Commun. Korean Math. Soc. 19 (2004), No. 1, pp. 35–51

FUZZY CLOSURE SYSTEMS AND FUZZY CLOSURE OPERATORS

YONG CHAN KIM AND JUNG MI KO

ABSTRACT. We introduce fuzzy closure systems and fuzzy closure operators as extensions of closure systems and closure operators. We study relationships between fuzzy closure systems and fuzzy closure spaces. In particular, two families $F(S)$ and $F(C)$ of fuzzy closure systems and fuzzy closure operators on X are complete lattice isomorphic.

1. Introduction and preliminaries

Closure systems and closure operators play an important role in topological spaces, lattices [2, 3, 8, 9, 11], Boolean algebras [1, 2, 8, 11], convex sets [1, 2, 8], deductive systems [3, 4, 6]. Recently, Gerla *et al.* [1, 6, 7] studied fuzzy closure operators and fuzzy closure systems as extensions of closure systems and closure operators. They have been developed in many view points (fuzzy logic [1, 3, 4, 6, 7, 9, 11], fuzzy subalgebra [3, 6, 7], fuzzy congruences [6, 7], fuzzy topologies [10, 12]).

In this paper, we will study relationships between (fuzzy) closure systems and (fuzzy) closure spaces in a sense Gerla *et al.* [1]. We define subspaces and products of two families of fuzzy closure systems and fuzzy closure spaces. In particular, two families $F(S)$ and $F(C)$ of fuzzy closure systems and fuzzy closure operators on X are complete lattice isomorphic.

Received February 4, 2003.
2000 Mathematics Subject Classification: 08A72, 06D72, 06A15, 54A40.
Key words and phrases: (fuzzy) closure systems, (fuzzy) closure spaces, (fuzzy) S-map, (fuzzy) closure maps.

2. Preliminaries

In this paper, let X be a nonempty set, $I = [0,1]$, 2^X a family of all subsets of X and I^X a family of all fuzzy sets of X.

If $A \subset X$, we define the *characteristic function* χ_A on X by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

A *closure operator* ([2]) is a map $cl : 2^X \to 2^X$ if it satisfies:
(cl1) $A \subset cl(A)$, for each $A \in 2^X$,
(cl2) $A \subset B \Rightarrow cl(A) \subset cl(B)$, for each $A, B \in 2^X$,
(cl3) $cl(cl(A)) = cl(A)$, for each $A \in 2^X$.

DEFINITION 2.1 ([1]). A function $C : I^X \to I^X$ is called a *fuzzy closure operator* on X if it satisfies the following conditions:
(C1) $C(\lambda) \geq \lambda$, for all $\lambda \in I^X$.
(C2) $C(\lambda_1) \leq C(\lambda_2)$, if $\lambda_1 \leq \lambda_2$.
(C3) $C(C(\lambda)) = C(\lambda)$ for all $\lambda \in I^X$.
The pair (X, C) is called *a fuzzy closure space.*

Let C_1 and C_2 be fuzzy closure operators on X. We say that C_1 is *finer* than C_2 (C_2 is *coarser* than C_1), denoted by $C_2 \ll C_1$, if and only if $C_1(\lambda) \leq C_2(\lambda)$, for all $\lambda \in I^X$.

A subset $\mathfrak{S}$ of 2^X is called a *closure system* [2] if it satisfies:
(s1) $X \in \mathfrak{S}$,
(s2) if $A_i \in \mathfrak{S}$ for all $i \in \Gamma$, then $\bigcap_{i\in\Gamma} A_i \in \mathfrak{S}$.

DEFINITION 2.2 ([1, 2]). A subset $\mathcal{S}$ of I^X is called a *fuzzy closure system* on X if it satisfies:
(S1) $\tilde{1} \in \mathcal{S}$,
(S2) if $\lambda_i \in \mathcal{S}$ for all $i \in \Gamma$, then $\bigwedge_{i\in\Gamma} \lambda_i \in \mathcal{S}$.
The pair $(X, \mathcal{S})$ is called *a fuzzy closure system.*

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be fuzzy closure systems on X. We say that $\mathcal{S}_1$ is *finer* than $\mathcal{S}_2$ ($\mathcal{S}_2$ is *coarser* than $\mathcal{S}_1$) if and only if $\mathcal{S}_2 \subset \mathcal{S}_1$.

THEOREM 2.3 ([1]). *Let $(X, \mathcal{S})$ be a fuzzy closure system. For each $\lambda \in I^X$, we define an operator $C_{\mathcal{S}} : I^X \to I^X$ as follows:*

$$C_{\mathcal{S}}(\lambda) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda, \;\; \mu \in \mathcal{S}\}.$$

Then $(X, C_{\mathcal{S}})$ is a fuzzy closure space.

THEOREM 2.4 ([1]). (1) *Let C be a fuzzy operator on X satisfying* (C1) *and* (C2). *Define*

$$\mathcal{S}_C(\lambda) = \{\lambda \in I^X \mid C(\lambda) = \lambda\}.$$

Then $\mathcal{S}_C$ is a fuzzy closure system on X such that $C = C_{\mathcal{S}_C}$.

(2) *If $(X, \mathcal{S})$ is a fuzzy closure system, then $\mathcal{S}_{C_\mathcal{S}} = \mathcal{S}$.*

REMARK 2.5. (1) A closure system is often called a *Moore family* (ref. [2, 9]).

(2) A fuzzy closure space (X, C) is called *topological* if $C(\tilde{0}) = \tilde{0}$ and $C(\lambda \vee \mu) = C(\lambda) \vee C(\mu)$, for all $\lambda, \mu \in I^X$(ref. [10]).

(3) Let $(X, \mathcal{T})$ be a fuzzy topological space (ref. [5]). Then $\mathcal{S}_\mathcal{T} = \{\lambda \in I^X \mid \tilde{1} - \lambda \in \mathcal{T}\}$ is a fuzzy closure system and $C_\mathcal{T}(\lambda) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \tilde{1} - \lambda \in \mathcal{T}\}$ is a topological fuzzy closure operator on X (ref. [10]).

(4) Let $(X, \mathfrak{S})$ be a closure system. For each $A \in 2^X$, we define an operator $cl_\mathfrak{S} : 2^X \to 2^X$ by $cl_\mathfrak{S}(A) = \bigcap\{B \in 2^X \mid A \subset B,\quad B \in \mathfrak{S}\}$. Then $(X, cl_\mathfrak{S})$ is a closure space.

(5) Let cl be an operator on X satisfying (cl1) and (cl2). Define $\mathfrak{S}_{cl} = \{A \in 2^X \mid cl(A) = A\}$. Then $\mathfrak{S}_{cl}$ is a closure system on X such that $cl = cl_{\mathfrak{S}_{cl}}$. Moreover, if $(X, \mathfrak{S})$ is a closure system, then $\mathfrak{S}_{cl_\mathfrak{S}} = \mathfrak{S}$.

EXAMPLE 2.6. Let $X = \{x, y, z\}$ be a set. Define $C : I^X \to I^X$ as follows:

$$C(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \chi_{\{x,y\}}, & \text{if } \tilde{0} \neq \lambda \leq \chi_{\{x\}}, \\ \chi_{\{z\}}, & \text{if } \tilde{0} \neq \lambda \leq \chi_{\{z\}}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Then (X, C) is not a fuzzy closure space. Since $C(\chi_{\{x\}}) = \chi_{\{x,y\}}$ and $C(\chi_{\{x,y\}}) = \tilde{1}$, we have

$$\tilde{1} = C(C(\chi_{\{x\}})) \neq C(\chi_{\{x\}}) = \chi_{\{x,y\}}.$$

But (X, C) satisfies (C1) and (C2), by Theorem 2.4, we can obtain the fuzzy closure system $\mathcal{S}_C$ as follows:

$$\mathcal{S}_C = \{\tilde{0}, \tilde{1}, \chi_{\{z\}}\}.$$

From Theorem 2.3, the fuzzy closure operator $C_{\mathcal{S}_C}$ on X by induced $(X, \mathcal{S}_C)$ is defined by

$$C_{\mathcal{S}_C}(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \chi_{\{z\}}, & \text{if } \tilde{0} \neq \lambda \leq \chi_{\{z\}} \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

It follows that $C(\chi_{\{x\}}) = \chi_{\{x,y\}}$ but $C_{\mathcal{S}_C}(\chi_{\{x\}}) = \tilde{1}$. Hence $C \neq C_{\mathcal{S}_C}$.

3. Fuzzy closure systems and fuzzy closure operators

For each $\lambda \in I^X$ and $r \in I$, we denote $\lambda_r = \{x \in X \mid \lambda(x) \geq r\}$. We obtain fuzzy closure operators and fuzzy closure systems from closure operators and closure systems.

THEOREM 3.1 ([1]). *Let $cl : 2^X \to 2^X$ be a closure operator. Define a fuzzy operator $C_{cl} : I^X \to I^X$ as follows:*

$$C_{cl}(\lambda)(x) = \bigvee \{r \in I \mid x \in cl(\lambda_r)\}.$$

Then it satisfies the following properties:
(1) *C_{cl} is a fuzzy closure operator.*
(2) *C_{cl} is a fuzzy closure operator if and only if cl is a closure operator.*

THEOREM 3.2 ([1]). *Let $\mathfrak{S}$ be a closure system on X. Define a fuzzy closure system $\mathcal{S}_{\mathfrak{S}}$ as follows:*

$$\mathcal{S}_{\mathfrak{S}} = \{\lambda \in I^X \mid \lambda_r \in \mathfrak{S}, \forall r \in I\}.$$

Then it satisfies the following properties:
(1) *$\mathcal{S}_{\mathfrak{S}}$ is a fuzzy closure system on X.*
(2) *$\mathcal{S}_{\mathfrak{S}}$ is a fuzzy closure system if and only if $\mathfrak{S}$ is a closure system.*

THEOREM 3.3 ([1]). (1) *Let $\mathfrak{S}$ be a closure system on X and $cl_{\mathfrak{S}}$ the closure operator associated with $\mathfrak{S}$. Then $C_{\mathcal{S}_{\mathfrak{S}}} = C_{cl_{\mathfrak{S}}}$.*

(2) *Let cl be a closure operator on X and $\mathfrak{S}_{cl}$ the closure system associated with cl. Then $\mathcal{S}_{\mathfrak{S}_{cl}} = \mathcal{S}_{C_{cl}}$.*

EXAMPLE 3.4. Let $X = \{a, b\}$ be a set. Define $cl : 2^X \to 2^X$ as follows:

$$cl(\emptyset) = \{a\},\ cl(\{a\}) = \{a\},\ cl(\{b\}) = X,\ cl(X) = X.$$

By Remark 2.5 (5), we obtain $\mathfrak{S}_{cl} = \{\{a\}, X\}$. For each $\lambda \in I^X$, put $\lambda_b \in I^X$ such that

$$\lambda_b(a) = 1, \lambda_b(b) = \lambda(b).$$

We obtain $C_{cl} : I^X \to I^X$ as follows:

$$C_{cl}(\lambda) = \begin{cases} \chi_{\{a\}}, & \text{if } \lambda(b) = 0, \\ \lambda_b, & \text{if } \lambda(b) \neq 0. \end{cases}$$

Then $\mathcal{S}_{C_{cl}} = \mathcal{S}_{\mathfrak{S}_{cl}} = \{\chi_{\{a\}}, \lambda_b\}$.

Let (X, cl_1) and (Y, cl_2) be closure spaces. A function $f : (X, cl_1) \to (Y, cl_2)$ is a *closure map* if for each $A \in 2^X$, $f(cl_1(A)) \subset cl_2(f(A))$.

DEFINITION 3.5. Let (X, C_1) and (Y, C_2) be fuzzy closure spaces. A function $f : (X, C_1) \to (Y, C_2)$ is called a *fuzzy closure map* if for each $\lambda \in I^X$, $f(C_1(\lambda)) \leq C_2(f(\lambda))$.

THEOREM 3.6. *Let (X, cl_1) and (Y, cl_2) be closure spaces. A function $f : (X, cl_1) \to (Y, cl_2)$ is a closure map if and only if*

$$f : (X, C_{cl_1}) \to (Y, C_{cl_2})$$

is a fuzzy closure map.

PROOF. ($\Rightarrow$) Let f be a closure map. Suppose there exists $\lambda \in I^X$ such that $f(C_{cl_1}(\lambda)) \not\leq C_{cl_2}(f(\lambda))$. Then there exists $y \in Y$ such that

$$f(C_{cl_1}(\lambda))(y) > C_{cl_2}(f(\lambda))(y).$$

Also, there exists $x \in X$ with $x \in f^{-1}(\{y\})$ such that

$$f(C_{cl_1}(\lambda))(f(x)) \geq C_{cl_1}(\lambda)(x) > C_{cl_2}(f(\lambda))(f(x)).$$

By the definition of $C_{cl_1}(\lambda)$, there exists $t \in I$ with $x \in cl_1(\lambda_t)$ such that

$$f(C_{cl_1}(\lambda))(f(x)) \geq C_{cl_1}(\lambda)(x) \geq t > C_{cl_2}(f(\lambda))(f(x)).$$

On the other hand, since f is a closure map, $f(cl_1(\lambda_t)) \subset cl_2(f(\lambda_t))$. It implies

$$f(x) \in f(cl_1(\lambda_t)) \subset cl_2(f(\lambda_t)) \subset cl_2(f(\lambda)_t).$$

Thus $C_{cl_2}(f(\lambda))(f(x)) \geq t$. It is contradiction.

($\Leftarrow$) Suppose f is not a closure map. Then there exists $A \subset X$ such that $f(cl_1(A)) \not\subset cl_2(f(A))$. Then there exists $y \in Y$ such that

$$y \in f(cl_1(A)), \ \ y \notin cl_2(f(A)).$$

Also, there exists $x \in X$ with $f(x) = y$ such that

$$x \in cl_1(A), \ \ y \notin cl_2(f(A)).$$

It implies

$$x \in cl_1((\chi_A)_t), \forall t \in I, \ \ y \notin cl_2((\chi_{f(A)})_s), \forall s \in (0,1].$$

So, $C_{cl_1}(\chi_A)(x) = 1, \quad C_{cl_2}(\chi_{f(A)})(f(x)) = 0$. Thus,

$$1 = f(C_{cl_1}(\chi_A))(f(x)) \not\leq C_{cl_2}(f(\chi_A))(f(x)) = C_{cl_2}(\chi_{f(A)})(f(x)) = 0.$$

Hence f is not a fuzzy closure map. □

Let $(X, \mathfrak{S}_1)$ and $(Y, \mathfrak{S}_2)$ be closure systems. A map $f : (X, \mathfrak{S}_1) \to (Y, \mathfrak{S}_2)$ is a S-map if $f^{-1}(B) \in \mathfrak{S}_1$ for all $B \in \mathfrak{S}_2$.

DEFINITION 3.7. Let $(X, \mathcal{S}_1)$ and $(Y, \mathcal{S}_2)$ be fuzzy closure systems. A function $f : (X, \mathcal{S}_1) \to (Y, \mathcal{S}_2)$ is called a fuzzy S-map if $f^{-1}(\lambda) \in \mathcal{S}_1$ for all $\lambda \in \mathcal{S}_2$.

THEOREM 3.8. *Let $(X, \mathfrak{S}_1)$ and $(Y, \mathfrak{S}_2)$ be fuzzy closure systems. A function $f : (X, \mathfrak{S}_1) \to (Y, \mathfrak{S}_2)$ is a S-map if and only if*

$$f : (X, \mathcal{S}_{\mathfrak{S}_1}) \to (Y, \mathcal{S}_{\mathfrak{S}_2})$$

is a fuzzy S-map.

PROOF. ($\Rightarrow$) For each $\lambda \in \mathcal{S}_{\mathfrak{S}_2}$, we have $\lambda_t \in \mathfrak{S}_2$ for all $t \in I$. Since f is a S-map, $f^{-1}(\lambda_t) = f^{-1}(\lambda)_t \in \mathfrak{S}_1$ for all $t \in I$. Hence $f^{-1}(\lambda) \in \mathcal{S}_{\mathfrak{S}_1}$.

($\Leftarrow$) For all $A \in \mathfrak{S}_2$, we have $(\chi_A)_t \in \mathfrak{S}_2$ for all $t \in I$. So, $\chi_A \in \mathcal{S}_{\mathfrak{S}_2}$. Since f is a fuzzy S-map, $f^{-1}(\chi_A) = \chi_{f^{-1}(A)} \in \mathcal{S}_{\mathfrak{S}_1}$. Hence, for all $t \in (0,1]$, $(\chi_{f^{-1}(A)})_t = f^{-1}(A) \in \mathfrak{S}_1$. □

4. Fuzzy closure systems and fuzzy closure operators

THEOREM 4.1. *Let $(X, \mathcal{S}_1)$ and $(Y, \mathcal{S}_2)$ be fuzzy closure systems. Then the following statements are equivalent:*
(1) *A function $f : (X, \mathcal{S}_1) \to (Y, \mathcal{S}_2)$ is a fuzzy S-map.*
(2) *$f : (X, C_{\mathcal{S}_1}) \to (Y, C_{\mathcal{S}_2})$ is a fuzzy closure map.*
(3) *$C_{\mathcal{S}_1}(f^{-1}(\lambda)) \leq f^{-1}(C_{\mathcal{S}_2}(\lambda))$, for each $\lambda \in I^Y$.*

PROOF. (1)$\Rightarrow$(2) Let f be a fuzzy S-map. For all $\lambda \in I^X$, we have the following:

$$C_{\mathcal{S}_2}(f(\lambda)) = \bigwedge\{\mu \in I^Y \mid \mu \geq f(\lambda), \mu \in \mathcal{S}_2\}$$

(Since $f^{-1}(\mu) \geq f^{-1}(f(\lambda)) \geq \lambda, f^{-1}(\mu) \in \mathcal{S}_1$,)

$$\begin{aligned} &\geq \bigwedge\{f(f^{-1}(\mu)) \mid f^{-1}(\mu) \geq \lambda, f^{-1}(\mu) \in \mathcal{S}_1\} \\ &\geq f(\bigwedge\{f^{-1}(\mu) \mid f^{-1}(\mu) \geq \lambda, f^{-1}(\mu) \in \mathcal{S}_1\}) \\ &\geq f(C_{\mathcal{S}_1}(\lambda)). \end{aligned}$$

(2)$\Rightarrow$(3) For all $\lambda \in I^X$, we have

$$\begin{aligned} f(C_{\mathcal{S}_1}(f^{-1}(\lambda))) &\leq C_{\mathcal{S}_2}(f(f^{-1}(\lambda))) \quad (f \text{ is a fuzzy closure map}) \\ &\leq C_{\mathcal{S}_2}(\lambda). \end{aligned}$$

It follows that

$$\begin{aligned} C_{\mathcal{S}_1}(f^{-1}(\lambda)) &\leq f^{-1}(f(C_{\mathcal{S}_1}(f^{-1}(\lambda)))) \\ &\leq f^{-1}(C_{\mathcal{S}_2}(\lambda)). \end{aligned}$$

(3)$\Rightarrow$(1) Let $\lambda \in \mathcal{S}_2$. Then $C_{\mathcal{S}_2}(\lambda) = \lambda$. By (3) and Definition 2.1 (C1),

$$C_{\mathcal{S}_1}(f^{-1}(\lambda)) = f^{-1}(\lambda).$$

Hence $f^{-1}(\lambda) \in \mathcal{S}_{C_{\mathcal{S}_2}} = \mathcal{S}_2$ from Theorem 2.4. □

The following corollary is similarly proved as in Theorem 4.1.

COROLLARY 4.2. *Let (X, C_1) and (Y, C_2) be fuzzy closure spaces. A map $f : (X, C_1) \to (Y, C_2)$ is a fuzzy closure map if and only if $f : (X, \mathcal{S}_{C_1}) \to (Y, \mathcal{S}_{C_2})$ is a fuzzy S-map.*

THEOREM 4.3. *Let X be a set and $\{(X_i, \mathcal{S}_i)\}_{i\in\Gamma}$ a family of fuzzy closure systems. Let $f_i : X \to X_i$ be a function. Define*

$$\mathcal{S} = \{\bigwedge_{i\in\Gamma} f_i^{-1}(\mu_i) \mid \mu_i \in \mathcal{S}_i\}.$$

Then:

(1) *$\mathcal{S}$ is the coarsest fuzzy closure system on X for which each f_i is a fuzzy S-map.*

(2) *A function $f : (Y, \mathcal{S}') \to (X, \mathcal{S})$ is a fuzzy S-map if and only if $f_i \circ f : (Y, \mathcal{S}') \to (X_i, \mathcal{S}_i)$ is a fuzzy S-map, for each $i \in \Gamma$.*

PROOF. (1) We easily prove that $\mathcal{S}$ is a fuzzy closure system on X. Let $f_i : (X, \mathcal{S}^*) \to (X_i, \mathcal{S}_i)$ be a fuzzy S-map. For $\mu = \bigwedge_{i\in\Gamma} f_i^{-1}(\mu_i) \in \mathcal{S}$ with $\mu_i \in \mathcal{S}_i$, then $f_i^{-1}(\mu_i) \in \mathcal{S}^*$. Since $\mathcal{S}^*$ is a fuzzy closure system on X, $\mu \in \mathcal{S}^*$, that is, $\mathcal{S} \subset \mathcal{S}^*$. Hence $\mathcal{S}$ is the coarsest fuzzy closure system on X for which each f_i is a fuzzy S-map.

(2) ($\Rightarrow$) It is trivial because the composition of fuzzy S-maps is a fuzzy S-map.

($\Leftarrow$) For $\mu = \bigwedge_{i\in\Gamma} f_i^{-1}(\mu_i) \in \mathcal{S}$ with $\mu_i \in \mathcal{S}_i$, then $f^{-1}(f_i^{-1}(\mu_i)) \in \mathcal{S}'$. It implies $f^{-1}(\mu) = \bigwedge_{i\in\Gamma} f^{-1}(f_i^{-1}(\mu_i)) \in \mathcal{S}'$. Thus, f is a fuzzy S-map.□

COROLLARY 4.4. *Let $F(S)$ be a family of fuzzy closure systems on X. In above theorem, each $f_i = id_X : X \to X$ is an identity function. For each $\{\mathcal{S}_i\}_{i\in\Gamma} \subset F(S)$, We can define*

$$\bigvee_{i\in\Gamma} \mathcal{S}_i = \{\bigwedge_{i\in\Gamma} \mu_i \mid \mu_i \in \mathcal{S}_i\}$$

$$\bigwedge_{i\in\Gamma} \mathcal{S}_i = \bigcap_{i\in\Gamma} \mathcal{S}_i.$$

Then $(F(S), \vee, \wedge, \subset, \{\tilde{1}\}, I^X)$ is a complete lattice where $\{\tilde{1}\}$ is the greatest lower bound and I^X is the least upper bound in $F(S)$.

Using Theorem 4.3, we can define subspaces and products in the obvious way.

DEFINITION 4.5. Let $(X, \mathcal{S})$ be a fuzzy closure system and A a subset of X. The pair $(A, \mathcal{S}_A)$ is said to be a *subspace* of $(X, \mathcal{S})$ if $\mathcal{S}_A$ is the coarsest fuzzy closure system on X which the inclusion function $i : A \to X$ is a fuzzy S-map.

DEFINITION 4.6. Let $\{(X_i, \mathcal{S}_i) \mid i \in \Gamma\}$ be a family of fuzzy closure systems. Let $X = \prod_{i\in\Gamma} X_i$ be a product set. The coarsest fuzzy closure system $\mathcal{S} = \otimes\mathcal{S}_i$ on X with respect to $(X, \pi_i, (X_i, \mathcal{S}_i))$ where $\pi_i : X \to X_i$ is projection map is called the *product fuzzy closure system* of $\{\mathcal{S}_i \mid i \in \Gamma\}$.

Using Theorem 4.3, we have the following corollary.

COROLLARY 4.7. *Let $\{(X_i, \mathcal{S}_i)\}_{i\in\Gamma}$ be a family of fuzzy closure systems, $X = \prod_{i\in\Gamma} X_i$ a product set and, for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection. The structure $\mathcal{S} = \otimes\mathcal{S}_i$ on X is defined by*

$$\mathcal{S} = \{\bigwedge_{i\in\Gamma} \pi_i^{-1}(\mu_i) \mid \mu_i \in \mathcal{S}_i\}.$$

Then:

(1) $\mathcal{S}$ is the coarsest fuzzy closure system on X which for each $i \in \Gamma$, π_i is a fuzzy S-map.

(2) A function $f : (Y, \mathcal{S}') \to (X, \mathcal{S})$ is a fuzzy S-map if and only if $\pi_i \circ f : (Y, \mathcal{S}') \to (X_i, \mathcal{S}_i)$ is a fuzzy S-map, for each $i \in \Gamma$.

THEOREM 4.8. *Let X be a set and $\{(X_i, C_i)\}_{i\in\Gamma}$ a family of fuzzy closure spaces. Let $f_i : X \to X_i$ be a function for each $i \in \Gamma$. For each $\lambda \in I^X$, we define the function $C : I^X \to I^X$ by*

$$C(\lambda) = \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda))).$$

Then we have the following statements:

(1) C is the coarsest fuzzy closure operator on X for which each f_i is a fuzzy closure map.

(2) A function $f : (Y, C') \to (X, C)$ is a fuzzy closure map if and only if $\pi_i \circ f : (Y, C') \to (X_i, C_i)$ is a fuzzy closure map, for each $i \in \Gamma$.

(3) $\mathcal{S}_C = \mathcal{S}$ where $\mathcal{S} = \{\bigwedge_{i\in\Gamma} f_i^{-1}(\mu_i) \mid \mu_i \in \mathcal{S}_{C_i}\}$.

PROOF. (1) We will show that C is the fuzzy closure operator on X.

(C1) Since $C(\lambda) = \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda))) \geq \bigwedge_{i\in\Gamma} f_i^{-1}(f_i(\lambda)) \geq \lambda$, we have $\lambda \leq C(\lambda)$.

(C2) It is trivial.

(C3) For each $\lambda \in I^X$, $C(C(\lambda)) = C(\lambda)$ from (C2) and

$$\begin{aligned} C(C(\lambda)) &= \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(C(\lambda)))) \\ &= \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda)))))) \\ &\leq \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(f_i^{-1}(C_i(f_i(\lambda)))))) \\ &\leq \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(C_i(f_i(\lambda)))) \\ &= \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda))) \\ &= C(\lambda). \end{aligned}$$

Let $f_i : (X, C^*) \to (X_i, C_i)$ be a fuzzy closure map for each $i \in \Gamma$. For each $\lambda \in I^X$,

$$\begin{aligned} & f_i(C^*(\lambda)) \leq C_i(f_i(\lambda)),\ \forall i \in \Gamma \\ \Longrightarrow\ & C^*(\lambda) \leq f_i^{-1}(f_i(C^*(\lambda))) \leq f_i^{-1}(C_i(f_i(\lambda))),\ \forall i \in \Gamma \\ \Longrightarrow\ & C^*(\lambda) \leq \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda))) \\ \Longrightarrow\ & C^*(\lambda) \leq C(\lambda). \end{aligned}$$

Hence C is the coarsest fuzzy closure operator on X for which each f_i is a fuzzy closure map.

(2) ($\Rightarrow$) It is trivial because the composition of fuzzy closure maps is a fuzzy closure map.

($\Leftarrow$) Let $f_i \circ f : (Y, C') \to (X_i, C_i)$ be a fuzzy closure map for each $i \in \Gamma$. For each $\lambda \in I^Y$,

$$\begin{aligned} & f_i \circ f(C'(\lambda)) \leq C_i(f_i \circ f(\lambda)),\ \forall i \in \Gamma \\ \Longrightarrow\ & f(C'(\lambda)) \leq f_i^{-1}(f_i(f(C'(\lambda)))) \leq f_i^{-1}(C_i(f_i(f(\lambda)))),\ \forall i \in \Gamma \\ \Longrightarrow\ & f(C'(\lambda)) \leq \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(f(\lambda)))) \\ \Longrightarrow\ & f(C'(\lambda)) \leq C(f(\lambda)). \end{aligned}$$

Hence $f : (Y, C') \to (X, C)$ is a fuzzy closure map.

(3) Let $\lambda \in \mathcal{S}_C$. Then $C(\lambda) = \lambda$, i.e. $\lambda = \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\lambda)))$. Since $C_i(C_i(f_i(\lambda))) = C_i(f_i(\lambda))$, then $C_i(f_i(\lambda)) \in \mathcal{S}_{C_i}$. Thus $\lambda \in \mathcal{S}$.

Let $\mu \in \mathcal{S}$. Then there exists $\rho_i \in \mathcal{S}_{C_i}$ with $C_i(\rho_i) = \rho_i$ such that $\mu = \bigwedge_{i\in\Gamma} f_i^{-1}(\rho_i)$. It implies

$$\begin{aligned} C(\mu) &= \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\mu))) \\ &= \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(\bigwedge_{i\in\Gamma} f_i^{-1}(\rho_i)))) \\ &\le \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(f_i(f_i^{-1}(\rho_i)))) \\ &\le \bigwedge_{i\in\Gamma} f_i^{-1}(C_i(\rho_i)) \\ &= \bigwedge_{i\in\Gamma} f_i^{-1}(\rho_i) \\ &= \mu. \end{aligned}$$

Then $C(\mu) = \mu$, i.e. $\mu \in \mathcal{S}_C$. □

THEOREM 4.9. *Let $\{(X_i, \mathcal{S}_i)\}_{i\in\Gamma}$ be a family of fuzzy closure systems. Let $f_i : X \to X_i$ be a function and $\mathcal{S}$ a fuzzy closure system on X as follows:*

$$\mathcal{S} = \{\bigwedge_{i\in\Gamma} f_i^{-1}(\mu_i) \mid \mu_i \in \mathcal{S}_i\}.$$

Then, for each $\lambda \in I^X$,

$$C_{\mathcal{S}}(\lambda) = \bigwedge_{i\in\Gamma} f_i^{-1}(C_{\mathcal{S}_i}(f_i(\lambda))).$$

PROOF. Let $C_{\mathcal{S}_i}(f_i(\lambda)) = \bigwedge\{\rho_i \mid \rho_i \ge f_i(\lambda), \quad \rho_i \in \mathcal{S}_i\}$. Since $f_i^{-1}(\rho_i) \ge f_i^{-1}(f_i(\lambda)) \ge \lambda$ and for each $\rho_i \in \mathcal{S}_i$,

$$f_i^{-1}(\rho_i) = (\bigwedge_{j\in\Gamma-\{i\}} f_j^{-1}(\tilde{1})) \wedge f_i^{-1}(\rho_i) \in \mathcal{S},$$

we have

$$\begin{aligned} f_i^{-1}(C_{\mathcal{S}_i}(f_i(\lambda))) &= \bigwedge\{f_i^{-1}(\rho_i) \mid \rho_i \ge f_i(\lambda), \ \ \rho_i \in \mathcal{S}_i\} \\ &\ge \bigwedge\{f_i^{-1}(\rho_i) \mid f_i^{-1}(\rho_i) \ge \lambda, \ \ f_i^{-1}(\rho_i) \in \mathcal{S}\}. \end{aligned}$$

It implies

$$\begin{aligned}&\bigwedge_{i\in\Gamma} f_i^{-1}(C_{\mathcal{S}_i}(f_i(\lambda)))\\ &\geq \bigwedge_{i\in\Gamma}\bigwedge\{f_i^{-1}(\rho_i)\mid f_i^{-1}(\rho_i)\geq\lambda,\ \ f_i^{-1}(\rho_i)\in\mathcal{S}\}\\ &\geq C_{\mathcal{S}}(\lambda).\end{aligned}$$

Suppose there exists $\lambda \in I^X$ such that

$$C_{\mathcal{S}}(\lambda) \not\geq \bigwedge_{i\in\Gamma} f_i^{-1}(C_{\mathcal{S}_i}(f_i(\lambda))).$$

Then there exists $x_0 \in X$ such that

$$C_{\mathcal{S}}(\lambda)(x_0) < \bigwedge_{i\in\Gamma} f_i^{-1}(C_{\mathcal{S}_i}(f_i(\lambda)))(x_0).$$

From the definition of $C_{\mathcal{S}}(\lambda)$, there exists $\mu \in I^X$ with $\mu \geq \lambda$ and $\mu \in \mathcal{S}$ such that

$$C_{\mathcal{S}}(\lambda)(x_0) \leq \mu(x_0) < \bigwedge_{i\in\Gamma} f_i^{-1}(C_{\mathcal{S}_i}(f_i(\lambda)))(x_0).$$

On the other hand, since $\mu \in \mathcal{S}$, there exists $\nu_i \in I^{X_i}$ such that

$$\nu_i \in \mathcal{S}_i \ \text{ and } \bigwedge_{i\in\Gamma} f_i^{-1}(\nu_i) = \mu.$$

Since $\bigwedge_{i\in\Gamma} f_i^{-1}(\nu_i) \geq \lambda$, we have

$$f_i(\lambda) \leq f_i(\bigwedge_{i\in\Gamma} f_i^{-1}(\nu_i)) \leq \nu_i.$$

Then $C_{\mathcal{S}_i}(f_i(\lambda)) \leq \nu_i$.

$$\bigwedge_{i\in\Gamma} f_i^{-1}(C_{\mathcal{S}_i}(f_i(\lambda))) \leq \bigwedge_{i\in\Gamma} f_i^{-1}(\nu_i) = \mu.$$

It is a contradiction. Hence $C_{\mathcal{S}}(\lambda) \geq \bigwedge_{i\in\Gamma} f_i^{-1}(C_{\mathcal{S}_i}(f_i(\lambda)))$. □

THEOREM 4.10. *Let $F(C)$ be a family of fuzzy closure operators on X. For each $\{C_i \mid i \in \Gamma\} \subset F(C)$, we define:*

$$\Cup_{i\in\Gamma} C_i(\lambda) = \bigwedge_{i\in\Gamma} C_i(\lambda)$$

$$\Cap_{i\in\Gamma} C_i = C_{\cap \mathcal{S}_{C_i}}$$

where $C_{\cap \mathcal{S}_{C_i}} = \bigcap\{\mu \mid \mu \geq \lambda, \ \ \mu \in \cap_{i\in\Gamma}\mathcal{S}_{C_i}\}$.

Then $(F(C), \Cup, \Cap, \ll, C_0, C_1)$ is a complete lattice where C_0 is the greatest lower bound and C_1 is the least upper bound in $F(C)$ defined as follows:

$$C_0(\lambda) = \tilde{1}, \ \ C_1(\lambda) = \lambda, \ \ \forall \lambda \in I^X.$$

PROOF. In Theorem 4.8, let $\{C_i\}_{i\in\Gamma}$ be a family of fuzzy closure operators on X and $f_i = id_X : X \to X$ identity function. Then $\Cup_{i\in\Gamma} C_i(\lambda) = \bigwedge_{i\in\Gamma} C_i(\lambda)$ is the least upper bound of $\{C_i\}_{i\in\Gamma}$.

Since $\cap \mathcal{S}_{C_i}$ is a fuzzy closure system, $C_{\cap \mathcal{S}_{C_i}}$ is a fuzzy closure operator. Since $\cap_{i\in\Gamma}\mathcal{S}_{C_i} \subset \mathcal{S}_{C_i}$, $id_X : (X, \mathcal{S}_{C_i}) \to (X, \cap_{i\in\Gamma}\mathcal{S}_{C_i})$ is a fuzzy S-map. By Theorem 4.1, $id_X : (X, C_{\mathcal{S}_{C_i}}) \to (X, C_{\cap_{i\in\Gamma}\mathcal{S}_{C_i}})$ is a fuzzy closure map. Hence

$$C_{\cap_{i\in\Gamma}\mathcal{S}_{C_i}} \ll C_{\mathcal{S}_{C_i}} = C_i.$$

If C is a fuzzy closure operator such that $C \ll C_i$ for all $i \in \Gamma$, then $C(\lambda) \geq C_i(\lambda)$ for all $i \in \Gamma$. By Theorem 4.1, $\mathcal{S}_C \subset \mathcal{S}_{C_i}$. Since $C(C(\lambda)) = C(\lambda)$, then $C(\lambda) \in \mathcal{S}_C \subset \mathcal{S}_{C_i}$, for each $i \in \Gamma$. Thus $C(\lambda) \in \cap_{i\in\Gamma}\mathcal{S}_{C_i}$. It implies

$$C_{\cap_{i\in\Gamma}\mathcal{S}_{C_i}}(\lambda) \leq C(\lambda).$$

Hence $C \ll C_{\cap_{i\in\Gamma}\mathcal{S}_{C_i}}$. So, $\Cap_{i\in\Gamma} C_i = C_{\cap \mathcal{S}_{C_i}}$ is the greatest lower bound of $\{C_i\}_{i\in\Gamma}$. □

EXAMPLE 4.11. Let $X = \{a, b, c\}$ be a set. Define $C_i : I^X \to I^X$ as follows:

$$C_1(\lambda) = \begin{cases} \chi_{\{a\}}, & \text{if } \tilde{0} \leq \lambda \leq \chi_{\{a\}}, \\ \tilde{1}, & \text{otherwise,} \end{cases}$$

$$C_2(\lambda) = \begin{cases} \chi_{\{b\}}, & \text{if } \tilde{0} \leq \lambda \leq \chi_{\{b\}}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

We define $(C_1 \vee C_2)(\lambda) = C_1(\lambda) \vee C_2(\lambda)$. Then

$$(C_1 \vee C_2)(\lambda) = \begin{cases} \chi_{\{a,b\}}, & \text{if } \tilde{0} \leq \lambda \leq \chi_{\{a\}}, \\ \chi_{\{a,b\}}, & \text{if } \tilde{0} \leq \lambda \leq \chi_{\{b\}}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

Since

$$\tilde{1} = (C_1 \vee C_2)(C_1 \vee C_2)(\chi_{\{a\}}) \neq (C_1 \vee C_2)(\chi_{\{a\}}) = \chi_{\{a,b\}},$$

$C_1 \vee C_2$ is not a fuzzy closure operator. We obtain $\mathcal{S}_{C_i}$ as follows:

$$\mathcal{S}_{C_1} = \{\chi_{\{a\}}, \tilde{1}\}, \ \ \mathcal{S}_{C_2} = \{\chi_{\{b\}}, \tilde{1}\}.$$

By Corollary 4.4, we obtain

$$\mathcal{S}_{C_1} \cap \mathcal{S}_{C_2} = \{\tilde{1}\}.$$

By Theorem 4.10, we have, for all $\lambda \in I^X$,

$$C_1 \Cap C_2(\lambda) = C_{\mathcal{S}_{C_1} \cap \mathcal{S}_{C_2}}(\lambda) = \tilde{1}.$$

Using Theorem 4.8, we can define subspaces and products in the obvious way.

DEFINITION 4.12. Let (X, C) be a fuzzy closure space and A a subset of X. The pair (A, C_A) is said to be a *subspace* of (X, C) if C_A is the coarsest fuzzy closure operator on X which the inclusion function i is a fuzzy closure map.

DEFINITION 4.13. Let $\{(X_i, C_i) \mid i \in \Gamma\}$ be the family of fuzzy closure spaces. Let $X = \prod_{i \in \Gamma} X_i$ be a product set. The coarsest fuzzy closure operator $C = \otimes C_i$ on X with respect to $(X, \pi_i, (X_i, C_i))$ where $\pi_i : X \to X_i$ is projection map is called the *product fuzzy closure operator* of $\{C_i \mid i \in \Gamma\}$.

Using Theorem 4.8, we have the following corollary.

COROLLARY 4.14. *Let $\{(X_i, C_i) \mid i \in \Gamma\}$ be a family of fuzzy closure spaces. Let $X = \prod_{i \in \Gamma} X_i$ be a product set and, for each $i \in \Gamma$, $\pi_i : X \to X_i$ a projection. The structure $C = \otimes C_i : I^X \to I^X$ is defined by*

$$C(\mu) = \bigwedge_{i \in \Gamma} \pi_i^{-1}(C_i(\pi_i(\mu))).$$

Then:

(1) C is the coarsest fuzzy closure operator on X which for each $i \in \Gamma$, π_i is a fuzzy closure map.

(2) A function $f : (Y, C') \to (X, C)$ is a fuzzy closure map if and only if $\pi_i \circ f : (Y, C') \to (X_i, C_i)$ is a fuzzy closure map, for each $i \in \Gamma$.

DEFINITION 4.15. Let $(L_1, \vee, \wedge, \leq)$ and $(L_2, \vee, \wedge, \leq)$ be complete lattices. A function $f : L_1 \to L_2$ is called a *complete lattice isomorphism* if it satisfies the following conditions:

(1) f is bijective,

(2) $f(\bigvee_{i\in\Gamma} a_i) = \bigvee_{i\in\Gamma} f(a_i)$ and $f(\bigwedge_{i\in\Gamma} a_i) = \bigwedge_{i\in\Gamma} f(a_i)$ for any $\{a_i \mid i \in \Gamma\} \subset L_1$,

(3) $f^{-1}(\bigvee_{i\in\Gamma} b_i) = \bigvee_{i\in\Gamma} f^{-1}(b_i)$ and $f^{-1}(\bigwedge_{i\in\Gamma} b_i) = \bigwedge_{i\in\Gamma} f^{-1}(b_i)$ for any $\{b_i \mid i \in \Gamma\} \subset L_2$.

Two lattices L_1 and L_2 are *complete lattice isomorphic* if there exists complete lattice isomorphism $f : L_1 \to L_2$.

THEOREM 4.16. *Let $(F(S), \vee, \wedge, \subset)$ and $(F(C), \Cup, \Cap, \ll)$ be complete lattices where $F(S)$ and $F(C)$ are two families of fuzzy closure systems and fuzzy closure operators on X. Define a function $f : F(S) \to F(C)$ by $f(\mathcal{S}) = C_{\mathcal{S}}$. Then f is a complete lattice isomorphism, that is, $F(S)$ and $F(C)$ are complete lattice isomorphic.*

PROOF. (1) Define a function $g : F(C) \to F(S)$ by $g(C) = \mathcal{S}_C$. Then $f(g(C)) = f(\mathcal{S}_C) = C_{\mathcal{S}_C} = C$ and $g(f(\mathcal{S})) = f(C_{\mathcal{S}}) = \mathcal{S}_{C_{\mathcal{S}}} = \mathcal{S}$ from Theorem 2.4. Hence f is bijective.

(2) In Theorem 4.9, let $f_i = id_X : X \to X$ be an identity map for each $i \in \Gamma$. Then $C_{\bigvee_{i\in\Gamma}\mathcal{S}_i} = \bigwedge_{i\in\Gamma} C_{\mathcal{S}_i}$. Since $f(\bigvee_{i\in\Gamma}\mathcal{S}_i) = C_{\bigvee_{i\in\Gamma}\mathcal{S}_i}$ and $\Cup_{i\in\Gamma} f(\mathcal{S}_i) = \Cup_{i\in\Gamma} C_{\mathcal{S}_i}$, for any $\{\mathcal{S}_i \mid i \in \Gamma\} \subset F(S)$, by Theorem 4.10,

$$f(\bigvee_{i\in\Gamma}\mathcal{S}_i) = C_{\bigvee_{i\in\Gamma}\mathcal{S}_i} = \bigwedge_{i\in\Gamma} C_{\mathcal{S}_i} = \Cup_{i\in\Gamma} C_{\mathcal{S}_i} = \Cup_{i\in\Gamma} f(\mathcal{S}_i).$$

Since $f(\bigwedge_{i\in\Gamma}\mathcal{S}_i) = C_{\cap_{i\in\Gamma}\mathcal{S}_i}$ and $\Cap_{i\in\Gamma} f(\mathcal{S}_i) = \Cap_{i\in\Gamma} C_{\mathcal{S}_i}$, by Theorem 4.10, for any $\{\mathcal{S}_i \mid i \in \Gamma\} \subset F(S)$,

$$\Cap_{i\in\Gamma} f(\mathcal{S}_i) = \Cap_{i\in\Gamma} C_{\mathcal{S}_i} = C_{\cap\mathcal{S}_{C_{\mathcal{S}_i}}} = C_{\cap\mathcal{S}_i} = f(\bigwedge_{i\in\Gamma}\mathcal{S}_i).$$

(3) In Theorem 4.8(3), let $f_i = id_X : X \to X$ be an identity map for each $i \in \Gamma$. For any $\{C_i \mid i \in \Gamma\} \subset F(C)$,

$$g(\Cup_{i\in\Gamma} C_i) = \mathcal{S}_{\Cup_{i\in\Gamma} C_i} = \bigvee_{i\in\Gamma} \mathcal{S}_{C_i} = \bigvee_{i\in\Gamma} g(C_i).$$

Let $g(\Cap_{i\in\Gamma} C_i) = \mathcal{S}_{\Cap_{i\in\Gamma} C_i}$ and $\bigwedge_{i\in\Gamma} g(C_i) = \bigwedge_{i\in\Gamma} \mathcal{S}_{C_i}$, for any $\{C_i \mid i \in \Gamma\} \subset F(C)$. Since $\Cap_{i\in\Gamma} C_i = C_{\cap\mathcal{S}_{C_i}}$ from Theorem 4.10,

$$g(\Cap_{i\in\Gamma} C_i) = \mathcal{S}_{\Cap_{i\in\Gamma} C_i} = \mathcal{S}_{C_{\cap\mathcal{S}_{C_i}}} = \bigcap_{i\in\Gamma} \mathcal{S}_{C_i} = \bigwedge_{i\in\Gamma} g(C_i).$$

By (1), (2) and (3), f is a complete lattice isomorphism. □

EXAMPLE 4.17. Let $X = \{a, b, c\}$ be a set. Define $\mathcal{S}_i$ as follows:

$$\mathcal{S}_1 = \{\chi_{\{a\}}, \chi_{\{a,b\}}, \tilde{1}\}, \ \ \mathcal{S}_2 = \{\chi_{\{b\}}, \chi_{\{a,b\}}, \tilde{1}\}.$$

By Corollary 4.4, we obtain

$$\mathcal{S}_1 \vee \mathcal{S}_2 = \{\tilde{0}, \chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{a,b\}}, \tilde{1}\},$$

$$\mathcal{S}_1 \wedge \mathcal{S}_2 = \{\chi_{\{a,b\}}, \tilde{1}\}.$$

Also, we obtain $C_{\mathcal{S}_i} : I^X \to I^X$ as follows:

$$C_{\mathcal{S}_1}(\lambda) = \begin{cases} \chi_{\{a\}}, & \text{if } \tilde{0} \leq \lambda \leq \chi_{\{a\}}, \\ \chi_{\{a,b\}}, & \text{if } \lambda \not\leq \chi_{\{a\}}, \ \ \lambda \leq \chi_{\{a,b\}}, \\ \tilde{1}, & \text{otherwise,} \end{cases}$$

$$C_{\mathcal{S}_2}(\lambda) = \begin{cases} \chi_{\{b\}}, & \text{if } \tilde{0} \leq \lambda \leq \chi_{\{b\}}, \\ \chi_{\{a,b\}}, & \text{if } \lambda \not\leq \chi_{\{b\}}, \ \ \lambda \leq \chi_{\{a,b\}}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

By Theorem 4.10, we have

$$C_{\mathcal{S}_1} \Cup C_{\mathcal{S}_2}(\lambda) = \begin{cases} \tilde{0}, & \text{if } \lambda = \tilde{0}, \\ \chi_{\{a\}}, & \text{if } \lambda \leq \chi_{\{a\}}, \ \ \lambda \not\leq \chi_{\{b\}}, \\ \chi_{\{b\}}, & \text{if } \lambda \leq \chi_{\{b\}}, \ \ \lambda \not\leq \chi_{\{a\}}, \\ \chi_{\{a,b\}}, & \text{if } \lambda \not\leq \chi_{\{a\}}, \ \lambda \not\leq \chi_{\{b\}}, \ \lambda \leq \chi_{\{a,b\}}, \\ \tilde{1}, & \text{otherwise,} \end{cases}$$

$$C_{\mathcal{S}_1} \Cap C_{\mathcal{S}_2}(\lambda) = \begin{cases} \chi_{\{a,b\}}, & \text{if } \tilde{0} \leq \lambda \leq \chi_{\{a,b\}}, \\ \tilde{1}, & \text{otherwise.} \end{cases}$$

References

[1] L. Biacino and G. Gerla, *An extension principle for closure operators*, J. Math. Anal. Appl. **198** (1996), 1–24.

[2] G. Birkhoff, *Lattice theory, 3rd Edition*, Amer. Math. Soc., Rhode Island (1967).

[3] N. Caspard and B. Monjardet, *The lattices of closure systems, closure operators, and implicational systems on a finite set: a survey*, Discrete Applied Mathematics (Article in press).

[4] M. K. Chakraborty and J. Sen, *MV-algebras embedded in a CL-algebra*, International Journal of Approximate Reasoning **18** (1998), 217–229.

[5] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.

[6] G. Gerla, *An extension principle for fuzzy logics*, Mathematical Logic Quarterly **40** (1994), 357–380.

[7] G. Gerla and L. Scarpati, *Extension principles for fuzzy set theory*, Journal of Information Sciences **106** (1998), 49–69.

[8] G. Grätzer, *Universal Algebra*, Springer-Verlag, New York (1978).

[9] A. Higuchi, *Lattices of closure operators*, Discrete Mathematics **179** (1998), 267–272.

[10] A. S. Mashhour and M. H. Ghanim, *Fuzzy closure spaces*, J. Math. Anal. Appl. **106** (1985), 154–170.

[11] F. Ranzato, *Psedocomplements of closure operators on posets*, Discrete Mathematics **248** (2002), 143–155.

[12] R. Srivastava, A. K. Srivastava and A. Choubey, *Fuzzy closure spaces*, J. Fuzzy Math. **2** (1994), 525–534.

Department of Mathematics
Kangnung National University
Kangwondo 210-702, Korea
E-mail: yck@kangnung.ac.kr
jmko@kangnung.ac.kr