Kangweon-Kyungki Math. Jour. **11** (2003), No. 2, pp. 137–146

THE WEAK DENJOY$_*$ EXTENSION OF THE BOCHNER, DUNFORD, PETTIS AND MCSHANE INTEGRALS

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Abstract. In this paper we introduce the concepts of the weak Denjoy$_*$ integral of real-valued functions and the weak Denjoy$_*$-Dunford, weak Denjoy$_*$-Pettis, weak Denjoy$_*$-Bochner, weak Denjoy$_*$-McShane integrals of Banach-valued functions and then investigate some of their properties.

1. Introduction

The Denjoy integral of real-valued functions which is an extension of the Lebesgue integral was studied by some authors ([4],[5],[7]). R. A. Gordon [5] and S. Saks [9] also studied the Denjoy$_*$ integral of real-valued functions. The McShane integral of real-valued functions is an extension of the Riemann integral. The McShane integral of real-valued functions is equivalent to the Lebesgue integral. D. H. Fremlin [1] and D. H. Fremlin, J. Mendoza [2] studied the McShane integral of Banach-valued functions. J. L. Gamez, J. Mendoza [3] and R. A. Gordon [4] studied the Denjoy extension of the Bochner, Pettis and Dunford integrals. J. M. Park and D. H. Lee [8] defined the Denjoy-McShane integral which is an extension of the McShane integral and obtained some of their properties.

In this paper we introduce the concepts of the weak Denjoy$_*$ integral of real-valued functions which is an extension of the Denjoy$_*$ integral of real-valued functions and the weak Denjoy$_*$-Dunford, weak

Received July 7, 2003.
2000 Mathematics Subject Classification: 28B05, 26A39, 46G10.
Key words and phrases: Denjoy integral, weak Denjoy$_*$ integral, weak Denjoy$_*$-Dunford integral, weak Denjoy$_*$-Pettis integral, weak Denjoy$_*$-Bochner integral, weak Denjoy$_*$-McShane integral.

Denjoy$_*$-Pettis, weak Denjoy$_*$-Bochner, weak Denjoy$_*$-McShane integrals of Banach-valued functions and then investigate some of their properties.

2. Preliminaries

Throughout this paper, X will denote a real Banach space and X^* its dual. Let $\omega(F, [c, d]) = \sup\{\|F(y) - F(x)\| : c \le x < y \le d\}$ denote the oscillation of a function $F : [a, b] \to X$ on an interval $[c, d]$.

Definition 2.1[9]. Let $F : [a, b] \to X$ and let $E \subset [a, b]$.

(a) The function F is absolutely continuous on E (F is AC on E) if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \|F(d_i) - F(c_i)\| < \epsilon$ whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$. The function F is absolutely continuous in the restricted sense on E (F is AC$_*$ on E) if F is bounded on an interval that contains E and for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < \epsilon$ whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$.

(b) The function F is generalized absolutely continuous on E (F is ACG on E) if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC. The function F is generalized absolutely continuous in the restricted sense on E (F is ACG$_*$ on E) if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC$_*$.

Definition 2.2[9]. Let $F : [a, b] \to X$ and let $t \in (a, b)$. A vector z in X is the approximate derivative of F at t if there exists a measurable set $E \subset [a, b]$ that has t as a point of density such that $\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z$. We will write $F'_{ap}(t) = z$.

A function $f : [a, b] \to \mathbb{R}$ is Denjoy (resp. Denjoy$_*$) integrable on $[a, b]$ if there exists an ACG (resp. ACG$_*$) function $F : [a, b] \to \mathbb{R}$ such that $F'_{ap} = f$ (resp. $F' = f$) almost everywhere on $[a, b]$. The function f is Denjoy (resp. Denjoy$_*$) integrable on a set $E \subset [a, b]$ if $f\chi_E$ is Denjoy (resp. Denjoy$_*$) integrable on $[a, b]$.

Definition 2.3[4]. (a) A function $f : [a, b] \to X$ is Denjoy-Dunford integrable on $[a, b]$ if for each $x^* \in X^*$ the function x^*f is Denjoy integrable on $[a, b]$ and if for every interval I in $[a, b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all $x^* \in X^*$.

(b) A function $f : [a, b] \to X$ is Denjoy-Pettis integrable on $[a, b]$ if f is Denjoy-Dunford integrable on $[a, b]$ and if $x_I^{**} \in X$ for every interval I in $[a, b]$.

(c) A function $f : [a, b] \to X$ is Denjoy-Bochner integrable on $[a, b]$ if there exists an ACG function $F : [a, b] \to X$ such that F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$.

A function $f : [a, b] \to X$ is integrable in one of the above senses on a set $E \subset [a, b]$ if the function $f\chi_E$ is integrable in that sense on $[a, b]$.

Definition 2.4[6]. A McShane partition of $[a, b]$ is a finite collection $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ such that $\{[c_i, d_i] : 1 \le i \le n\}$ is a non-overlapping family of subintervals of $[a, b]$ covering $[a, b]$ and $t_i \in [a, b]$ for each $i \le n$. A gauge on $[a, b]$ is a function $\delta : [a, b] \to (0, \infty)$. A McShane partition $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ is subordinate to a gauge δ if $[c_i, d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $i \le n$. If $f : [a, b] \to X$ and if $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ is a McShane partition of $[a, b]$, we will denote $f(\mathcal{P})$ for $\sum_{i=1}^{n} f(t_i)(d_i - c_i)$. A function $f : [a, b] \to X$ is McShane integrable on $[a, b]$, with McShane integral z, if for each $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, \infty)$ such that $\|f(\mathcal{P}) - z\| < \varepsilon$ whenever $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \le i \le n\}$ is a McShane partition of $[a, b]$ subordinate to δ.

Definition 2.5[8]. A function $f : [a, b] \to X$ is Denjoy-McShane integrable on $[a, b]$ if there exists a continuous function $F : [a, b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG on $[a, b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable almost everywhere on $[a, b]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a, b]$.

3. The weak Denjoy$_*$ extension

In this section, we introduce the concepts of the weak Denjoy$_*$ integral of real-valued functions and the weak Denjoy$_*$-Dunford, weak Denjoy$_*$-Pettis, weak Denjoy$_*$-Bochner, weak Denjoy$_*$-McShane integrals of Banach-valued functions and then investigate some of their properties.

Definition 3.1. A function $f : [a,b] \to \mathbb{R}$ is weak Denjoy$_*$ integrable on $[a,b]$ if there exists an ACG$_*$ function $F : [a,b] \to \mathbb{R}$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$. The function f is weak Denjoy$_*$ integrable on a set $E \subset [a,b]$ if $f\chi_E$ is weak Denjoy$_*$ integrable on $[a,b]$.

Since every ACG$_*$ function is ACG, from [4, Theorem 24] we obtain the following:

Let $F : [a,b] \to \mathbb{R}$ be ACG$_*$ on $[a,b]$. If $F'_{ap} = 0$ almost everywhere on $[a,b]$, then F is constant on $[a,b]$.

This guarantees the uniqueness of the weak Denjoy$_*$ integral.

Note that Denjoy$_*$ integrability $\Longrightarrow$ weak Denjoy$_*$ integrability $\Longrightarrow$ Denjoy integrability.

Definition 3.2. (a) A function $f : [a,b] \to X$ is weak Denjoy$_*$-Dunford integrable on $[a,b]$ if for each $x^* \in X^*$ the function x^*f is weak Denjoy$_*$ integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all $x^* \in X^*$.

(b) A function $f : [a,b] \to X$ is weak Denjoy$_*$-Pettis integrable on $[a,b]$ if f is weak Denjoy$_*$-Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$.

(c) A function $f : [a,b] \to X$ is weak Denjoy$_*$-Bochner integrable on $[a,b]$ if there exists an ACG$_*$ function $F : [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$.

A function $f : [a,b] \to X$ is integrable in one of the above senses on a set $E \subset [a,b]$ if the function $f\chi_E$ is integrable in that sense on $[a,b]$.

Definition 3.3. A function $f : [a,b] \to X$ is weak Denjoy$_*$-McShane integrable on $[a,b]$ if there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG$_*$ on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is approximately differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$.

Note that if $f : [a,b] \to X$ is weak Denjoy$_*$-McShane integrable on $[a,b]$ then f is Denjoy-McShane integrable on $[a,b]$.

THEOREM 3.4. *If $f : [a,b] \to X$ is Bochner integrable on $[a,b]$, then f is weak Denjoy$_*$-Bochner integrable on $[a,b]$.*

Proof. If $f : [a,b] \to X$ is Bochner integrable on [a,b], then there exists an AC function $F : [a,b] \to X$ such that F is differentiable almost everywhere on $[a,b]$ and $F' = f$ almost everywhere on $[a,b]$. By [5, Theorem 6.4], $F : [a,b] \to X$ is AC$_*$. Hence F is ACG$_*$ and approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$. Thus f is weak Denjoy$_*$-Bochner integrable on $[a,b]$. □

THEOREM 3.5. *If $f : [a,b] \to X$ is Dunford integrable on $[a,b]$, then f is weak Denjoy$_*$-Dunford integrable on $[a,b]$.*

Proof. If $f : [a,b] \to X$ is Dunford integrable on [a,b], then x^*f is Lebesgue integrable on $[a,b]$ for each $x^* \in X^*$ and for every measurable subset E in $[a,b]$ there exists a vector x^{**}_E in X^{**} such that $x^{**}_E(x^*) = \int_E x^*f$ for all $x^* \in X^*$. Since x^*f is weak Denjoy$_*$ integrable on $[a,b]$ for each $x^* \in X^*$, f is weak Denjoy$_*$-Dunford integrable on $[a,b]$. □

THEOREM 3.6. *If $f : [a,b] \to X$ is Pettis integrable on $[a,b]$, then f is weak Denjoy$_*$-Pettis integrable on $[a,b]$.*

Proof. If $f : [a,b] \to X$ is Pettis integrable on [a,b], then f is Dunford integrable on $[a,b]$ and $x^{**}_E \in X$ for every measurable set E in $[a,b]$. Hence f is weak Denjoy$_*$-Dunford integrable on $[a,b]$ by Theorem 3.5 and $x^{**}_I \in X$ for every interval I in $[a,b]$. Thus f is weak Denjoy$_*$-Pettis integrable on $[a,b]$. □

Since every weak Denjoy$_*$ integrable function is Denjoy integrable, we can obtain the following theorem.

THEOREM 3.7. *(a) If $f : [a, b] \to X$ is weak Denjoy$_*$-Bochner integrable on $[a, b]$, then f is Denjoy-Bochner integrable on $[a, b]$.*

(b) If $f : [a, b] \to X$ is weak Denjoy$_*$-Dunford integrable on $[a, b]$, then f is Denjoy-Dunford integrable on $[a, b]$.

(c) If $f : [a, b] \to X$ is weak Denjoy$_*$-Pettis integrable on $[a, b]$, then f is Denjoy-Pettis integrable on $[a, b]$.

THEOREM 3.8. *If $f : [a, b] \to X$ is McShane integrable on $[a, b]$, then f is weak Denjoy$_*$-McShane integrable on $[a, b]$.*

Proof. Let $f : [a, b] \to X$ be McShane integrable on $[a, b]$. Then for each $x^* \in X^*$ x^*f is McShane integrable on $[a, b]$ and hence x^*f is Lebesgue integrable on $[a, b]$. Let $F(t) = (M) \int_a^t f$. Then F is continuous on $[a, b]$ by [6, Theorem 8] and for each $x^* \in X^*$ $x^*F(t) = (M) \int_a^t x^*f = (L) \int_a^t x^*f$. Hence x^*F is AC and so x^*F is ACG$_*$ and $(x^*F)' = x^*f$ almost everywhere on $[a, b]$. Thus f is weak Denjoy$_*$-McShane integrable on $[a, b]$. □

THEOREM 3.9. *If $f : [a, b] \to X$ is weak Denjoy$_*$-Bochner integrable on $[a, b]$, then f is weak Denjoy$_*$-McShane integrable on $[a, b]$.*

Proof. Let $f : [a, b] \to X$ be weak Denjoy$_*$-Bochner integrable on $[a, b]$. Then there exists an ACG$_*$ function $F : [a, b] \to X$ such that F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$. For each $x^* \in X^*$ x^*F is also ACG$_*$ on $[a, b]$ and x^*F is approximately differentiable almost everywhere on $[a, b]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a, b]$. Hence f is weak Denjoy$_*$-McShane integrable on $[a, b]$. □

THEOREM 3.10. *If $f : [a, b] \to X$ is weak Denjoy$_*$-McShane integrable on $[a, b]$, then f is weak Denjoy$_*$-Pettis integrable on $[a, b]$.*

Proof. Suppose that $f : [a, b] \to X$ is weak Denjoy$_*$-McShane integrable on $[a, b]$. Let $F(t) = (wD_*M) \int_a^t f$. Since x^*F is ACG$_*$ on $[a, b]$ and x^*F is approximately differentiable almost everywhere on $[a, b]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a, b]$ for each $x^* \in X^*$, x^*f is weak Denjoy$_*$ integrable on $[a, b]$ for each $x^* \in X^*$. For every interval $[c, d]$ in $[a, b]$, we have

$$\begin{aligned} x^*(F(d) - F(c)) &= x^*F(d) - x^*F(c) \\ &= (wD_*)\int_a^d x^*f - (wD_*)\int_a^c x^*f \\ &= (wD_*)\int_c^d x^*f. \end{aligned}$$

Since $F(d) - F(c) \in X$, f is weak Denjoy$_*$-Pettis integrable on $[a, b]$.□

Since every weak Denjoy$_*$-Pettis integrable function $f : [a, b] \to X$ is Denjoy-Pettis integrable, we can obtain the following theorem from [4, Theorem 38].

THEOREM 3.11. *Suppose that X contains no copy of c_0 and let $f : [a, b] \to X$. If f is weak Denjoy$_*$-Pettis integrable on $[a, b]$, then every perfect set in $[a, b]$ contains a portion on which f is Pettis integrable.*

THEOREM 3.12. *Suppose that X contains no copy of c_0 and let $f : [a, b] \to X$ be measurable. If f is weak Denjoy$_*$-McShane integrable on $[a, b]$, then every perfect set in $[a, b]$ contains a portion on which f is McShane integrable.*

Proof. Let f be weak Denjoy$_*$-McShane integrable on $[a, b]$. Since f be weak Denjoy$_*$-Pettis integrable on $[a, b]$ by Theorem 3.10, every perfect set in $[a, b]$ contains a portion on which f is Pettis integrable by Theorem 3.11. Since f is measurable, f is McShane integrable on that portion by [6, Theorem 17]. □

EXAMPLE 3.13. A weak Denjoy$_*$-Bochner integrable function that is not Bochner integrable.

Let X be an infinite dimensional Banach space. By the Dvoretsky-Rogers Theorem there exists a series $\sum_n x_n$ in X that converges unconditionally but not absolutely. For each positive integer n let $I_n = (\frac{1}{n+1}, \frac{1}{n})$ and define $f : [0, 1] \to X$ by $f(t) = \frac{1}{\mu(I_n)}\, x_n$ for t in I_n and $f(t) = 0$ for all other values of t. The function f is measurable since it is countably valued, but f is not Bochner integrable on $[0, 1]$ since

$$\int_0^1 \|f\| = \sum_n \int_{I_n} \frac{1}{\mu(I_n)} \|x_n\| = \sum_n \|x_n\| = \infty.$$

Define $F : [0,1] \to X$ by

$$F(t) = \frac{t - \frac{1}{n+1}}{\mu(I_n)}\, x_n + \sum_{k=n+1}^{\infty} x_k \text{ for } t \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

and $F(0) = 0$. Then F is continuous on $[0,1]$ and $F' = f$ almost everywhere on $[0,1]$ and hence $F'_{ap} = f$ almost everywhere on $[0,1]$. Furthermore, the function F is ACG$_*$ on $[0,1]$ since F is AC$_*$ on $\{0\}$ and on each of the intervals $[\frac{1}{n+1}, \frac{1}{n}]$. Hence the function f is weak Denjoy$_*$-Bochner integrable on $[0,1]$.

Example 3.14. A weak Denjoy$_*$-McShane integrable function that is not weak Denjoy$_*$-Bochner integrable.

Let $\{r_k\}$ be a listing of the rational numbers in $[0,1)$ and for each pair of positive integers n and k let

$$I_n^k = \left(r_k + \frac{1}{n+1}, r_k + \frac{1}{n}\right).$$

For each k define $f_k : [0,1] \to l_2$ by $f_k(t) = \{(n+1)\chi_{I_n^k}(t)\}$. Then the series $\sum_k 4^{-k} f_k$ is l_2-valued almost everywhere on $[0,1]$. For each positive integer j let $A_j = \cup_k \{t \in [0,1] : |t - r_k| < 2^{-j-k}\}$ and let $A = \cap_j A_j$. Then $\mu(A) = 0$ and $\{r_k\} \subset A$.

Define $g : [0,1] \to l_2$ by $g(t) = \sum_k 4^{-k} f_k(t)$ for t in $[0,1] - A$ and $g(t) = 0$ for t in A. Then g is measurable and Pettis integrable on $[0,1]$, but not Denjoy-Bochner integrable on $[0,1]$ by [4, Example 42]. By [6, Theorem 17], g is McShane integrable on $[0,1]$, but not weak Denjoy$_*$-Bochner integrable on $[0,1]$. By Theorem 3.8, g is weak Denjoy$_*$-McShane integrable on $[0,1]$.

Example 3.15. A weak Denjoy$_*$-Pettis integrable function that is not weak Denjoy$_*$-McShane integrable.

For each positive integer n let

$$I'_n = \left(\frac{1}{n+1}, \frac{n+\frac{1}{2}}{n(n+1)}\right), \quad I''_n = \left(\frac{n+\frac{1}{2}}{n(n+1)}, \frac{1}{n}\right)$$

and define $f_n = [0,1] \to \mathbb{R}$ by $f_n(t) = 2n(n+1)(\chi_{I'_n}(t) - \chi_{I''_n}(t))$. Define $f : [0,1] \to c_0$ by $f(t) = \{f_n(t)\}$. Then by [4, Example 44]

x^*f is Lebesgue integrable on $[0,1]$ for each $x^* = \{\alpha_n\} \in l_1$. Hence x^*f is weak Denjoy$_*$ integrable on $[0,1]$ for each $x^* = \{\alpha_n\} \in l_1$ and $\int_E f = \{\int_E f_n\}$ for every measurable set $E \subset [0,1]$. For each interval $I \subset [0,1]$ we have $\int_I f \in c_0$ by the choice of $\{f_n\}$ and it follows that f is weak Denjoy$_*$-Pettis integrable on $[0,1]$. By [8, Example 3.8], f is not Denjoy-McShane integrable on $[0,1]$. Hence f is not weak Denjoy$_*$-McShane integrable on $[0,1]$.

THEOREM 3.16. *If $f : [a,b] \to X$ is weak Denjoy$_*$-Dunford integrable on $[a,b]$, then f is weakly measurable.*

Proof. If $f : [a,b] \to X$ is weak Denjoy$_*$-Dunford integrable on $[a,b]$, then x^*f is weak Denjoy$_*$ integrable on $[a,b]$ for all $x^* \in X^*$. Hence x^*f is measurable for all $x^* \in X^*$ by [4, Theorem 12]. Thus f is weakly measurable. □

THEOREM 3.17. *If $f : [a,b] \to X$ is bounded and weak Denjoy$_*$-Dunford integrable on $[a,b]$, then f is Dunford integrable on $[a,b]$.*

Proof. If $f : [a,b] \to X$ is bounded and weak Denjoy$_*$-Dunford integrable on $[a,b]$, then x^*f is bounded and weak Denjoy$_*$ integrable on $[a,b]$ for all $x^* \in X^*$. Hence x^*f is Lebesgue integrable on $[a,b]$ for all $x^* \in X^*$ by [5, Theorem 15.9]. Thus f is Dunford integrable on $[a,b]$. □

References

1. D. H. Fremlin, *The Henstock and McShane integrals of vector-valued functions*, Illinois J. Math. **38** (1994), 471-479.
2. D. H. Fremlin and J. Mendoza, *On the integration of vector-valued functions*, Illinois J. Math. **38** (1994), 127-147.
3. J. L. Gamez and J. Mendoza, *On Denjoy-Dunford and Denjoy-Pettis integrals*, Studia Math. **130** (1998), 115-133.
4. R. A. Gordon, *The Denjoy extension of the Bochner, Pettis and Dunford integrals*, Studia Math. **92** (1989), 73-91.
5. ———, *The integrals of Lebesgue, Denjoy, Perron and Henstock*, Grad. Stud. Math. 4, Amer. Math. Soc., Providence, R.I., 1994.
6. ———, *The McShane integral of Banach-valued functions*, Illinois J. Math. **34** (1990), 557-567.

7. S. Hu and V. Lakshmikantham, *Some remarks on generalized Riemann integral*, J. Math. Anal. Appl. **137** (1989), 515-527.
8. J. M. Park and D. H. Lee, *The Denjoy extension of the McShane integral*, Bull. Korean Math. Soc. **33 No. 3** (1996), 411-417.
9. S. Saks, *Theory of the integral*, Dover, New York, 1964.

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