사각형 박판의 비선형 열탄성 응력 수치해석

김치경† · 김성중
인천대학교 안전공학과
(2003. 3. 22. 접수 / 2004. 10. 4. 채택)

Numerical Analysis of Nonlinear Thermoelastic Stress for Rectangular Thin Plate

Chi-Kyung Kim† · Sung-Jung Kim
Department of Safety Engineering, University of Incheon
(Received March 22, 2003 / Accepted October 4, 2004)

Abstract : A simply supported rectangular thin plate with temperature distribution varying over the thickness is analyzed. Since the thermal deflections are large compared to the plate thickness during bending and membrane stresses are developed and as such a nonlinear stress analysis is necessary. For the geometrically nonlinear, large deflection behavior of the plate, the classical von Karman equations are used. These equations are solved numerically by using the finite difference method. An iterative technique is employed to solve these quasi-linear algebraic equations. The results obtained from the suggested method are presented and discussed.

요 약 : 판의 두께에 선형적으로 변화하는 온도분포의 열하중을 받는 단순지지의 사각형 박판을 해석하였다. 열에 의한 판의 처짐이 판두께에 비해 상대적으로 과대하여 막응력이 부수적으로 발생하여 문제는 비선형 해석이 된다. 큰 처짐을 가지는 기하학적 비선형 문제를 지배하는 기본방정식은 von Karman 방정식이 사용되며 차분법으로 수치해석 한다. 차분화 하여 얻어지는 유사선형 대수방정식은 반복법을 도입하여 해석하고 결과치를 해석적으로 얻은 해와 비교 검토한다.
Key Words : thermoelastic stress, finite difference method, nonlinear equation, iteration

1. Introduction

One of the causes of stress in a body is nonuniform heating. With rising temperature the elements of a body expand. Such an expression generally cannot proceed freely in a continuous body, and stresses due to the heating are set up.

Fracture of glass when a surface is rapidly heated is attributable to such stress.

Fatigue failure can occur as a result of temperature fluctuations. The consequences of such thermal stress are important in many aspects of engineering design, as in turbines, jet engines, and nuclear reactors. Also the determination of thermal and shrinkage stresses in restrained plates subjected to arbitrary temperature variation and shrinkage is of practical importance and often arises in many concrete constructions. The magnitudes of thermal and shrinkage stresses are necessary for determining the size of concrete block in massive concrete construction or for designing the shear connectors in case of composite construction to prevent cracking of concrete. A number of classical articles on the thermal stress analysis are presented in the literature[2,3,6,7]. Approximate solutions for a rectangular plate restrained along an edge have been given by Aleck[1] using energy methods, in which both normal and shear stresses along the edge have been considered. For the same conditions of restraint, Kobotake and Inoue[5] also

†To whom correspondence should be addressed.
kimchi@incheon.ac.kr

have been approximate solutions for a rectangular plate restrained along an edge and two opposite edges. In most of these investigations, either boundary conditions on the free sides or clamping conditions are not completely satisfied. Further, most of these investigators obtain the solution to the linear problem by converting the thermal stress problem into one of specified surface traction. Zienkiewicz[11)] proposed the numerical solutions using a finite difference procedures for the problem of elastically restrained plate. He first employed a finite difference procedure for linear problems to take advantage of the special characteristics of the matrices. Vallabhan[10)] solved the nonlinear plate problem using finite difference method in which the large deflected plate was solved using a direct method. The primary objective of this study is to develop a finite difference formulation of thin plate with temperature distribution varying over the thickness so that a iterative technique can be used to solve the plate numerically. A brief derivation of the nonlinear thermoelastic equations is given here, where the thermoelastic conditions are added on to the classical von Karman equations[8,9)]. A iterative procedure for nonlinear problems to take advantage of the characteristics of the matrices will be employed. These general equations are used for computing thermal stresses in the plate by using the finite difference method[4)].

2. Nonlinear Thermoelastic Equations of Plate

For a thin plate of thickness t for large deflections, if u, v, w are the displacements along the x, y, z axes for the middle surface of the plate, according to von Karman, the equations for plate is governed by the following differential equations

$$D \nabla^4 w = p + tL(w,\phi) - \nabla^2 M^* \tag{1}$$

$$\nabla^4 \phi = -\frac{E}{2} L(w,w) - \frac{1-\mu}{t} \nabla^2 N^* \tag{2}$$

Eq.(1) and Eq.(2) are the two major domain equations to be solved satisfying the necessary boundary conditions. The general boundary conditions for bending behavior at the edge $y=a$ are represented by

$$w = \overline{w}$$
$$w_{,x} = \overline{g}_x$$
$$V_x = -D\,[w_{,xxx} + (2-\mu) w_{,xyy}] = \overline{V_x}$$
$$M_x = -D\,[w_{,xx} + \mu\, w_{,yy}] - M^* = \overline{M_x}$$
$$-M_x^* = \overline{M^*}_x$$

Similar equations can be developed for the edge representing $y=b$. For membrane behavior the boundary conditions at $y=a$ are represented by

$$\sigma_x = t\,\phi_{,yy} = \overline{N}_x$$
$$\tau_{xy} = -t\,\phi_{,xy} = \overline{N}_{xy}$$

Similar equations can be developed for the other edge. Here the quantities

$$M^* = \frac{\alpha E}{1-\mu} \int_{-t/2}^{t/2} (\triangle T)\, z\, dz \tag{3}$$

$$N^* = \frac{\alpha E}{1-\mu} \int_{-t/2}^{t/2} (\triangle T)\, dz \tag{4}$$

are termed thermal stress resultants. The differential operator L, applied to w and φ, is defined as

$$L(w,\phi) = [w_{,xx}\phi_{,yy} - 2\phi_{,xy}w_{,xy} + w_{,yy}\phi_{,xx}] \tag{5}$$

Here $w(x,y)$ is the transverse displacement, $D=Et^3/12(1-\mu^2)$ is the flexural rigidity of the plate, t is the plate thickness, φ is the membrane stress function, $p(x, y)$ is the lateral pressure, α is the coefficient of thermal expansion, $\triangle T$ is the change in temperature, E is the Young's modulus or the modulus of elasticity, μ is the Poisson's ratio, ∇^4 is the Biharmonic operator, $w_{,x} = \partial w/\partial x$ and $\varphi_{,xx} = \partial^2 \varphi/\partial x^2$. These two equations are nonlinear and coupled. Eq.(1) represents the bending behavior of the plate due to the combined effects of lateral load, in-plane membrane stresses and temperature gradients across the thickness. Eq.(2) represents the membrane behavior of the plate due to lateral displacements and in-plane temperature distributions.

3. Mathematical Model for the Temperature and Thermal Stress

The temperature condition on the plate is modeled by considering temperature distributions on the top and bottom of the plate. By the temperatures T_t and T_b at all nodal points at top and bottom respectively, the mean temperatures T_m at the middle surface and temperature gradients $(T_t - T_b) / t$ across the thickness of the plate are calculated for each nodal point. A linear variation of temperature across the thickness of the plate is assumed. The difference in temperature $\triangle T$ for use in Eq.(3) and (4) is given by

$$\triangle T = T_m + \frac{(T_t - T_b)}{t} z$$

Substituting $\triangle T$ into Eq.(3) and (4), we get

$$N^* = \frac{E\alpha t}{(1-\mu)} N^*$$

and $$M^* = \frac{E\alpha t^2}{12(1-\mu)} (T_t - T_b)$$

Values of N^* and M^* are evaluated at each and every nodal point using the above equations. Using the finite difference models, values of $\nabla^2 M^*$ and $\nabla^2 N^*$ at each node are calculated from the Eq.(1) and (2), and these values represent the effects of the thermal loading on the plate. From the displacements and membrane functions obtained for the temperature condition, the thermal stresses are computed from the following expressions:

At the top of the plate,

$$\sigma_x = \phi_{,yy} + \frac{6}{t^2}[-D(w_{,xx} + \mu w_{,yy}) - M^*] \tag{6}$$

$$\sigma_y = \phi_{xx} + [-D(w_{,yy} + \mu w_{,xx}) - M^*] \tag{7}$$

$$\tau_{xy} = -\phi_{xy} - \frac{6}{t^2} D(1-\mu) w_{,xy} \tag{8}$$

Similar expressions can be written for the stresses at the bottom of the plate by changing the signs of the bending stresses.

4. Finite Difference Model

The method of finite differences replaces the plate differential equation and the expressions defining the boundary conditions with equivalent difference equation of a set of algebraic equations, written for every nodal point within the plate. The differential equations to be solved in this problem are Eq.(1) and (2). Since the left hand sides of the equations contain a biharmonic operator, standard patterns for developing the algebraic finite difference equations are represented by finite difference models. Fig. 1. shows the computational models for nodes close to the boundary.

Using central difference models with equal increments along the x and y axes, Eq.(1) and (2) are replaced by two sets of algebraic equations, and the nonlinear differential equations have been transformed into two sets of coupled equations as

$$[M]\{w\} = \{p\} + L_1(w, \varphi) + \{D_b\} \tag{9}$$

and

$$[N]\{\varphi\} = L_2(w) + \{D_m\} \tag{10}$$

where

$[M]$ and $[N]$ = Biharmonic operators
w = vector representing lateral displacement
p = vector representing load
φ = vector representing membrane function
L_1 and L_2 = nonlinear functions representing part of right side of von Karman's equations.

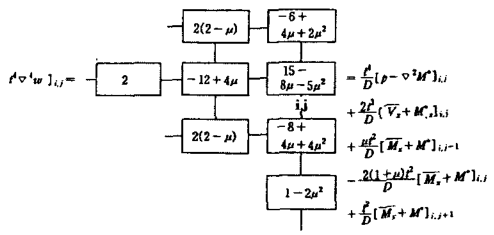

Fig. 1. Biharmonic equation on the edge near the corner of the plate

Vectors D_b and D_m correspond to the thermal terms associated with the bending and membrane equations. Both matrices M and N are symmetric and positive definite. It can be seen that Eq.(9) represents lateral deflection, while Eq.(10) represents the membrane stress function.

5. Iterative Procedure

The iterative procedure is a sequence of calculations in which the body is fully loaded in each iteration. Since the thermal deflections are large compared to the plate thickness during bending and membrane stresses, a nonlinear stress analysis is necessary. Using values of w and φ from the i-th iteration the L_1 function can be calculated numerically from the expression for $L(w, \varphi)$. The first von Karman's equation for the $(i+1)$-th iteration becomes

$$[M]\{w^{i+1}\} = \{p\} + L_1(w^i, \varphi^i) + \{D_b\} \tag{11}$$

From this, w^{i+1} can be determined. Now that w^{i+1} is known, it can be substituted into the right hand side of the second von Karman's equation such as that Eq.(10) becomes

$$[N]\{\varphi^{i+1}\} = L_2(w^{i+1}) + \{D_m\}\{D_b\} \tag{12}$$

and from this φ^{i+1} can be obtained. The iteration is repeated until a satisfactory convergence is reached for the displacement vector w^i. An error term is used to end the iteration when convergence is reached in the computation of w

$$\delta^{i+1} = \frac{\sum_{k=1}^{N} | w^{i+1,k} - w^{i,k} |}{N} \le \beta (w_{\max})^{i+1} \tag{13}$$

where

i = iteration number
k = node number
N = number of nodes in the grid
β = iterative tolerance number

The above iterative scheme will converge only for very small deflections. When deflections become large, the scheme will diverge. For the solution technique the steps during a typical load increment can be summarized as follows:

1. Assume initial values of w and φ. Let w^i and φ^i be the value for the i-th iteration.
2. Use w^i and φ^i to determine the values of the vector $L_1(w, \varphi)$ in Eq.(9)
3. Solve for w^{i+1}
4. Use w^{i+1} to obtain the value of vector $L_2(w)$ in Eq.(10)
5. Solve for φ^{i+1}
6. Check the convergence and repeat the 2 through 6, if the results are not satisfied.

The cycling is terminated when the nodal displacement of step 3 reach sufficiently small values. If this is not achieved in a predetermined number of cycles, collapse conditions are deemed and the process is stopped.

6. Numerical implementation and discussions

To illustrate the formulation of the method and its efficiency the solution of a simply supported nonuniformly loaded rectangular thin aluminium plate is presented. Due to the symmetry of the problem only one quarter of the plate is considered in the analysis(see Fig. 1), and it is modelled by 12×10 grid sizes. The dimensions and material properties of the plate used are $a = 60$cm, $b = 72$cm, $t = 1.0$cm, $E = 200$GPa, $a =$

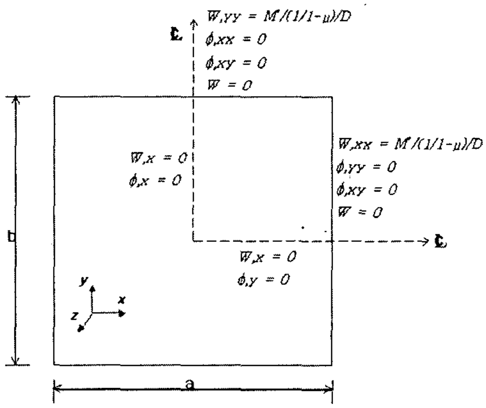

Fig. 2. Geometry, computational domain and boundary conditions for the thermoelastic stress analysis of a rectangular plate.

0.000012, and μ = 0.28. The plate is subjected to uniform temperature 100℃ and uniform temperature 30℃ at the upper and the lower surface, respectively. The boundary conditions for a rectangular plate at simply supported edges are

For the purpose of comparing the numerical results of the present method, a linear plate with same conditions is solved first. The Fourier series solution for displacement $w(x, y)$ is obtained as

$$w(x,y) = \frac{16M^*}{(1-\mu)D\pi^4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin(m\pi x/a)\ \sin(n\pi y/b)}{mn[(m/a)^2+(n/b)^2]}$$

m,n=1,3,5,..., ∞

Fig. 3 shows the temperature deflection curve of the plate for thermal loads. These results give close agreement to the analytical solution for a nonuniform heating. The maximum deflection at the center is found to be 2.94cm, which is within 2.2% of the exact solution. The variation of the displacements in the plate due to thermal pressures is shown in Fig.4. This demonstrates the nonlinearity of the displacement patterns and the migration of the maximum displacement from the support towards the center of plate. A set of computer generated maximum principle stress contours and minimum principal stress contours are also presented here in Fig. 5 and 6. These figures show the variation of the maximum principal stresses from the center towards the corners of the plate. For analysis and design of plate, the values of maximum principle stresses and their location are valuable information.

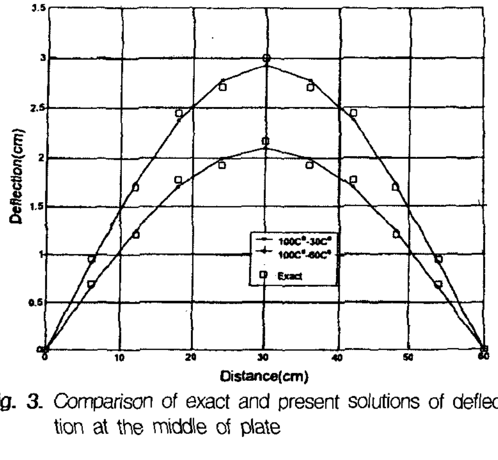

Fig. 3. Comparison of exact and present solutions of deflection at the middle of plate

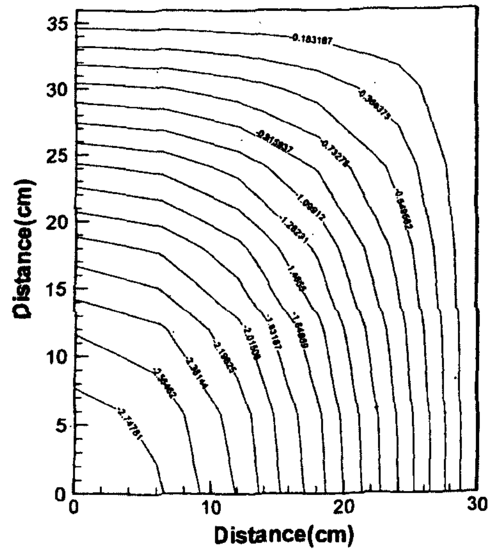

Fig. 4. Displacement distributions with $\triangle T$=70℃

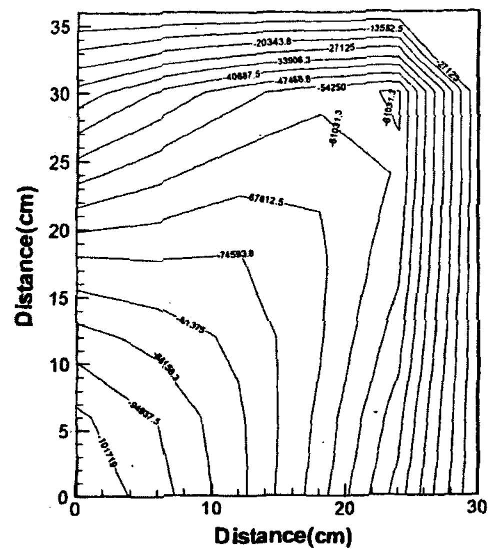

Fig. 5. Contours of maximum principle tensile stresses due to $\triangle T$=70℃

7. Conclusions

For a simply supported rectangular plate with temperature distribution varying over the thickness is analyzed. Since the thermal deflections are large compared to the plate thickness during loading, both bending and membrane stresses are developed and as such a nonlinear stress analysis is necessary, accounting for the effects of large deflection. Thermal stresses in plates

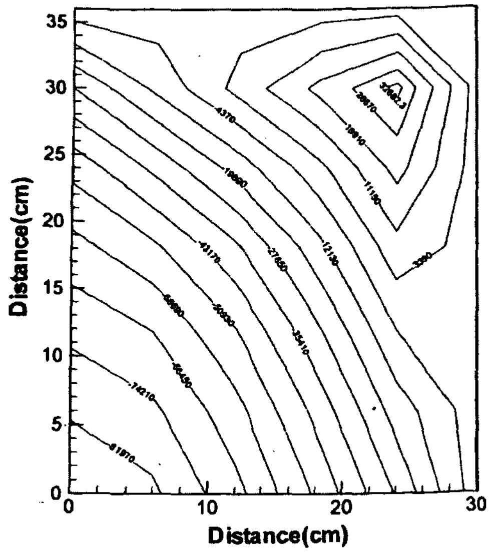

Fig. 6. Contours of minimum principle tensile stresses due to $\triangle T = 70℃$

may be divided into membrane and flexural stresses. In both cases thermal stresses are produced only if restrictions on free expansions or contractions are imposed, either by the boundary conditions or by continuity requirements of the material. The mathematical formulation of thermoelastic plate problems shows a close resemblance to the corresponding isothermal cases. For the geometrically nonlinear large deflection behavior of the plate, the classical von Karman equations are used. These nonlinear equations are solved numerically by using the finite difference method. This method essentially consists of large sets of algebraic equations in terms of discrete values of the functions at discrete nodes. An iterative scheme is employed to solve these two sets of quasi-linear algebraic equations. This iterative technique appears to be suitable for general nonlinear behavior because it relies on the fact that a unique deflection exists for an increment of load. The study in the thermal stress analysis can be directed towards analyzing various solar panels with different sizes, thickness and coefficients of thermal expansion.

Acknowledgement

This work was supported by University of Inchon Research Fund in 2004.

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