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ON RIGHT(LEFT) DUO PO-SEMIGROUPS

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Abstract. We investigate some properties on right(resp. left) duo *po*-semigroups.

1. Introduction

Kehayopulu([6]) prove that every ideal of an $\mathcal{N}$-class of an ordered semigroup does not contain proper prime ideals. As a consequence, each prime ideal of an ordered semigroup is decomposable into its $\mathcal{N}$-classes.

In this paper, we give the relation between the left(resp. right) filters and the prime left(resp. right) ideals. We define a semilattice congruence $\mathcal{N}_l$(resp. $\mathcal{N}_r$) generated by the left(resp. right) filter on a right(resp. left) duo *po*-semigroup and investigate some properties on the right(resp. left) duo *po*-semigroups. Also we prove that every left(resp. right) ideal of $\mathcal{N}_l$-classresp. $\mathcal{N}_r$-class) of a right(resp. left) duo *po*-semigroup does not contain the proper prime left(resp. right) ideals. As a consequence, each prime left(resp. right) ideal of a right(resp. left) duo *po*-semigroup is decomposable into its $\mathcal{N}_l$-classes(resp. $\mathcal{N}_r$-classes).

A *po-semigroup*(: ordered semigroup) is an ordered set S at the same time a semigroup such that $a \leq b \Longrightarrow xa \leq xb$ and $ax \leq bx$ for all $x \in S$.

Let S be a *po*-semigroup. A nonempty subset A of S is called a *left*(resp. *right*) *ideal* of S if (1) $SA \subseteq A$(resp. $SA \subseteq A$), (2) $a \in A$ and $b \leq a$ for $b \in S \Longrightarrow b \in A$([3,5]). A is called an *ideal* of S if it is a left and right ideal of S.

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A *po*-semigroup S is said to be *right*(resp. *left*) *duo* if every right(resp. left) ideal is a left(resp. right) ideal([4,5]).

A non-empty subset T of a *po*-semigroup S is said to be *prime* if $AB \subseteq T \Longrightarrow A \subseteq T$ or $B \subseteq T$ for subsets A, B of S([8]). Equivalent Definition: For elements a, b in a subset T $ab \in T \Rightarrow a \in T$ or $b \in T$. T is called a *prime left*(resp. *right*) *ideal* if T is prime as a left(resp. right) ideal([2]).

A non-empty subsemigroup F of a *po*-semigroup S is called a *left*(resp. *right*) *filter* of S if (1) $ab \in F$ for $a, b \in S \Longrightarrow b \in F$(resp. $a \in F$), (2) $a \in F$, $a \leq c$ for $c \in S \Longrightarrow c \in F$([9]). A subsemigroup F of S is called a *filter* of S if F is a left and right filter ([2,4,5]).

An equivalence relation σ on S is called a *left congruence*(resp. *right congruence*) on S if $(a, b) \in \sigma \Longrightarrow (ac, bc) \in \sigma$(resp. $(ca, cb) \in \sigma$) for all $c \in S$. An equivalence relation σ on S is called a *congruence* it is a left and right congruence. A relation σ is called a *semilattice congruence* on S if σ is a congruence such that $(x^2, x) \in \sigma$ and $(xy, yx) \in \sigma$([1,2,4]).

Notation. For a semilattice congruence σ, $(z)_\sigma$ is a class of the semilattice congruence σ containing an element z in a *po*-semigroup S.

2. Main Results.

LEMMA ([9]). *Let S be a po-semigroup and F a nonempty subset of S. The following are equivalent:*

1) *F is a left(resp. right) filter of S.*
2) *$S \setminus F = \emptyset$ or $S \setminus F$ is a prime left(resp. right) ideal of S.*

From Lemma, we get the following corollary.

COROLLARY 1([2]). *Let S be a po-semigroup and F a nonempty subset of S. The following are equivalent:*

1) *F is a filter of S.*
2) *$S \setminus F = \emptyset$ or $S \setminus F$ is a prime ideal of S.*

PROPOSITION 1. *A po-semigroup S does not contain proper left(resp. right) filters if and only if S does not contain proper prime left(resp. right) ideals.*

Proof. $\Rightarrow$. Assume that S contains a proper prime left ideal L of S. Then $\emptyset \neq S \setminus L \subset S$. Since $S \setminus (S \setminus L) = L$, we note that $S \setminus (S \setminus L)$

is a prime left ideal of S. By Lemma 1, $S \setminus L$ is a proper left filter of S. It is impossible. Hence S does not contain proper prime left ideals.

$\Leftarrow$. Suppose that F is a proper left filter of S. Then $S \setminus F \neq \emptyset$. By Lemma 1, $S \setminus F$ is a proper prime left ideal of S. It is impossible. Hence S does not contain proper prime left filters. $\square$

By Proposition 1, we have the following corollary.

Corollary 2([6, Remark 2]). *A po-semigroup S does not contain proper filters if and only if S does not contain proper prime ideals.*

Now we define a relation "$\mathcal{N}_l$" on a *po*-semigroup S as follows:

$$\mathcal{N}_l := \{(x, y)|N_l(x) = N_l(y)\}, \quad \mathcal{N}_r := \{(x, y)|N_r(x) = N_r(y)\}$$

where $N_l(x)$ (resp. $N_r(x)$) is the left (resp. right) filter of S generated by $x \in S$.

Proposition 2. *$\mathcal{N}_l$ (resp. $\mathcal{N}_r$) is a semilattice congruence on a right(resp. left) duo po-semigroup S.*

Proof. It is easy to check that $\mathcal{N}_l$ is an eguivalence relation on S.

Let $(x, y) \in \mathcal{N}_l$. Then $N_l(x) = N_l(y)$. Since $xz \in N_l(xz)$ for all $z \in S$ and $N_l(xz)$ is a left filter, we get $x \in N_l(xz)$ and $z \in N_l(xz)$. Thus $N_l(x) \subseteq N_l(xz)$ and so $y \in N_l(y) = N_l(x) \subseteq N_l(xz)$. Since $y, z \in N_l(xz)$ and $N_l(xz)$ is a subsemigroup of S, we get $yz \in N_l(xz)$. Therefore $N_l(yz) \subseteq N_l(xz)$. By symmetry, we get $N_l(xz) \subseteq N_l(yz)$. Hence $N_l(xz) = N_l(yz)$. Therefore $\mathcal{N}_l$ is a right congruence.

Now we shall show that $(x^2, x) \in \mathcal{N}_l$. Let $x \in S$. Since $x^2 \in N_l(x^2)$ and $N_l(x^2)$ is a left filter, we get $x \in N_l(x^2)$. Thus $N_l(x) \subseteq N_l(x^2)$. Since $x \in N_l(x)$ and $N_l(x)$ is a subsemigroup of S, we get $x^2 \in N_l(x)$. Hence $N_l(x^2) \subseteq N_l(x)$. Therefore $N_l(x^2) = N_l(x)$, and so $(x^2, x) \in \mathcal{N}_l$.

Next we shall show that $(xy, yx) \in \mathcal{N}_l$. Let $x, y \in S$. Since $xy \in N_l(xy)$ and $N_l(xy)$ is a left filter, we have $x \in N_l(xy)$. Suppose that $y \notin N_l(xy)$. Then $y \in S \setminus N_l(xy)$. Since $S \setminus N_l(xy)$ is a prime right ideal and S is a right duo, $xy \in S(S \setminus N_l(xy)) \subseteq S \setminus N_l(xy)$. It is impossible. Thus $y \in N_l(xy)$. Since $N_l(xy)$ is a filter, $yx \in N_l(xy)$. Thus $N_l(yx) \subseteq N_l(xy)$. By symmetry, $N_l(xy) \subseteq N_l(yx)$. Therefore $N_l(xy) = N_l(yx)$ and so $(xy, yx) \in \mathcal{N}_l$.

Finally, we shall show that $\mathcal{N}_l$ is a left congruence. Let $(x, y) \in \mathcal{N}_l$, and $z \in S$. Then $N_l(zx) = N_l(xz) = N_l(yz) = N_l(zy)$.
Therefore $\mathcal{N}_l$ is a left congruence. It follows that $\mathcal{N}_l$ is a semilattice congruence. □

PROPOSITION 3. *Let S be a po-semigroup. If F is a left filter of S and $F \cap (z)_{\mathcal{N}_l} \neq \emptyset$ for $z \in S$, then $(z)_{\mathcal{N}_l} \subseteq F$.*

Proof. Assume that F is a left filter of S and $a \in F \cap (z)_{\mathcal{N}_l}$ for $z \in S$. If $y \in (z)_{\mathcal{N}_l}$ then $(y)_{\mathcal{N}_l} = (z)_{\mathcal{N}_l} = (a)_{\mathcal{N}_l}$. Thus $(y, a) \in \mathcal{N}_l$, and so $N_l(y) = N_l(a)$. Since F is a left filter of S and $a \in F$, we have $N_l(a) \subseteq F$. Thus $y \in N_l(y) = N_l(a) \subseteq F$. Hence $(z)_{\mathcal{N}_l} \subseteq F$. □

PROPOSITION 4. *For a po-semigroup S, $a \leq b$ implies $(a, ba) \in \mathcal{N}_l$ and $(a, ab) \in \mathcal{N}_r$.*

Proof. Suppose that $a \leq b$. Since $a \in N_l(a)$ and $N_l(a)$ is a left filter, we get $b \in N_l(a)$. Thus $ba \in N_l(a)$, and so $N_l(ba) \subseteq N_l(a)$. Since $ba \in N_l(ba)$ and $N_l(ba)$ is a left filter, we have $a \in N_l(ba)$. Thus $N_l(a) \subseteq N_l(ba)$. Hence $N_l(a) = N_l(ba)$, and so $(a, ba) \in \mathcal{N}_l$.

By symmetry, we can prove that $(a, ab) \in \mathcal{N}_r$. □

PROPOSITION 5. *Let S be a right duo po-semigroup. If L is a left ideal of $(z)_{\mathcal{N}_l}$ for $z \in S$ then L does not contain proper prime left ideals.*

Proof. From Proposition 1, it is sufficient to prove that L does not contain proper left filters (of L). Let F be a left filter of L and $a \in F$. Now we define $T := \{x \in S \mid a^2x \in F\}$. Then T is a nonempty set, since $a^2a = a^3 \in F$.

Now we show that $F = T \cap L$. If $y \in F$, then $a^2y \in F$. Thus $y \in T$. Since F is a left filter of L, $F \subseteq L$. Hence $y \in T \cap L$, and so $F \subseteq T \cap L$. Conversely, if $y \in T \cap L$, then $a^2y \in F$. Since F is a left filter of L, we get $y \in F$. Therefore $F = T \cap L$.

Next we show that T is a left filter of L. If $x \in T$ and $y \in T$, then $a^2x, a^2y \in F$. Since F is a left filter, we have $x, y \in F$. Since $a \in F$, $a^2xy \in F$. Thus $xy \in T$. If $xy \in T$ for $x, y \in L$, then $(a^2x)y = a^2(xy) \in F$. Since F is a left filter of L, we get $y \in F$. If $x \in T$ and $x \leq y$ for $y \in L$, then $a^2x \in F$. Since $x \leq y$, we get $a^2x \leq a^2y$. Since F is a left filter, $a^2y \in F$. Thus $y \in T$. Therefore T is a left filter of L.

We note that $a \in F = T \cap L \subseteq L \subseteq (z)_{\mathcal{N}_l}$, and so $T \cap (z)_{\mathcal{N}_l} \neq \emptyset$. Since T is a left filter of L, we have $(z)_{\mathcal{N}_l} \subseteq T$ by Proposition 3. Thus $L = (z)_{\mathcal{N}_l} \cap L \subseteq T \cap L = F \subseteq L$, and so $F = L$. Hence L does not contain proper left filters (of L). Therefore by Proposition 1, L does not contain proper prime right ideals. □

PROPOSITION 6. *Let S be a right duo po-semigroup and L a prime left ideal of S. Then $L = \cup\{(x)_{\mathcal{N}_l} \mid x \in L\}$.*

Proof. Let $t \in (x)_{\mathcal{N}_l}$ for some $x \in L$. Since $(x)_{\mathcal{N}_l}$ is a left ideal of $(x)_{\mathcal{N}_l}$, $(x)_{\mathcal{N}_l}$ does not contain proper prime left ideals by Proposition 5. If we prove that $(x)_{\mathcal{N}_l} \cap L$ is a prime left ideal of $(x)_{\mathcal{N}_l}$ then $(x)_{\mathcal{N}_l} \cap L = (x)_{\mathcal{N}_l}$.

We first show that $(x)_{\mathcal{N}_l} \cap L$ is a left ideal of $(x)_{\mathcal{N}_l}$. We note that $(x)_{\mathcal{N}_l} \cap L \neq \emptyset$ since $x \in (x)_{\mathcal{N}_l} \cap L$. And $(x)_{\mathcal{N}_l}((x)_{\mathcal{N}_l} \cap L) = {(x)_{\mathcal{N}_l}}^2 \cap (x)_{\mathcal{N}_l}L \subseteq (x)_{\mathcal{N}_l} \cap SL \subseteq (x)_{\mathcal{N}_l} \cap L$. Let $a \in (x)_{\mathcal{N}_l} \cap L$ and $b \leq a$ for $b \in (x)_{\mathcal{N}_l}$. Since L is a left ideal of S, b is contained in L. Thus $b \in (x)_{\mathcal{N}_l} \cap L$. Hence $(x)_{\mathcal{N}_l} \cap L$ is a left ideal of $(x)_{\mathcal{N}_l}$.

Finally, we show that $(x)_{\mathcal{N}_l} \cap L$ is prime in $(x)_{\mathcal{N}_l}$. Let $yz \in (x)_{\mathcal{N}_l} \cap L$ for $y, z \in (x)_{\mathcal{N}_l}$. Since $yz \in L$ and L is a prime left ideal of S, y is contained in L or z is contained in L. Hence $y \in (x)_{\mathcal{N}_l} \cap L$ or $z \in (x)_{\mathcal{N}_l} \cap L$. Therefore $(x)_{\mathcal{N}_l} \cap L$ is a prime left ideal of $(x)_{\mathcal{N}_l}$.

It follows that

$$L \subseteq \cup\{(x)_{\mathcal{N}_l} \mid x \in L = \cup\{(x)_{\mathcal{N}_l} \cap L \mid x \in L\} \subseteq L.$$

Therefore $L = \cup\{(x)_{\mathcal{N}_l} \mid x \in L\}$. □

B. y similar methods of Proposition 3, 5 and 6, we have the followings:

(1) If F is a right filter of a *po*-semigroup S and $F \cap (z)_{\mathcal{N}_r} \neq \emptyset$ for $z \in S$, then $(z)_{\mathcal{N}_r} \subseteq F$.

(2) If R is a right ideal of $(z)_{\mathcal{N}_r}$ of left duo *po*-semigroups then R does not contain proper prime right ideals.

(3) If R is a prime right ideal of left duo *po*-semigroups, then

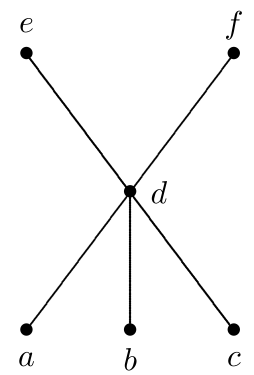
$$R = \cup\{(x)_{\mathcal{N}_r} | x \in R\}.$$

3. Examples

Now we give an example of a left filter which is not a right filter in *po*-semigroups and an example of a left and right filter in a *po*-semigroup.

EXAMPLE 1([7]). Let $S := \{a, b, c, d, e, f\}$ be a *po*-semigroup with Cayley table and Hasse diagram on S as follows:

$\cdot$	a	b	c	d	e	f
a	b	c	d	d	d	d
b	c	d	d	d	d	d
c	d	d	d	d	d	d
d	d	d	d	d	d	d
e	e	e	e	e	e	e
f	f	f	f	f	f	f



The set $A := \{e, f\}$ is a left filter, but not a right filter of S. Thus A is not a filter of S.

EXAMPLE 2([8]). Let $S := \{a, b, c, d, e, f\}$ be a *po*-semigroup with Cayley table (Table 2) and Hasse diagram (Figure 2) on S as follows:

$\cdot$	a	b	c	d	e	f
a	a	b	b	d	e	f
b	b	b	b	b	b	b
c	b	b	b	b	b	b
d	d	b	b	d	e	f
e	e	f	f	e	e	f
f	f	f	f	f	f	f

Table 2

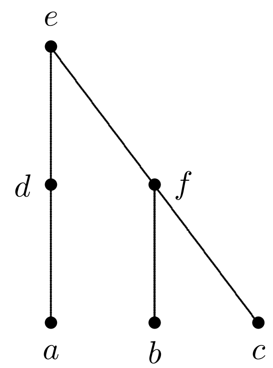


Figure 2

The set $B := \{a, d, e\}$ is a left and right filter of S, and so B is a filter of S.

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