Commun. Korean Math. Soc. 19 (2004), No. 1, pp. 1–10

ON THE SEMIATOMICITY FOR COMPLETELY RIGHT INJECTIVE SEMIGROUPS

EUNHO L. MOON

ABSTRACT. We here consider necessary and sufficient conditions for a completely right injective semigroup S whose lattice $L(S)$ of right congruences on S is semiatomic. These are preceded by a number of results on the characterization of a semigroup S in which every automaton over S is injective(called a completely right injective semigroup).

1. Introduction

A right congruence ρ on a semigroup S is an equivalence relation on S such that if $(a,b) \in \rho$ and $s \in S$, then $(as, bs) \in \rho$. Let $L(S)$ be the set of all right congruences on S. If α and β are two elements of $L(S)$, then the right congruence $\alpha \vee \beta$, called the *join* of α and β, is the smallest right congruence that contains both α and β. Also the right congruence $\alpha \wedge \beta$, called the *meet* of α and β, is the largest right congruence contained in both α and β.

Each of the right congruences defined in the above is well-defined. The unique largest element of $L(S)$ is the *universal congruence* $v = \{(s,t) \mid \forall s,t \in S\}$. The unique smallest element of $L(S)$ is the *identity congruence* $\iota = \{(t,t) \mid \forall t \in S\}$. Thus $L(S)$ becomes a complete lattice.

A right congruence ρ is said to be *minimal* if $\iota \neq \tau$ and if $\iota \leq \rho \leq \tau$ implies $\iota = \rho$ or $\rho = \tau$. We have seen [8] that there is a type of minimal right congruences of interest to us in this paper. We describe it here.

Let ρ be a right congruence on S. Let U be an equivalence class of ρ containing one element e. If for every $d \in S$ such that $Ud \subseteq U$ we have $ed = e$, then e is called *a zero of* ρ. If $U = \{e\}$ then e is called *a trivial zero of* ρ; otherwise, e is called *a nontrivial zero* of ρ.

Received April 15, 2002.
2000 Mathematics Subject Classification: 20M10.
Key words and phrases: injective S-automaton, completely right injective semigroup, semiatomic, essential right congruence.

DEFINITION 1. A minimal right congruence ρ is of *Type 2* if every equivalence class contains exactly one element z such that if whenever $(z, zx) \in \rho$ we have $z = zx$.

Throughout this paper our minimal right congruences are of *Type 2*.

DEFINITION 2. As a lattice, $L(S)$ is said to be *semiatomic* if the universal congruence υ is the join of its minimal right congruences on S.

DEFINITION 3. A right congruence ρ is said to be *essential* if for every right congruence α we have the implication $\alpha \cap \rho = \iota \Longrightarrow \alpha = \iota$.

2. Completely right injective semigroups

DEFINITION 4. A (deterministic) *automaton over* S(or S*-automaton*), $A = (A, S, \delta)$, is a triple where A is a nonempty set, S is a nonempty semigroup, δ is a function mapping $A \times S$ into A. We shall assume the useful property that

$$\delta(a, st) = \delta(\delta(a, s), t), i.e., a(st) = (as)t$$

for a $\in$ A and s, t $\in$ S.

REMARK 1. If M, N are automata over a semigroup S, we have $A \subseteq M$ is a subautomaton of M if and only if $AS \subseteq A$, the map $\phi : M \to N$ is a homomorphism(or S-homomorphism) if and only if $\phi(ms) = \phi(m)s$ for all $m \in M$ and all $s \in S$. Similarly S-epimorphism, S-isomorphism are defined.

DEFINITION 5. An automaton J over a semigroup S is called *injective* if for every monomorphism $\alpha : L \to M$ and a homomorphism $\beta : L \to J$ where L, M are automata over S, there is a homomorphism $f : M \to J$ such that $f\alpha = \beta$.

THEOREM 1. *Let S be a semigroup having a zero element and J an automaton over S. Then J is injective if and only if for every right ideal I of S and a S-homomorphism $\phi : I \to J$ there is an element y in J such that $\phi(s) = ys$ for all s in I.(Clearly, then ϕ can be extended to all elements t of S by $\phi(t) = yt$)*

PROOF. Assume that an automaton J over S is injective and we have the diagram

$$\begin{array}{ccc} I & \xrightarrow{i} & S^1 \\ \downarrow{\scriptstyle\phi} & & \\ J & & \end{array}$$

where i is the injection of I into S^1. We can complete the diagram with $h : S^1 \to J$ so that the diagram commutes. Set $h(1) = y$. Then $h(s) = ys$ for all s in S^1. But then $\phi(s) = ys$ for all s in I. Conversely assume that we have the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow \alpha & & \\ J & & \end{array}$$

where $A \subseteq B$ are automata over S, and α is an S-homomorphism of A into J. Consider the set of all pairs (h, C) such that C is a subautomaton of B containing A and $h : C \to J$ is a S-homomorphism which extends α. We partially order this set by the relation $(h, C) \leq (h', C')$ if and only if h' extends h and $C \subseteq C'$. Since any totally ordered subset has an upper bound in the set, we can use Zorn's Lemma to get a maximal pair (h, C). To prove that J is an injective automaton over S, we show $C = B$. Suppose that $C \subset B$ and let b be an element of $B \setminus C$. Set $I = \{s \mid bs \in C\}$. We first assume that I is empty. Since $bS^1 \cap C$ is empty, we define a map $k : bS^1 \to J$ by $k(x) = y0$ for all x in bS^1, where y is arbitrary but fixed element of J. Since $k(x)t = (y0)t = y0 = k(xt)$ for all t in S, k is clearly a S-homomorphism. Assume that I is nonempty, and let $\lambda(s) = h(bs)$ for all s in I. Note that I is a right ideal of S and λ is an S-homomorphism from I into J. Thus there is an element y in J such that $\lambda(s) = ys$ for all s in I and hence $h(bs) = ys$ for all s in I. Define $k : bS^1 \to J$ by $k(bs) = ys$ for all s in S^1. Then k is a well-defined S-homomorphism. Let $x \in C \cap bS^1$. Since $x = ba \in C$, where $a \in I$, $k(x) = k(ba) = ya = h(ba) = h(x)$. Hence $k = h$ on $C \cap bS^1$. Set $C^* = C \cup bS^1$, and let $h^* : C^* \to J$ be the map defined by $h^*(x) = h(x)$ for x in C and $h^*(x) = k(x)$ for x in bS^1. Then h^* is clearly an S-homomorphism of C^* into J which extends h. Since it contradicts to the maximality of (h, C), $C = B$ and J is injective. □

LEMMA 1. ([2]) *Every right ideal of a completely right injective semigroup S is generated by an idempotent element.*

LEMMA 2. ([2]) *The set of right ideals of a completely right injective semigroup S is linearly ordered by inclusion.*

LEMMA 3. *Let S be a completely right injective semigroup with a left identity and I be a right ideal of S. If I is not minimal, then the set K of nongenerators of I is the largest right ideal contained in I.*

PROOF. For any right ideal I, we let $U(I)$ be the set of all generators of I, and denote $K = I \setminus U(I)$. Since $U(I)$ is nonempty, K is a proper subset of I. Assume that I is a right ideal that is not minimal. If K were empty, then for every element $a \in S$, $I = aS$. Let J be a right ideal of S such that $J \subseteq I$, let x be an element of J. Since $x \in I$ implies that $J \subseteq I = xS \subseteq J$, K is nonempty. Let $x \in K$ and $s \in S$. If xs were not in K, then I is generated by xs. But then $x \in I$, hence it follows $x \in U(I)$ which contradicts to the choice of x. Thus K is a right ideal contained in I properly. If J is any right ideal of S such that $K \subset J \subseteq I$ and x is in $J \setminus K$, then x is in $U(I)$ so that $I = J$. Thus there are no other right ideals strickly between I and K. □

LEMMA 4. *If S is a completely right injective semigroup with a left identity, then S has a zero element.*

PROOF. Let S^0 be the semigroup adjoined with a zero element 0. Since S^0 is injective as an S-automaton by assumption, there exists an S-epimorphism $f : S^0 \to S$. Set $z = f(0)$ in S. Then $zs = f(0)s = f(0) = z$ for all s in S so that z is a left zero element of S. But since sz is also a left zero of S, we have that there is either $\{sz\} \subseteq \{z\}$ or $\{z\} \subseteq \{sz\}$, *i.e.*, $sz = z = zs$. Thus z is a zero element of S. □

THEOREM 2. *If a semigroup S has a left identity, then S is completely right injective if and only if S has a zero and every right ideal of S is generated by an idempotent element.*

PROOF. Let S be a semigroup with a left identity. We have seen that for a completely right injective semigroup S it has a zero and every right ideal of S is generated by an idempotent. To show that S is completely right injective we now assume that S has a zero and every right ideal of S is generated by an idempotent. If I is a right ideal of S, J is an S-automaton, and $\phi : I \to J$ is an S-homomorphism, then $I = eS$ for some idempotent e in S. Let $\phi(e) = x$ in J. Then for every s in I, $\phi(s) = \phi(es) = \phi(e)s = xs$. If S has a zero, it suffices that J is injective by Theorem 2. □

REMARK 2. Let U_0 be the set of generators of S, and I_1 be the set of nongenerators of S. Since U_0 is nonempty, it is easy to see that I_1 is a proper right ideal of S. Let U_1 be the set of generators of I_1, and $I_2 = I_1 \setminus U_1$. Then I_2 is the largest right ideal in I_1 by the same argument. Continuing this process, we have a chain

$$S = I_0 \supset I_1 \supset I_2 \supset \cdots$$

where U_i is a nonempty set of generators of a right ideal I_i for all i. Thus every right ideal of S is in this chain . Since every right ideal is generated by an idempotent, every U_i contains an idempotent. Also since S has a zero, this chain of right ideals must be finite so that $U_t = I_t$ and $S = U_0 \cup I_1 = U_0 \cup U_1 \cup I_2 = \cdots = U_0 \cup U_1 \cup \cdots \cup U_t$ for some t.

As defined in ([1], pp.47–48), $\mathcal{H}, \mathcal{R}$ and $\mathcal{L}, \mathcal{J}$ will denote Green's equivalence relations on the semigroup S. L_a(resp. R_a, H_a) denotes the $\mathcal{L}-$ (resp. $\mathcal{R}-, \mathcal{H}-$) class of S containing the element a.

THEOREM 3. (Green's theorem) *If a, b and ab all belong to the same $\mathcal{H}$-class H of a semigroup S, then H is a subgroup of S. In particular any $\mathcal{H}$-class containing an idempotent is a subgroup of S.*

DEFINITION 6. A semigroup S is called a *prisemigroup* if every right ideal of S has a generator.

THEOREM 4. *If S is a completely right injective semigroup with a left identity, then it is a prisemigroup which is a union of groups.*

PROOF. Let S be a completely right injective semigroup having a left identity. Since every right ideal is generated by an idempotent, it is clear that S is a prisemigroup and hence a regular semigroup. If a is an element in S, there exists an inverse b of a such that $a = aba, b = bab$. Define a map $f : baS \to aS$ by $f(bas) = as$ for all s in S. Then f is clearly an S-isomorphism. Since all right ideals of S are linearly ordered by inclusion, it suffices that $baS = aS = abS$ where both elements ab and ba are idempotent. If we let $a^{'} = bba$, then $aa^{'}a = a$ and $a^{'}aa^{'} = a^{'}$. But then $aa^{'} = a(bba) = ba = b(baa) = (bba)a = a^{'}a$, it suffices that $a^{'}$ is too an inverse of a with $aa^{'} = a^{'}a$. Since a and $aa^{'}$ are related by $\mathcal{H}$-class, the $\mathcal{H}$-class H_a containing a is a group by Green's theorem. Since S is a union of $\mathcal{H}$-classes, we have that S is a union of groups. □

LEMMA 5. ([1]) *If all right ideals of a semigroup S are linearly ordered by inclusion, then each $\mathcal{L}$-class contains only one idempotent generator.*

THEOREM 5. *If S is a completely right injective semigroup with a left identity, then every right ideal of S is two-sided.*

PROOF. Let I be a right ideal of S and $a \in I$. For an element s not in I, we claim that sa is in I. Since s is an element of S, there is some idempotent e such that s belongs to $\mathcal{H}$-class H_e and $se = s = es$, so it implies $ea = a$. Let t be an element of S such that $ts = e$. Then $a \in Ssa$ and $sa \in L_a$ where L_a is a $\mathcal{L}$-class containing a.

Suppose that f is an idempotent in L_a. If $b \in L_a$, then $H_a \subseteq L_a = L_f$ and $H_b \subseteq L_b = L_f$. Thus both H_a and H_b are contained in L_f. But Since each $\mathcal{H}$-class is a group, the idempotent element f is in both H_a

and H_b so that b is in H_a. It suffices that L_a is a group and there is an element u in L_a such that $sa = au$ in I. Thus I is a two-sided ideal. □

THEOREM 6. *Let U_i, U_j be the sets we mentioned in Remark 2. Then for every i, j with $i \leq j$, $U_iU_j \subseteq U_j$ and $U_jU_i \subseteq U_j$.*

PROOF. Let a in U_i and b in U_j and a', b' be an inverse of a, b such that $aa'a = a, a'aa' = a'$ and $bb'b = b, b'bb' = b'$ respectively. If $e = aa'$ and $f = bb'$, then $ea = a$ and $fb = b$. But then $a'f$ in fS and $f(a'f) = a'f$, it follows that f is in $(af)S$. Since af is also in fS, $af \in U_j$ and $ab \in U_j$. Next if ba were not in U_j, then there is some $k > j$ such that $ba \in U_k$. But then $b' = b'(bb') = b'f \in fS \subseteq eS$ and hence $eb' = b'$. It follows that f is in baS which contradicts to the choice of k. □

THEOREM 7. *For each i, U_i in Remark 2 is a right group.*

PROOF. Let a be an element in U_i and e be an idempotent element in U_i. If b is an element such that $ab = e$, then $b = be$ and $e = ab \in bS$ so that b is in U_i. Let u be an element of U_i. But then $u = eu = (ab)u = a(bu) \in aU_i$, so $U_i = aU_i$ for all a in U_i. Hence U_i is a right group. □

In this section we have seen that completely right injective semigroups having a left identity can be decomposed into disjoint right groups.

3. Semiatomicity for completely right injective semigroups

In [6], a characterization is given for those semigroup S whose lattice $L(S)$ of right congruences is semiatomic. If I is any right ideal of S, we denote by $U(I)$ the set of all generators of I.

We first state eight conditions we can place on a semigroup S.

1. S has no proper essential right congruences.
2. The right ideals of S form a descending sequence
$$S = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots .$$
3. Every right ideal of S is two-sided and has an idempotent generator.
4. For every right ideal I of S we have that $U(I)$ is a right group.
5. For every pair of right ideals I and J of S such that $I \subseteq J$ we have $U(I)U(J)$ and $U(J)U(I)$ to be subsets of $U(I)$.
6. If I and J are right ideals of S such that I is properly contained in J, then for $a \in U(J)$ and $b \in U(I)$ we have $ab = b$.
7. For every right ideal I of S with a and b in $U(I)$ and f in S we have $fa = fb$ implies $a = b$.

8. If S has a minimal right ideal I then $I = G \times K$ where G is a group generated by its minimal subgroups and K is a right zero semigroup.

THEOREM 8. ([6]) *Let S be a semigroup with a minimal right ideal. Then $L(S)$ is semiatomic if and only if S satisfies conditions* (1) *through* (8).

THEOREM 9. ([5]) *Let S be a semigroup with identity and with DCC on right ideals. Then S has no proper essential right congruences if and only if the followings hold:*

1. *A sequence of two-sided ideals $S = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_t$.*
2. *Every right ideal of S appears in this sequence.*
3. *$U_i = G_i \times K_i$ is a right group.*
4. *For each $i \leq j$, there is a homomorphism $\phi_{ij} : G_i \to G_j$ such that $\phi_{jk}\phi_{ij} = \phi_{ik}$ and a map $\psi_{ij} : (G_i \times K_i) \times K_j \to K_j$ such that $\psi_{ij}(d_k, \psi_{ij}(c_i, k_j)) = \psi_{kj}(c_i d_k, k_j)$ for all $d_k \in U_k, c_i \in U_i, k_j \in K_j$ and where $k \leq j$ and where $l = max\{k, l\}$.*
5. *For $i \leq j$, defined as $(g_i, k_i)(g_j, k_j) = (\phi_{ij}(g_i)g_j, k_j)$ and $(g_j, k_j)(g_i, k_i) = (g_j\phi_{ij}(g_i), \psi_{ij}((g_i, k_i), k_j))$.*
6. *For each $1 \leq r \leq t$ and every idempotent $x \in U_{r-1}$, there is an idempotent $y \in U_r$ such that $yx = y$ and if $a, b \in U_{r-1}$, then $ya = yb$ implies $a = b$.*
7. *G_t has no proper essential subgroup.*
8. *If Γ is S-admissible, U_{t-1}-transitive proper partition on K_t, then there is an S-admissible partition π on K_t such that $\Gamma \cap \pi = \iota$.*
9. *If $a, b \in U_{t-1}$ and $f \in U_t$, then $fa = fb$ implies $a = b$.*

NOTE. Let S be a semigroup with a left identity and with DCC on right ideals and assume that S has no proper essential right congruences. It is not hard to see that S satisfies all nine conditions in the previous theorem even though S has a left identity.

THEOREM 10. *Assume that S is a completely right injective semigroup with a left identity. Then the lattice $L(S)$ of right congruences on S is semiatomic if and only if S has no proper essential right congruences.*

PROOF. We shall first assume that the lattice $L(S)$ of right congruences on S is semiatomic. Let α be an essential right congruence on S and ρ in Ω where Ω is the set of all proper minimal right congruences on S. Since $\alpha \cap \rho \subseteq \rho, \alpha \cap \rho$ is either ρ or ι by the minimality of ρ. If the right congruence α is essential, then $\alpha \cap \rho = \rho, i.e., \rho \subseteq \alpha$, and hence the join, $\vee\rho$, of minimal right congruences is contained in α. Thus

$\upsilon = \vee\rho \subseteq \alpha$ so that $\alpha = \upsilon$. Therefore S has no proper essential right congruences. □

To show that the converse also holds, we assume that S is a completely right injective semigroup which has no proper essential right congruences. If S is completely right injective, it trivially holds dcc condition on right ideals and then the last right group U_t in the decomposition of S in characterization theorem of semigroups which has no proper essential right congruences contains a zero element e_t . If $a, b \in U_{t-1}$ with $a \neq b$, then condition 6 implies $e_t a \neq e_t b$ which contradicts to the fact that e_t is a zero of S . Hence $| U_{t-1} |= 1$. Set $U_{t-1} = \{e_{t-1}\}$. If $a, b \in U_{t-2}$ with $a \neq b$, then condition 6 also shows that for every idempotent x in U_{t-1} , $e_{t-1}x = e_{t-1}$ and $e_{t-1}a \neq e_{t-1}b$, but it contradicts to the fact that both $e_{t-1}a$ and $e_{t-1}b$ are in the singleton U_{t-1} . Hence $| U_{t-2} |= 1$. Continuing this process, we see that each right group U_i is a singleton(actually an idempotent) and then $S = \{e_0, e_1, \cdots, e_t\}$ where $e_i e_i = e_i$ for $i = 0, 1, \cdots, t$. Since all elements of S are dually well-ordered, we can list them as

$$1 = e_0 > e_1 > \cdots > e_t = 0.$$

Define a right congruence ρ_i on S generated by (e_{i-1}, e_i) for $i = 1, \cdots, t-1$. Since ρ_i is right compatible, $(e_{i-1}s, e_i s) \in \rho_i$ for all s in S. If $s = e_j$ for $j \leq i$, then $se_{i-1} = e_{i-1}s = e_{i-1}$ and $se_i = e_i s = e_i$ so that $(e_{i-1}, e_i) = (e_{i-1}s, e_i s) \in \rho_i$. Since $\rho_i = \{(e_{i-1}, e_i), (e_i, e_{i-1})\} \cup \iota$ for all i, it is easily seen that any two elements of S are connected by the join of those minimal right congruences ρ_i . Therefore the join of minimal right congruences is the universal congruence. It suffices to prove that $L(S)$ is semiatomic.

LEMMA 6. *If S is a group having more than one element, then the lattice $L(S)$ of right congruences on S can not be semiatomic.*

PROOF. Let Ω be the set of minimal right congruences on a group S. If $\rho \in \Omega$, let x be a non-trivial zero of ρ and let y be in the equivalence class which contains x. Then $x(x^{-1}y)\rho x$ implies $y = x(x^{-1}y) = x$. Since this contradicts the assumption that x was a nontrivial zero of ρ, the set Ω is empty and $\cup\Omega = \iota \neq \upsilon$ unless $| S |= 1$. □

We also observe the semiatomicity for semigroups for which every automaton is projective(called completely right projective semigroups).

DEFINITION 7. An automaton P is called *projective* if every diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow g \\ L & \xrightarrow{f} & M \end{array}$$

where f is onto can be completed (to yield a commutative diagram) by a homomorphism $h : P \to L$ such that $fh = g$.

THEOREM 11. ([4]) *Let S be a completely right projective semigroup. Then S is either a trivial group or a trivial group adjoined with 0.*

THEOREM 12. *Let $L(S)$ be the lattice of right congruences on a semigroup S with a left identity. Then the followings are equivalent.*

1. *For the completely right injective semigroup S, $L(S)$ is semiatomic;*
2. *For the completely right projective semigroup S, $L(S)$ is semiatomic;*
3. $\mid S \mid = 1$.

PROOF. (1) $\Leftrightarrow$ (3) If $\mid S \mid = 1$, then a right congruence ρ on a semigroup S is both the identity and the universal congruence. If Ω is the set of minimal right congruences on S, then Ω is empty and $\cup\Omega = \iota = \upsilon$ so that $L(S)$ is semiatomic.

We assume that S is a completely right injective semigroup whose lattice $L(S)$ of right congruences on S is semiatomic. If a semigroup S is completely right injective, then S satisfies conditions 1 - 7 in the characterization for those semigroups whose lattice of right congruences is seatomic. But since S has a zero, the condition 7 can not be satisfied unless $\mid S \mid = 1$. Thus the lattice $L(S)$ of the completely right injective semigroup S is semiatomic if and only if $\mid S \mid = 1$.

(2) $\Leftrightarrow$ (3) If S is the singleton, we have seen that $L(S)$ is semiatomic. Hence assume that S is a completely right projective semigroup whose lattice $L(S)$ of right congruences on S is semiatomic. If S is a completely right projective semigroup, then S is either $\{1\}$ or $\{1, 0\}$. Since S have a zero in either case, it is easily seen that the lattice $L(S)$ of this type of semigroups can not be semiatomic unless $\mid S \mid = 1$.

COROLLARY 1. *Assume that S is a completely right injective semigroup with a left identity. Then S has no proper essential right congruences if and only if $\mid S \mid = 1$.*

References

[1] A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups*, vol. 1, Math. Surveys, No. 7, A.M.S., 1961.

[2] E. H. Feller and R. L. Gantos, *Completely injective semigroups with central idempotents*, Glasgow Math. J. **10** (1969), 16–20.

[3] J. M. Howie, *An introduction to semigroup theory*, Academic Press, London, 1976.

[4] E. H. L. Moon, *Completely right projective semigroups*, J. KSME **9** (2002), no. 2, 119–128.

[5] R. H. Oehmke, *On essential right congruences of semigroup*, Acta Math. Hung. **57** (1991), no. (1-2), 73–83.

[6] ———, *Semigroups whose lattices of right congruences are semiatomic*, Comm. Algebra **27** (1999), no. 5, 2317–2329.

[7] ———, *On the semiatomicity for semigroups*, Algebras Groups Geom. **16** (1999), no. 4, 481–486.

[8] ———, *On minimal right congruences of a semigroup*, Tamkang J. Math. **16** (1985), no. 2, 29–35.

Center for Liberal Arts & Instructional Development
Myongji University
Kyunggido 449-728, Korea
E-mail: ehlmoon@mju.ac.kr