ is continuous. This completes the proof. $\square$

Using the exactly same argument as for the case f and $*$ are both increasing, the case where f is decreasing and $*$ is hybrid can be proved.

In the following theorem, we introduce the algebraic operation on fuzzy numbers using piecewise linear functions and maximum($\vee$).

THEOREM 2. 2. *[Lmap-Max Method] Let $A, B \in F(R)$ and $f{:}S(A) \rightarrow S(B)$ a piecewise linear function. Then, we define a fuzzy set on R, $A*_jB$, as follows: $(A*_jB)(z) = A(x) \vee B(f(x))$ if $z = x * f(x)$ for some $x \in S(A)$ and $(A*_jB)(z) = 0$ if $z \neq x * f(x)$ for any $x \in S(A)$. If $(f, *)$ is an available pair, then $A*_pB$ is a fuzzy number.*

Proof. 1) Suppose f and $*$ are both increasing. Since $(A*_jB)^{+0} \subseteq i(A*B)$ and $i(A*B)$ is bounded, $(A*_jB)^{+0}$ is bounded. Since $(A*_jB)^1 \neq 0$ and for each $\alpha \in (0,1]$. $(A*_jB)^1 \subseteq (A*_jB)^\alpha$, $(A*_pB)^\alpha \neq 0$ for each $\alpha \in (0,1]$. Suppose $z_1, z_2 \in (A*_jB)^\alpha$ and $z_1 \leq z \leq z_2$. Since $(A*_jB)^\alpha \subseteq i(A*B)$ and $i(A*B)$ is an interval, $z \in i(A*B)$ and so there is $x \in S(A)$ such that $z = x * f(x)$. Since $z_1, z_2 \in (A*_jB)^\alpha$, there are $x_1, x_2 \in S(A)$ such that $z_1 = x_1 * f(x_1)$ and $z_2 = x_2 * f(x_2)$. Since f and $*$ are increasing, $x_1 \leq x \leq x_2$.

Case 1. $x_1 \leq x \leq m_A \leq x_2$: Then $A(x_1) \leq A(x)$ and $B(f(x_1)) \leq B(f(x))$. Hence $A(x_1) \vee B(f(x_1)) \leq A(x) \vee B(f(x))$ and so $(A*_jB)(z) \geq \alpha$.

Case 2. $x_1 \leq m_A \leq x \leq M_A \leq x_2$: Then $A(x_1) = 1$ and $B(f(x_1)) = 1$ and so $(A*_jB)(z) \geq \alpha$.

Case 3. $x_1 \leq M_A \leq x \leq x_2$: Then $A(x_2) \leq A(x)$ and $B(f(x_2)) \leq B(f(x))$. Hence $A(x_2) \vee B(f(x_2)) \leq A(x) \vee B(f(x))$ and so $(A*_jB)(z) \geq \alpha$.

Thus $(A*_jB)(z) = A(x) \vee B(f(x)) \geq \alpha$ and hence $z \in (A*_jB)^\alpha$. Therefore $(A*_jB)^\alpha$ is an interval. Let $z_0 = inf(A*_jB)^\alpha$. Then there is a decreasing sequence $< z_n >$ in $(A*_jB)^\alpha$ such that $z_n \rightarrow z_0$. Then for each $n \in N$ there is $x_n \in S(A)$ such that $z_n = x_n * f(x_n)$ and $A(x) \vee B(f(x)) \geq \alpha$ Since f and $*$ are increasing, $< x_n >$ is a decreasing sequence in S(A). Since $S(A)$ is bounded, there is $x_0 \in S(A)$ such that $x_n \rightarrow x_0$. Since A, B, f, $\vee$ and $*$ are continuous, $A(x_n) \vee B(f(x_n)) \rightarrow A(x_0) \vee B(f(x_0))$ and $z_n \rightarrow x_0 * f(x_0) = z_0$. Since $A(x_n) \vee B(f(x_n)) \geq \alpha$

for each $n \in N$, $A(x_0) \vee B(f(x_0)) \geq \alpha$ and so $z_0 \in (A*_j B)^\alpha$. Similarly we have $sup(A*_j B)^\alpha \in (A*_j B)^\alpha$. In all, $(A*_j B)^\alpha$ is a non-empty closed interval. Since f, $*$, A, B and $\vee$ are continuous, $(A*_j B)$ is continuous. This completes the proof. □

Using the exactly same argument as for the case f and $*$ are both increasing, the case where f is decreasing and $*$ is hybrid can be proved.

In the following two theorems, we show that $\{*_m, *_p, *_e$ and $*_j\}$ is a lattice.

THEOREM 2. 3. *Let* $A, B \in F(R)$. *If* $(f, *)$ *is an available pair, then* $A*_m B \leq A*_p B \leq A*_j B$.

Proof. Straightforward. □

THEOREM 2. 4. *Let* $A, B \in F(R)$. *If* $(f, *)$ *is an available pair, then* $A*_m B \leq A*_e B \leq A*_j B$.

Proof. Let (x_0, y_0) be the critical point of $A*_e B$ with respect to $z \in i(A * B)$ and $x \in S(A)$ such that $(A*_j B)(z) = A(x) \vee B(f(x))$. Suppose that f and $*$ are both increasing. If $x = x_0$, then $(A*_e B)(z) \leq (A*_j B)(z)$. Suppose $x_0 < x$. Since f and $*$ are both increasing, $y_0 \geq f(x)$ and so $A(x_0) < A(x)$ or $B(y_0) < B(f(x))$. Thus $(A*_e B)(z) \leq (A*_j B)(z)$. Suppose $x_0 > x$. Since f and $*$ are both increasing, $y_0 \leq f(x)$ and so $A(x_0) < A(x)$ or $B(y_0) < B(f(x))$. Thus $(A*_e B)(z) \leq (A*_j B)(z)$. Therefore $A*_e B \leq A*_j B$. □

Using the exactly same argument as for the case f and $*$ are both increasing, the case where f is decreasing and $*$ is hybrid can be proved.

COROLLARY 2. 5. *Let* $A, B \in F(R)$. *Then one has the following:*

1) If $A \sim B$ *and* $*$ *are increasing, then* $A*_m B = A*_p B = A*_e B = A*_j B$.

2) 1) If $A \simeq B$ *is decreasing and* $*$ *are hybrid, then* $A*_m B = A*_p B = A*_e B = A*_j B$.

Proof. Straightforward. □

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