HOLOMORPHIC FUNCTIONS AND THE BB-PROPERTY ON PRODUCT SPACES

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Abstract. In [25] Taskinen shows that if \( \{E_n\}_n \) and \( \{F_n\}_n \) are two sequences of Fréchet spaces such that \( (E_m, F_n) \) has the BB-property for all \( m \) and \( n \) then \( (\prod_m E_m, \prod_n F_n) \) also has the BB-property. Here we investigate when this result extends to
(i) arbitrary products of Fréchet spaces,
(ii) countable products of \( DFN \) spaces,
(iii) countable direct sums of Fréchet nuclear spaces.
We also look at topologies properties of \( (\mathcal{H}(U), \tau) \) for \( U \) balanced open in a product of Fréchet spaces and \( \tau = \tau_0, \tau_\omega \text{ or } \tau_\beta \).

1. Introduction

Our main motivation in this paper is in studying holomorphic functions on products of \( DFN \) and Fréchet spaces. For Fréchet spaces our main interest is in the case of uncountable products while for \( DFN \) spaces countable products are sufficient to uncover new interesting results. If \( B_1 \) (resp. \( B_2 \)) is a bounded subset of the locally convex space \( E \) (resp. \( F \)) then \( \bar{\Gamma}(B_1 \otimes \pi B_2) \) is a bounded subset of \( E \otimes \pi F \). We shall say that the pair of spaces \( (E, F) \) has the BB-property if every bounded subset of \( E \otimes \pi F \) is contained in a set of this form. The BB-property is fundamental in the study of equivalence of certain topologies on certain locally topologies (see [16]) and in the establishment of exponential laws for certain spaces of holomorphic functions (see [8]). In our investigation of holomorphic functions on products of \( DFN \) and Fréchet spaces we discover that the problems under consideration are closely related to extending the following result of Taskinen [25]: if \( \{E_m\}_m \) and \( \{F_n\}_n \) are two sequences of Fréchet spaces such that \( (E_m, F_n) \) has the

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BB-property for all \( m \) and \( n \) then \( \prod_{m} E_m, \prod_{n} F_n \) also has the BB-property. In fact we characterise arbitrary products of Fréchet spaces and products of \( DFN \) spaces for which various extensions of Taskinen’s result hold. The crucial examples which arise in our study are the spaces \( C^I \) (I uncountable), \( (C^N)^{(N)} \) or \( \phi \omega \) and \( (C^{(N)})^N \) or \( \omega \phi \).

In section 1 we collect the main known results for the spaces \( (C^N)^{(N)} \) and \( (C^{(N)})^N \) and prove some new results. In section 2 we discuss the BB-property and a weaker concept — the BB-property for points. This section contains our main results and include the following:

1. if \( E \) is a countable product of \( DFN \) spaces then \( (E, E) \) has the BB-property if and only if \( E \) is isomorphic to \( C^N \),
2. uncountable products do not have the BB-property,
3. if \( E \) is a direct sum of Fréchet Nuclear spaces then \( (E, E) \) has the BB-property if and only if it admits a continuous norm.

In the final section we prove various results for holomorphic functions on products. We refer the reader to [20] for further information on locally convex spaces and to [14] for further information on infinite dimensional holomorphy.

2. Holomorphic functions on the spaces \( (C^N)^{(N)} \) and \( (C^{(N)})^N \)

The aim of this section is to investigate holomorphic functions on the spaces \( (C^N)^{(N)} \) and \( (C^{(N)})^N \). These space have proved useful in the linear theory and in constructing counterexamples to many conjectures. They are considered by Köthe in [22, § 13.5] where he uses \( \omega \) to denote \( C^N \), \( \phi \) to denote \( C^{(N)} \), \( \phi \omega \) to denote \( (C^N)^{(N)} \) and \( \omega \phi \) to denote \( (C^{(N)})^N \). The spaces \( (C^N)^{(N)} \) and \( (C^{(N)})^N \) are both reflexive A-nuclear spaces with a basis and hence are fully nuclear spaces. It is clear that \( (C^N)^{(N)} \) is the strong dual of \( (C^{(N)})^N \).

Since \( C \) is a closed subspace of \( C^N \) it will follow that \( C^N \times C^N \cong C^{(N)} \times C \times \cdots \times C \cdots \) is a closed complemented subspace of \( C^{(N)} \times C^{(N)} \times \cdots \times C^{(N)} \cdots \cong (C^{(N)})^N \). As \( C^N \times C^{(N)} \) is isomorphic to its dual we have that \( C^{(N)} \times C^{(N)} \) is a closed complemented subspace of \( C^N \oplus C^N \oplus \cdots \oplus C^N \cdots \cong (C^N)^{(N)} \).

Let \( E \) a locally convex space and \( n \) a positive integer. In [16], four topologies are considered on \( P(nE) = (\bigotimes E)' \). We use \( \tau_\alpha \) (resp. \( \tau_\beta \)) to denote the compact-open (resp. strong) topology — the topology of uniform convergence on compact (resp. bounded) subsets of \( E \) and \( \tau_\omega \) to denote the Nachbin ported topology. We define the topology \( \beta \)
by \((P^n E), \beta := \left( \bigotimes_{s,n, \pi} E \right)_b'\). We always have that \(\tau_0 \leq \tau_b \leq \beta \leq \tau_\omega\) on \(P^n E\). We know that \(\tau_0 = \tau_b\) on \(P^n E\) if and only if \(E\) is semi-Montel and that \(\beta = \tau_b\) on \(P^n E\) if and only if each bounded subset of \(\bigotimes_{s,n, \pi} E\) is contained in a set of the form \(\bar{\Gamma} \left( \bigotimes_{s,n, \pi} B \right)\) for \(B\) a bounded subset of \(E\). When this happens we shall say that \(E\) has \((BB)_n\). In particular, as noted in \([16]\), if \((E, E)\) has \(BB\) then \(\beta = \tau_b\) on \(P^2 E\).

Since \((P^n E), \tau_\omega = \left( \bigotimes_{s,n, \pi} E \right)_i'\), we see that \(\beta = \tau_\omega\) on \(P^n E\) if and only if \(\left( \bigotimes_{s,n, \pi} E \right)_b' = \left( \bigotimes_{s,n, \pi} E \right)_i'\).

We now list properties of spaces of holomorphic functions on \((C^N)^N\) and \((C^N)^N\).

**Theorem 1.**

1. For \(E = (C^N)^N\) or \((C^N)^N\) and \(n \geq 2\), we have \(\tau_0 \neq \tau_\omega\) on \(P^n E\). For \(U\) open in \((C^N)^N\) or \((C^N)^N\) we have \(\tau_\omega \neq \tau_b\) on \(\mathcal{H}(U)\).
2. For any open subset of \((C^N)^N\) or \((C^N)^N\) we have \(\mathcal{H}_{HY}^n(U) = \mathcal{H}_{M}(U)\). However for \(E = (C^N)^N\) or \((C^N)^N\) and \(n \geq 2\) we have \(P^n E \neq P_{HY}^n E\). In particular neither \((C^N)^N\) or \((C^N)^N\) are \(k\)-spaces.
3. For \(U\) an open polydisc in \((C^N)^N\) or \((C^N)^N\), \((\mathcal{H}(U), \tau_0)\) is not complete and the \(\tau_0\) bounded sets of \(\mathcal{H}(U)\) are not locally bounded.
4. We have the identity

\[
(\mathcal{H}((C^N)^N), \tau_{\delta}) = \text{ind}_\alpha (\mathcal{H}((C^N)^n), \tau_0)
\]

and \((\mathcal{H}((C^N)^N), \tau_{\delta}))\) and \((\mathcal{H}((C^N)^N), \tau_\omega)\) are both complete.

**Proof.**

1. Since \(\tau_0 \neq \tau_\omega\) on \(P^2 C^N \times C^N\) it will follow, by a modification of \([15, \text{Theorems 5 and 9}]\), that \(\tau_0 \neq \tau_\omega\) on \(P^2 E\) for \(E = (C^N)^N\) or \((C^N)^N\).

Since \(C^N\) contain a nontrivial very strongly convergent sequence it follows that \((C^N)^N\) and \((C^N)^N\) both contain a nontrivial very strongly convergent sequences. Therefore from \([13, 2.52]\) we have that \(\tau_\omega \neq \tau_\delta\) on \(\mathcal{H}(U)\) for \(U\) open in either \((C^N)^N\) or \((C^N)^N\).
2. For every compact subset $K$ of $(C^{(N)})^N$ (resp. $(C^N)^{(N)}$) there is a bounded subset $B$ such that $K$ is contained and compact in $E_B$. It follows from [13, Example 2.19] that for any open subset $U$ of either of these spaces $\mathcal{H}_{HY}(U) = \mathcal{H}_M(U)$. $(\mathcal{H}_M(U))$ is the space of $G$-holomorphic functions on $U$ such that $\frac{d^n f(x)}{n!}$ sends bounded sets to bounded sets for each $x$ in $U$ and each integer $n$. From [7, Corollary] and [24, Proposition 2] we conclude that for $E$ equal to $(C^{(N)})^{(N)}$ or $(C^{(N)})^N$ we have that $P(nE) \neq P_{HY}(nE)$.

3. Apply (2) and [6, Proposition 9].

4. Since $(C^{(N)})^N$ is an open and compact surjective limit of $DFN$ spaces we may apply [13, Proposition 6.28] to conclude that

$$\mathcal{H}((C^{(N)})^N), \tau_d) = \text{ind}_n(\mathcal{H}((C^{(N)})^n), \tau_d)$$

and that $(\mathcal{H}((C^{(N)})^N), \tau_d)$ is complete. By [3, Theorem 4.1] the $\tau_d$-bounded sets of $\mathcal{H}(U)$ are locally bounded for every open subset $U$ of $(C^{(N)})^N$. Applying [13, Proposition 5.37] we see that $(\mathcal{H}((C^{(N)})^N), \tau_d)$ is complete. This means in particular that the $\tau_d$-bounded sets of $\mathcal{H}((C^{(N)})^N)$ are not $\tau_d$-bounded.

In [9] the authors show that $\mathcal{H}(U)$ is complete and the $\tau_d$-bounded subsets of $\mathcal{H}(U)$ are locally bounded for $U$ an open subset of $(C^{(N)})^N$. In the class of direct sums of Fréchet nuclear space, spaces with this properties are the exception. In fact $(C^{(N)})^{(N)}$, $C^N$ and $C^N \times C^{(N)}$ are the only three spaces in this class with the property that the $\tau_d$-bounded sets are locally bounded (see [9]).

3. The $BB$-property for points

Let us recall that the pair of locally convex space $E$ and $F$ has the $BB$-property if each bounded set in $E \underset{\pi}{\otimes} F$ is contained in a set of the form $\tilde{\Gamma}(B_1 \otimes B_2)$ for some bounded sets $B_1$ in $E$ and $B_2$ in $F$.

We shall say that a locally convex space $E$ has the $BB$-property for points if each point in $E \otimes E$ is contained in a set of the form $\tilde{\Gamma}(B \otimes B)$ for some bounded set $B$ in $E$.

Given locally convex spaces $E$ and $F$ we define

$$\Sigma(E_0 \underset{\pi}{\otimes} F_0)$$

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\[
\sum_{n=1}^{\infty} \lambda_n x'_n \otimes y'_n \in E'_b \bigotimes_{\pi} F'_b : \lambda \in \ell_1,
\]

\[
\{x'_n\}_n, \{y'_n\}_n \text{ are equicontinuous of } E'_b \text{ and } F'_b \text{ respectively}.
\]

Defant and Floret [12] call \(\Sigma(E'_b \bigotimes_{\pi} F'_b)\) the set of all points of \(E'_b \bigotimes_{\pi} F'_b\) which have series representations. Mangino, [23], characterises when each point in the projective tensor product of two complete countable inductive limits of Banach spaces with the approximation property has a series representation. We note that if every point of \(E'_b \bigotimes_{\pi} F'_b\) has a series representation then \((E'_b, F'_b)\) has the BB-property for points.

In [25] Corollary 3.6, Taskinen shows that if \(\{E_n\}_{n \in \mathbb{N}}\) and \(\{F_n\}_{n \in \mathbb{N}}\) are two sequences of Fréchet spaces such that \((E_m, F_n)\) has BB for each pair of integers \((m, n)\) then \((\prod_{m \in \mathbb{N}} E_m, \prod_{n \in \mathbb{N}} F_n)\) has BB. We will now show that this result does not extend to countable products of DFN-spaces, or to uncountable products of Fréchet spaces. We prove that a countable product of DFN spaces has the BB-property for points if and only if the space is isomorphic to \(\mathbb{C}^\mathbb{N}\).

**Proposition 2.** Let \(E = \prod_{n=0}^{\infty} E_n\) be a product of DFN spaces, then \(\tau_0 = \tau_\omega\) on \(P(2E)\) if and only if \(E = \mathbb{C}^\mathbb{N}\).

**Proof.** If \(E = \mathbb{C}^\mathbb{N}\) then \(E\) is a Fréchet nuclear space and hence \(\tau_0 = \tau_\omega\) on \(P(2E)\). If \(E \neq \mathbb{C}^\mathbb{N}\) then we may assume without loss of generality that \((E_0)'\) is not a normed space. By [7, Corollary 2] it follows that \(P(2E'_b) \neq P_{HY}(2E'_b)\) and therefore since \(E\) is fully nuclear, \(P(2E'_b) \neq P_{M}(2E'_b)\). Applying [13, Corollary 1.50] we get \(\tau_0 \neq \tau_\omega\) on \(P(2E)\).

We now show that we cannot replace the Fréchet spaces in [25, Corollary 3.6] by DFN spaces.

The \(\tau_0\) topology on \(P(\mathcal{N})\) may be defined in terms of the duality between \(P(\mathcal{N})\) and \(\bigotimes_{s,n,\pi} E\) as the topology on \(\bigotimes_{s,n,\pi} E\) of uniform convergence on sets of the form \(\Gamma\left(\bigotimes_{n,s} K\right)\) where \(K\) is a compact subset of \(E\). It is therefore weaker than the topology on \(P(\mathcal{N})\) of uniform convergence on all compact subsets of \(\bigotimes_{s,n,\pi} E\) and hence is weaker than the Mackey topology \(\tau(P(\mathcal{N}), \bigotimes_{s,n,\pi} E)\). If every \(x \in \bigotimes_{s,n,\pi} E\) is contained
in a set of the form $\hat{\Gamma}\left(\bigotimes_{n,s} K\right)$ for $K$ compact in $E$, then the $\tau_0$-topology on $P\left(^nE\right)$ is finer than the $\sigma(P\left(^nE\right), \bigotimes_{s,n,\pi} E)$-topology on $P\left(^nE\right)$. This
implies that $\tau_0$ is compatible with the duality between $P\left(^nE\right)$ and $\bigotimes_{s,n,\pi} E$
and hence we have that $(P\left(^nE\right), \tau_0)' = \bigotimes_{s,n,\pi} E$.

Let us now suppose that $E$ is a fully nuclear space in which the $\tau_\omega$-bounded sets are locally bounded. An adaptation of the proof of [17, Theorem 8] gives us that $\bigotimes_{s,n,\pi} E = (P\left(^nE\right), \tau_\omega)'$. Therefore, if every point
in $\bigotimes_{s,n,\pi} E$ is contained in a set of the form $\hat{\Gamma}\left(\bigotimes_{n,s} K\right)$ for $K$ compact
in $E$ we have that $(P\left(^nE\right), \tau_0)' = (P\left(^nE\right), \tau_\omega)'$. Applying [13, Propositions 1.56 and 1.67] this implies that $P\left(^nE'_b\right) = P_{HY}\left(^nE'_b\right)$.

Let $E = \prod_n E_n$ be a product of $DFN$ spaces at least one of which is infinite dimensional. It follows from [3, Theorem 4.1] that the $\tau_\omega$-bounded subsets of $P\left(^nE\right)$ are locally bounded. Therefore, if $E$ has the $BB$-property for points the above paragraph implies that $P\left(^nE'_b\right) = P_{HY}\left(^nE'_b\right)$. Since $E'_b$ is a direct sum of Fréchet nuclear spaces at least one of which is infinite dimensional this contradicts [7, Corollary 2] and we obtain the following result.

**Theorem 3.** Let $E$ be a countable product of $DFN$ spaces, then $(E, E)$ has the $BB$ property if and only if $E = C^N$.

**Remarks.**

1. This means that $\left((C^N)^N, (C^N)^{N}\right)$ does not have the $BB$-property for points, to see this set $E_n = C^N$.
2. We denote by $D'$ the space of distributions on the real line. Since $D' = \prod_n s'$ (s the space of rapidly decreasing sequences) we see that $(D', D')$ does not have the $BB$-property for points.
3. If we set $E_0 = C^{(N)}$ and $E_n = C$ for $n \geq 1$ we get that $(C^{(N)} \times C^N, C^{(N)} \times C^N)$ does not have the $BB$-property for points.

If $E$ and $F$ are a pair of Fréchet or $DF$ spaces, one of which is nuclear, then it follows from [19] that $\left(E \bigotimes r, F\right)'_b = E'_b \bigotimes r F'_b$. In [11, Theorem 3] Defant shows that if $E$ and $F$ are two locally convex spaces, such that the dual of one of these spaces has the local RNP and if $E'_b$ has the
approximation property then
\[(E \bigotimes_c E)' = \Sigma(E_b \bigotimes \pi E_b').\]
(For the definition of what it means that a space to have dual with the local RNP see [10]).

If \( E = \coprod_{n=1}^{\infty} E_n \) is a direct sum of Fréchet nuclear spaces, \( E_b' \) is a locally convex space whose dual has the local RNP and the approximation property. If at least one of these spaces is infinite dimensional then it follows from the above that
\[\Sigma(E_b' \bigotimes \pi E_b') \neq E_b' \bigotimes \pi E_b'.\]
Hence we have
\[(E \bigotimes_c E)' = E_b' \bigotimes \pi E_b'\]
if and only if \( E = C^{(N)} \).

**Theorem 4.** \((C^I, C^I)\) has the BB-property for points if and only if \( I \) is countable.

**Proof.** If \( I = N \) then \( C^N \) is a Fréchet nuclear space and therefore has the (BB) property. As we shall see in Proposition 10, \((P^n C^I), \tau_n\) is a Montel space. It follows by the argument used in [17, Theorem 8] that \( \bigotimes_{s,2,\pi}^c C^I = (P^2 C^I), \tau_\omega \)'. Therefore, if every point in \( C^I \bigotimes \pi C^I \) is contained in a set of the form \( \bigotimes_{\pi \omega}^c K \) for \( K \) compact in \( C^I \) then \((P^2 C^I), \tau_\omega)' = (P^2 C^I), \tau_\omega)' \). If \( I \) is uncountable this contradicts [2, Lemma 13] and so \((C^I, C^I)\) does not have the BB-property for points when \( I \) is uncountable. ([2, Lemma 13] says that the two topologies are not compatible for \( I \) having cardinality at least equal to the continuum. By [21] the same result holds for \( I \) uncountable.) \( \Box \)

Since the space \( C^I \) is complemented in \( \prod_{i \in I} E_i \) for any choice of \( E_i \), we also have:

**Proposition 5.** Let \( E = \prod_{i \in I} E_i \) be a product of Fréchet space. Then \((E, E)\) has the BB-property if and only if \( I \) is countable and \((E_i, E_j)\) has the BB-property for every pair \((i, j)\) in \( I^2 \).

To finish this section we consider the BB-property on countable direct sums of Fréchet nuclear spaces.

**Theorem 6.** Let \( E = \coprod_{n \in N} E_n \) be a direct sum of Fréchet nuclear spaces, the following are equivalent:
1. $\tau_0 = \tau_\omega$ on $\mathcal{H}(E)$,
2. $\tau_n = \tau_\omega$ on $P^n(E)$ for each integer $n$,
3. $E$ has $(BB)_n$ for each integer $n$,
4. each $E_n$ admits a continuous norm.

Proof. Clearly (1) implies (2) and (2) implies (3). If each $E_n$ admits a continuous norm then $E$ admits a continuous norm. Applying [24, Corollary 2 to Proposition 6] we see that $\tau_0 = \tau_\omega$ on $\mathcal{H}(E)$ and hence (4) implies (1). If (4) is not true, then we may suppose without loss of generality that $E_1$ does not admit a continuous norm. By [5, Theorem 1] $E_1$ contains $\mathbb{C}^N$ as a complemented subspace and therefore $E$ contains $\mathbb{C}^N \times \mathbb{C}^{(N)}$ as a complemented subspace. Since $\mathbb{C}^N \times \mathbb{C}^{(N)}$ does not have $(BB)_2$, it will follow that $E$ does not have $(BB)_2$ and therefore we see that (3) implies (4). \qed

4. Topological properties of spaces of holomorphic functions on Products of Fréchet spaces

In [1] it is shown that the spaces $(\mathcal{H}(U), \tau_0)$, $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(U), \tau_\delta)$ are complete for $U$ a balanced open subset of $\mathbb{C}^l$. In this section we extend this result to arbitrary products of Fréchet spaces and look at further topological properties of these spaces. We begin by looking at completeness for the $\tau_\omega$-topology. The proof of this result is an adaptation of [1, Proposition 3].

Proposition 7. Let $E = \prod_{i \in I} E_i$ be a product of Fréchet spaces and $U$ be a balanced open subset of $E$ then $(\mathcal{H}(U), \tau_\omega)$ is complete.

For $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(U), \tau_\delta)$ we have.

Proposition 8. Let $E = \prod_{i \in I} E_i$ be a product of Fréchet spaces $E_i$ and $U$ be a balanced open subset of $E$, then $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(U), \tau_\delta)$ are complete.

Proof. By Proposition 7 $\mathcal{H}(U)$ is both $\tau_\omega$ and $\tau_\delta$-T.S. complete. It suffices to complete the proof to show that $(\mathcal{P}^n E), \tau_\omega$ is complete for each integer $n$.

By [18, Lemma 2] we have

$$
\left(\bigotimes_{n, \pi} E\right)_{i} = \prod_{i_1, \ldots, i_n} \left(E_{i_1} \otimes \pi \cdots \otimes \pi E_{i_n}\right)_{i}.
$$
Since each $E_{i_1} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_{i_n}$ is a Fréchet space, using [22, p.400], we see that $(\hat{\bigotimes}_{n,\pi} E)$ is complete and is equal to

$$
\left( \left( \hat{\bigotimes}_{n,\pi} E \right)' , \beta \left( \left( \hat{\bigotimes}_{n,\pi} E \right)' , \left( \hat{\bigotimes}_{n,\pi} E \right)'' \right) \right).
$$

We denote by $C^{(n)}E$ the complement of $\hat{\bigotimes}_{n,\pi} E$ in $\hat{\bigotimes}_{n,\pi} E$. We have

$$
\left( \left( \hat{\bigotimes}_{n,\pi} E \right)' , \beta \left( \left( \hat{\bigotimes}_{n,\pi} E \right)' , \left( \hat{\bigotimes}_{n,\pi} E \right)'' \right) \right)
= \left( \left( \hat{\bigotimes}_{s,n,\pi} E \right)' , \beta \left( \left( \hat{\bigotimes}_{s,n,\pi} E \right)' , \left( \hat{\bigotimes}_{s,n,\pi} E \right)'' \right) \right)
\bigoplus \left( \left( C^{(n)}E \right)' , \beta \left( \left( C^{(n)}E \right)' , \left( C^{(n)}E \right)'' \right) \right).
$$

It follows from [18, Lemma 1] that

$$
\left( P^{(n)}E , \tau_\omega \right) = \left( \hat{\bigotimes}_{s,n,\pi} E \right)' = \left( \left( \hat{\bigotimes}_{s,n,\pi} E \right)' , \beta \left( \left( \hat{\bigotimes}_{s,n,\pi} E \right)' , \left( \hat{\bigotimes}_{s,n,\pi} E \right)'' \right) \right)
$$

is complete. \hfill \square

To consider Montelness and the approximation property on these spaces we need the following Theorem.

**Proposition 9.** Let $E = \prod_{i \in I} E_i$ where each $E_i$ is a Fréchet space and let $n$ be a positive integer such that $E_{i_1} \hat{\otimes}_\pi E_{i_2} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_{i_n}$ is distinguished for each $(i_1, \ldots, i_n) \in I^n$. Then $\beta = \tau_\omega$ on $P^{(n)}E$.

**Proof.** By [20, Theorem 15.4.1] we have that

$$
\hat{\bigotimes}_{n,\pi} E = \prod_{i_1, \ldots, i_n} E_{i_1} \hat{\otimes}_\pi E_{i_2} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_{i_n}.
$$

Taking strong duals we get that

$$
\left( \hat{\bigotimes}_{n,\pi} E \right)' = \prod_{i_1, \ldots, i_n} \left( E_{i_1} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_{i_n} \right)'.
$$
As in Proposition 8 we also have
\[
\left(\bigotimes_{n,\pi} E\right)' = \prod_{i_1, \ldots, i_n} (E_{i_1} \otimes \cdots \otimes E_{i_n})'_i.
\]

Hence
\[
\left(\bigotimes_{n,\pi} E\right)'_b = \left(\bigotimes_{s,n,\pi} E\right)'_i.
\]

In particular \(\bigotimes_{n,\pi} E\)' is bornological. Since \(\bigotimes_{s,n,\pi} E\) is complemented in \(\bigotimes_{n,\pi} E\) we have that \(\bigotimes_{s,n,\pi} E\)' is complemented in \(\bigotimes_{n,\pi} E\)' and therefore it also is bornological. Since \(\bigotimes_{s,n,\pi} E\) is barrelled we can apply Proposition 3.3 of [4] to conclude that
\[
(P^n E, \beta) = \left(\bigotimes_{s,n,\pi} E\right)'_b = \left(\bigotimes_{s,n,\pi} E\right)'_i = (P^n E, \tau_\omega).
\]

\[\square\]

Remark. Any product of Fréchet Schwartz spaces will satisfy the above condition as will an arbitrary product of Banach spaces.

Using Proposition 9 we can easily show the following results.

Proposition 10. Let \(E = \prod_{i \in I} E_i\) be a product of Fréchet spaces such that \(E_{i_1} \otimes \cdots \otimes E_{i_n}\) is reflexive (resp. Montel) for each \((i_1, \ldots, i_n) \in I^n\) and let \(U\) be a balanced open subset of \(E\) then \((\mathcal{H}(U), \tau_\omega)\) is semi-reflexive (resp. semi-Montel) and \((\mathcal{H}(U), \tau_\delta)\) is reflexive (resp. Montel).

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References

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