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CONFORMAL CHANGE OF THE VECTOR S_ω IN 5-DIMENSIONAL g-UFT

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Abstract. We investigate change of the vector S_ω induced by the conformal change in 5-dimensional g-unified field theory. These topics will be studied for the second class in 5-dimensional case

1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVAT Ý([8],1957). CHUNG([6],1968) also investigated the same topic in 4-dimensional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional $^*g$-UFT, and for the second class in 5-dimensional g-UFT were investigated by CHO([1], 1992, [2],1994, [3],1996, [4],1998).

In the present paper, we investigate change of the vector S_ω induced by the conformal change in 5-dimensional g-unified field theory. These topics will be studied for the second class in 5-dimensional case.

2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be referred to CHUNG([5],1988; [3],1988), CHO([1],1992; [2],1994; [3],1996; [4],1998).

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2.1. n-dimensional g-unified field theory. The n-dimensional g-unified field theory (n-g-UFT hereafter) was originally suggested by HLAVATÝ([8],1957) and systematically introduced by CHUNG([7],1963).

Let X_n[1] be an n-dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \to x^{\nu'}$, for which

$$Det\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[2]:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$Det((g_{\lambda\mu})) \neq 0 \qquad Det((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.4}$$

In our n-g-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma^\nu_{\omega\mu}$ with the following transformation rule :

$$\Gamma^{\nu'}_{\omega'\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x^{\omega'}\partial x^{\mu'}}\right) \tag{2.5}$$

and satisfies the system of Einstein's equations

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha} \tag{2.6}$$

where D_ω denotes the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^\nu = \Gamma^\nu_{[\lambda\mu]} \tag{2.7}$$

is the *torsion tensor* of $\Gamma^\nu_{\lambda\mu}$. The connection $\Gamma^\nu_{\lambda\mu}$ satisfying (2.6) is called the *Einstein's connection.*

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[1]Throughout the present paper, we assumed that $n \geq 2$.

[2]Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \cdots, n$ and follow the summation convention.

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

$$\begin{aligned} &\mathfrak{g} = Det((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = Det((h_{\lambda\mu})) \neq 0, \\ &\mathfrak{k} = Det((k_{\lambda\mu})), \end{aligned} \tag{2.8a}$$

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}}, \tag{2.8b}$$

$$K_p = k_{[\alpha_1}{}^{\alpha^1} \cdots k_{\alpha_p]}{}^{\alpha^p}, \quad (p = 0, 1, 2, \cdots) \tag{2.8c}$$

$$^{(0)}k_\lambda{}^\nu = \delta_\lambda{}^\nu, \quad ^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \quad ^{(p)}k_\lambda{}^\alpha = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu, \tag{2.8d}$$

$$K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu}, \tag{2.8e}$$

$$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{2.8f}$$

where ∇_ω is the symbolic vector of the convariant derivative with respect to the Christoffel symbols $\{{}^{\,\nu}_{\lambda\mu}\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

$$K_0 = 1; K_n = k \quad \text{if } n \text{ is even}; \quad K_p = 0 \quad \text{if } p \text{ is odd}, \tag{2.9a}$$

$$g = 1 + K_2 + \cdots + K_{n-\sigma}, \tag{2.9b}$$

$$^{(p)}k_{\lambda\mu} = (-1)^{p(p)}k_{\mu\lambda}, \quad ^{(p)}k^{\lambda\mu} = (-1)^{p(p)}k^{\nu\lambda}. \tag{2.9c}$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T:

$$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\nu} = T_{\alpha\beta\gamma}{}^{(p)}k_\omega{}^{\alpha(q)}k_\mu{}^{\beta(r)}k_\nu{}^\gamma, \tag{2.10a}$$

$$T = T_{\omega\mu\nu} = \overset{000}{T}, \tag{2.10b}$$

$$2\overset{pqr}{T}{}_{\omega[\lambda\mu]} = \overset{pqr}{T}{}_{\omega\lambda\mu} - \overset{pqr}{T}{}_{\omega\mu\lambda}, \tag{2.10c}$$

$$2\overset{(pq)r}{T}{}_{\omega\lambda\mu} = \overset{pqr}{T}{}_{\omega\lambda\mu} + \overset{qpr}{T}{}_{\omega\lambda\mu}. \tag{2.10d}$$

We then have

$$\overset{pqr}{T}{}_{\omega\lambda\mu} = -\overset{qpr}{T}{}_{\lambda\omega\mu}. \tag{2.11}$$

If the system (2.6) admits $\Gamma^\nu_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

$$\Gamma^\nu_{\omega\mu} = \{{}^{\,\nu}_{\omega\mu}\} + S_{\omega\mu}{}^\nu + U^\nu_{\omega\mu} \tag{2.12}$$

where

$$U_{\nu\omega\mu} = 2\overset{001}{S}{}_{\nu(\omega\mu)}. \tag{2.13}$$

The above two relations show that our problem of determining $\Gamma^{\nu}_{\omega\mu}$ in terms of $g_{\lambda\mu}$ is reduced to that of studying the tensor $S_{\omega\mu}{}^{\nu}$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

$$S = B - 3\overset{(110)}{S} \tag{2.14}$$

where

$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k^{\alpha}_{\omega]}k^{\beta}_{\nu}. \tag{2.15}$$

DEFINITION 2.1. The vector S_ω defined by

$$S_\omega = S_{\omega\alpha}{}^{\alpha}. \tag{2.16}$$

2.2. Some results for the second class in 5-g-UFT. In this section, we introduce some results of 5-g-UFT without proof, which are needed in our subsequent considerations.

They may be referred to CHO([1],1992).

DEFINITION 2.2. In 5-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class, if $K_2 \neq 0$, $K_4 = 0$.

THEOREM 2.3 (Main Recurrence Relations). *For the second class in 5-UFT, the following recurrence relation hold*

$$^{(p+3)}k_{\lambda}{}^{\nu} = -K_2^{(p+1)}k_{\lambda}{}^{\nu}, \quad (p = 0, 1, 2, \cdots). \tag{2.17}$$

THEOREM 2.4 (For The Second Class In 5-g-UFT). *A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is*

$$1 - (K_2)^2 \neq 0. \tag{2.18}$$

If the condition (2.18) is satisfied, the unique solution of (2.14) is given by

$$(1 - K_2^2)(S - B) = -2\overset{(10)1}{B} + (K_2 - 1)\overset{110}{B} + 2\overset{(20)2}{B} + 2\overset{112}{B}. \tag{2.19}$$

3. Conformal change of the 5-dimensional vector S_ω for the second class

In this final chapter we investigate the change $S_\omega \to \overline{S}_\omega$ of the vector induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and $\overline{X}_n$ are conformal if and only if

$$\overline{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x) \tag{3.1}$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the vector S_ω. An explicit representation of the change of 5-dimensional vector S_ω for the second class will be exhibited in this chapter.

Agreement 3.1. Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of $T(\overline{T})$ will be raised and/or lowered by means of $h^{\lambda\nu}(\overline{h}^{\lambda\nu})$ and/or $h_{\lambda\nu}(\overline{h}_{\lambda\nu})$.

The results in the following theorems are needed in our further considerations. They may be referred to CHO([1],1992, [2],1994, [3],1996).

Theorem 3.2. *In n-g-UFT, the conformal change (3.1) induces the following changes:*

$$^{(p)}\overline{k}_{\lambda\mu} = e^{\Omega(p)} k_{\lambda\mu}, \quad ^{(p)}\overline{k}_{\lambda} = {}^{(p)}k_{\lambda}^{\nu}, \quad ^{(p)}\overline{k}^{\lambda\mu} = e^{-\Omega(p)} k^{\lambda\mu}, \tag{3.2a}$$

$$\overline{g} = g, \quad \overline{K}_p = K_p, \qquad (p = 1, 2, \cdots). \tag{3.2b}$$

Theorem 3.3 (For all classes in 5-g-UFT). *The change of the tensor $B_{\omega\mu\nu}$ induced by the conformal change (3.1) may be given by*

$$\begin{aligned} \overline{B}_{\omega\mu\nu} &= e^{\Omega}(B_{\omega\mu\nu} + k_{\nu[\omega}\Omega_{\mu]} - k_{\omega\mu}\Omega_\nu \\ &\quad -h_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_\delta + 2^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta}\Omega_\delta + k_{\omega\mu}{}^{(2)}k_\nu{}^{\delta}\Omega_\delta). \end{aligned} \tag{3.3}$$

Now, we are ready to derive representations of the changes $S_\omega \to \overline{S}_\omega$ in 5-g-UFT for the second class induced by the conformal change (3.1).

Theorem 3.4. *The conformal change (3.1) induces the following change :*

$$\begin{aligned} 2\overline{\overset{(10)1}{B}}_{\omega\mu\nu} = e^{\Omega}[\ & 2\overset{(10)1}{B}_{\omega\mu\nu} + (-2^{(4)}k_{\nu[\omega}k_{\mu]}{}^{\delta} \\ & +2^{(2)}k_{\nu[\omega}k_{\mu]}{}^{\delta} - k_{\nu[\omega}{}^{(2)}k_{\mu]}{}^{\delta})\Omega_\delta - {}^{(3)}k_{\nu[\omega}\Omega_{\mu]}]. \end{aligned} \tag{3.4}$$

Theorem 3.5. *The conformal change (3.1) induces the following change:*

$$\begin{aligned} \overline{\overset{ppq}{B}}_{\omega\mu\nu} &= e^{\Omega}[\overset{ppq}{B}_{\omega\mu\nu} + (-1)^p\{2^{(p+q+2)}k_{\nu[\omega}{}^{(p+1)}k_{\mu]}{}^{\delta} \\ &\quad +{}^{(2p+1)}k_{\omega\mu}{}^{(2+q)}k_\nu{}^{\delta} - {}^{(2p+1)}k_{\omega\mu}{}^{(q)}k_\nu{}^{\delta} \\ &\quad +{}^{(p+q+1)}k_{\nu[\omega}{}^{(p)}k_{\mu]}{}^{\delta} - {}^{(p+q)}k_{\nu[\omega}{}^{(p+1)}k_{\mu]}{}^{\delta}\}\Omega_\delta]. \end{aligned} \tag{3.5}$$

$$\left(\begin{array}{c} p = 0, 1, 2, 3, 4, \cdots \\ q = 0, 1, 2, 3, 4, \cdots \end{array} \right)$$

Theorem 3.6. *The change $S_\omega \to \overline{S}_\omega$ induced by conformal change (3.1) may be represented by*

$$\begin{aligned} \overline{S}_\omega &= S_\omega + \tfrac{1}{2C}[(-7K_2{}^3 + 11K_2{}^2 - 8K_2 + 1)k_\omega{}^\delta \Omega_\delta \\ &\quad -(3 - K_2 + K_2{}^2)^{(2)} k_\alpha{}^\alpha k_\omega{}^\delta \Omega_\delta] \end{aligned} \tag{3.6}$$

where $C = K_2{}^2 - 1$.

Proof. In virtue of (2.19) and Agreement (3.1), we have

$$(1 - \overline{K}_2{}^2)(\overline{S} - \overline{B}) = -2 \overset{\overline{(10)1}}{B} + (\overline{K}_2 - 1)\overset{\overline{110}}{B} + 2 \overset{\overline{(20)2}}{B} + 2 \overset{\overline{112}}{B}. \tag{3.7}$$

The relation(3.6) follows by substituting (3.2), (3.3), (3.4), (3.5), (2.17), Definition (2.1), Definition (2.2) into (3.7). □

Remark 3.7. The results (3.6) can also be obtained by using Theorem 3.6 of [4] and (2.16).

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