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ITERATIVE APPROXIMATION OF FIXED POINTS FOR ϕ-HEMICONTRACTIVE OPERATORS IN BANACH SPACES

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Abstract. Suppose that X is a real Banach space, K is a nonempty closed convex subset of X and $T : K \to K$ is a uniformly continuous ϕ-hemicontractive operator or a Lipschitz ϕ-hemicontractive operator. In this paper we prove that under certain conditions the three-step iteration methods with errors converge strongly to the unique fixed point of T. Our results extend the corresponding results of Chang [1], Chang et al. [2], Chidume [3]-[7], Chidume and Osilike [9], Deng [10], Liu and Kang [13], [14], Osilike [15], [16] and Tan and Xu [17].

1. Introduction

For a real Banach space, we denote by J the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X,$$

where X^* denotes the dual space of X and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between X and X^*. In the sequel, I denotes the identity operator on X. An operator T with domain $D(T)$ and range $R(T)$ in X is called *strongly pseudocontractive* if there exists a constant $t > 1$ such that for any given $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ satisfying

$$\langle Tx - Ty, j(x-y)\rangle \le t^{-1}\|x-y\|^2. \tag{1.1}$$

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T is called *ϕ-strongly pseudocontractive* if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for any given $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ satisfying

$$(1.2) \qquad \langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|.$$

T is called *ϕ-hemicontractive* if $F(T) = \{x \in D(T) : Tx = x\} \ne \emptyset$ and if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for any given $x \in D(T)$ and $q \in F(T)$ there exists $j(x - q) \in J(x - q)$ satisfying

$$(1.3) \qquad \langle Tx - q, j(x - q)\rangle \le \|x - q\|^2 - \phi(\|x - q\|)\|x - q\|.$$

In [8], Chidume and Osilike proved that the class of ϕ-strongly pseudocontractive operators with a nonempty fixed point set is a proper subset of the class of ϕ-hemicontractive operators. In [3], Chidume obtained that if K is a nonempty closed convex bounded subset of X and $T : K \to K$ is a Lipschitz strongly pseudocontractive operator, then the Mann iteration sequence converges strongly to the fixed point of T. In [10], Deng generalized the result to the Ishikawa iteration sequence. Tan and Xu [18] extended the results of Chidume [3] and Deng [10] to real q-uniformly smooth Banach spaces, where $1 < q < 2$. Osilike [15] improved the results of Chidume [3], Deng [10], Tan and Xu [18] to both real q-uniformly smooth Banach spaces, where $q > 1$ and ϕ-hemicontractive operator. Recently, Osilike [16] generalized the above results to a real Banach space, and Liu and Kang [13], [14] extended Osilike's results to the Ishikawa iteration sequence with errors in a real Banach space.

In this paper our purpose is to show that the three-step iteration sequences with errors converge strongly to the unique fixed point of T if $T : K \to K$ is a Lipschitz ϕ-hemicontractive operator or $T : K \to K$ is a uniformly continuous ϕ-hemicontractive operator, where X is a real Banach space and K is a nonempty closed convex subset of X. Our results extend, improve and unify the results in Chang [1], Chang et al. [2], Chidume [3]-[7], Chidume and Osilike [9], Deng [10], Liu and Kang [13], [14], Osilike [15], [16], Tan and Xu [17].

The following lemmas are of importance in the proofs of our results.

LEMMA 1.1. [13] *Suppose that $\phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $\phi(0) = 0$. If $\{r_n\}_{n=0}^{\infty}$, $\{s_n\}_{n=0}^{\infty}$, $\{k_n\}_{n=0}^{\infty}$ and*

$\{t_n\}_{n=0}^{\infty}$ *are sequence of nonnegative numbers satisfying the following conditions:*

$$\sum_{n=0}^{\infty} k_n < \infty, \quad \sum_{n=0}^{\infty} t_n < \infty, \quad \sum_{n=0}^{\infty} s_n = \infty; \tag{1.4}$$

$$r_{n+1} \leq (1+k_n)r_n - s_n r_n \frac{\phi(r_{n+1})}{1+\phi(r_{n+1})+r_{n+1}} + t_n, \quad n \geq 0, \tag{1.5}$$

then $\lim_{n\to\infty} r_n = 0$.

LEMMA 1.2. [12] *Let* $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta\}_{n=0}^{\infty}$ *and* $\{\gamma_n\}_{n=0}^{\infty}$ *be three nonnegative real sequences satisfying the inequality*

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \beta_n\omega_n + \gamma_n \quad \text{for } n \geq 0,$$

where $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty}\omega_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n < \infty$. *Then* $\lim_{n\to\infty}\alpha_n = 0$.

2. Main results

THEOREM 2.1. *Let* X *be a real Banach space,* K *be a nonempty closed convex subset of* X *and* $T : K \to K$ *be a Lipschitz* ϕ*-hemicontractive operator with the Lipschitz constant* $L \geq 1$. *For any* $x_0, u_0, v_0, w_0 \in K$, *the three-step iteration sequence with errors* $\{x_n\}_{n=0}^{\infty}$*defined by*

$$\begin{aligned} z_n &= a''_n x_n + b''_n T x_n + c''_n w_n, \\ y_n &= a'_n x_n + b'_n T z_n + c'_n v_n, \\ x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \quad n \geq 0, \end{aligned} \tag{2.1}$$

where $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ *and* $\{w_n\}_{n=0}^{\infty}$ *are arbitrary bounded sequences in* K, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$ *and* $\{c''_n\}$ *are real sequences in* $[0,1]$ *satisfying the following conditions:*

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1, \quad n \geq 0; \tag{2.2}$$

$$\sum_{n=0}^{\infty} c_n < \infty, \quad \sum_{n=0}^{\infty} b_n b'_n < \infty, \quad \sum_{n=0}^{\infty} b_n c'_n < \infty, \quad \sum_{n=0}^{\infty} b_n^2 < \infty; \tag{2.3}$$

$$\sum_{n=0}^{\infty} b_n = \infty. \tag{2.4}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* T.

PROOF. Since T is a ϕ-hemicontractive operator, $F(T)$ is a nonempty set. We claim that $F(T)$ is a singleton. Otherwise, for any different $p, q \in F(T)$ we conclude that

$$\begin{aligned}\|p-q\|^2 &= \langle Tp - q, j(p-q)\rangle \\ &\leq \|p-q\|^2 - \phi(\|p-q\|)\|p-q\| \\ &< \|p-q\|^2,\end{aligned}$$

which is a contradiction and hence $F(T)$ is a singleton. Let $F(T) = \{q\}$. Since T is a ϕ-hemicontractive operator, we know that for $x \in K$

$$\begin{aligned}\langle (I-T)x - (I-T)q, j(x-q)\rangle &\geq \phi(\|x-q\|)\|x-q\| \\ &\geq A(x,q)\|x-q\|^2,\end{aligned}$$

where $A(x,q) = \frac{\phi(\|x-q\|)}{1+\phi(\|x-q\|)+\|x-q\|} \in [0,1)$. Lemma 1.1 of Kato [11] ensures that for any $x \in K$ and $r > 0$

$$\|x-q\| \leq \|x - q + r[(I-T-A(x,q))x - (I-T-A(x,q))q]\|. \tag{2.5}$$

Put $d_n = b_n + c_n$, $d'_n = b'_n + c'_n$ and $d''_n = b''_n + c''_n$ for $n \geq 0$. According to (2.1) we have that

$$\begin{aligned}x_n &= x_{n+1} + d_n x_n - d_n T y_n + c_n(Ty_n - u_n) \\ &= (1+d_n)x_{n+1} + d_n(I - T - A(x_{n+1}, q))x_{n+1} \\ &\quad - (1 - A(x_{n+1}, q))d_n x_n + (2 - A(x_{n+1}, q))d_n^2(x_n - Ty_n) \\ &\quad + d_n(Tx_{n+1} - Ty_n) + c_n[1 + (2 - A(x_{n+1}, q))d_n] \\ &\quad \times (Ty_n - u_n)\end{aligned} \tag{2.6}$$

and

$$q = (1+d_n)q + d_n(I - T - A(x_{n+1}, q))q - (1 - A(x_{n+1}, q))d_n q \tag{2.7}$$

for $n \geq 0$. It follows from (2.5)-(2.7) that

$$\begin{aligned}&\|x_n - q\| \\ &\geq (1+d_n)\|x_{n+1} - q + \frac{d_n}{1+d_n}[(I - T - A(x_{n+1}, q))x_{n+1} \\ &\quad - (I - T - A(x_{n+1}, q))q]\| - d_n(1 - A(x_{n+1}, q))\|x_n - q\| \\ &\quad - (2 - A(x_{n+1}, q))d_n^2\|x_n - Ty_n\| - d_n\|Tx_{n+1} - Ty_n\| \\ &\quad - c_n[1 + (2 - A(x_{n+1}, q))d_n]\|Ty_n - u_n\| \\ &\geq (1+d_n)\|x_{n+1} - q\| - d_n(1 - A(x_{n+1}, q))\|x_n - q\| \\ &\quad - (2 - A(x_{n+1}, q))d_n^2\|x_n - Ty_n\| - d_n\|Tx_{n+1} - Ty_n\| \\ &\quad - c_n[1 + (2 - A(x_{n+1}, q))d_n]\|Ty_n - u_n\|,\end{aligned}$$

which implies that

$$
\begin{aligned}
&\|x_{n+1}-q\| \\
&\le \frac{1+(1-A(x_{n+1},q))d_n}{1+d_n}\|x_n-q\| \\
&\quad + (2-A(x_{n+1},q))d_n^2\|x_n-Ty_n\| + d_n\|Tx_{n+1}-Ty_n\| \\
&\quad + c_n[1+(2-A(x_{n+1},q))d_n]\|Ty_n-u_n\| \\
&\le (1-A(x_{n+1},q)d_n+d_n^2)\|x_n-q\| + 2d_n^2\|x_n-Ty_n\| \\
&\quad + d_n\|Tx_{n+1}-Ty_n\| + c_n(1+2d_n)\|Ty_n-u_n\|
\end{aligned} \tag{2.8}
$$

for $n \ge 0$. Set $B = \sup\{\|u_n-q\|, \|v_n-q\|, \|w_n-q\| : n \ge 0\}$. By virtue of (2.1) we get that

$$
\begin{aligned}
&\|z_n-q\| \\
&\le (1-d_n'')\|x_n-q\| + d_n''L\|x_n-q\| \\
&\quad + c_n''(\|w_n-q\| + L\|x_n-q\|) \\
&\le (1-d_n''+d_n''L+Lc_n'')\|x_n-q\| + c_n''\|w_n-q\| \\
&\le 2L\|x_n-q\| + c_n''B
\end{aligned} \tag{2.9}
$$

for $n \ge 0$. In view of (2.1) and (2.9) we obtain that

$$
\begin{aligned}
&\|y_n-q\| \\
&\le (1-d_n')\|x_n-q\| + d_n'L\|z_n-q\| \\
&\quad + c_n'(\|v_n-q\| + L\|z_n-q\|) \\
&\le [(1-d_n') + 2L^2(d_n'+c_n')]\|x_n-q\| + L(d_n'+c_n')c_n''B + c_n'B \\
&\le [1+2L^2(d_n'+c_n')]\|x_n-q\| + L(d_n'+c_n')c_n''B + c_n'B
\end{aligned} \tag{2.10}
$$

for $n \ge 0$. According to (2.1), (2.9) and (2.10) we conclude that

$$
\begin{aligned}
&\|x_n-Ty_n\| \\
&\le \|x_n-q\| + \|Ty_n-q\| \\
&\le \|x_n-q\| + L\|y_n-q\| \\
&\le [1+L+2L^3(d_n'+c_n')]\|x_n-q\| + [L^2(d_n'+c_n')c_n'' + Lc_n']B
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
&\|x_n-y_n\| \\
&\le d_n'\|x_n-Tz_n\| + c_n'\|Tz_n-v_n\| \\
&\le [d_n'+2L^2(d_n'+c_n')]\|x_n-q\| + [L(d_n'+c_n')c_n''+c_n']B
\end{aligned} \tag{2.12}
$$

for $n \geq 0$. It follows from (2.1) and (2.9)-(2.12) that

$$
\begin{aligned}
&\|Tx_{n+1} - Ty_n\| \\
&\leq L(1-d_n)\|x_n - y_n\| + Ld_n\|Ty_n - y_n\| + Lc_n\|Ty_n - u_n\| \\
&\leq L\|x_n - y_n\| + (L^2 d_n + L^2 c_n + Ld_n)\|y_n - q\| + Lc_n B \\
&\leq [Ld'_n + 2L^3(d'_n + c'_n) + Ld_n + L^2(d_n + c_n) \\
&\quad + 2L^3 d_n(d'_n + c'_n) + 2L^4(d_n + c_n)(d'_n + c'_n)]\|x_n - q\| \\
&\quad + [L^2(d'_n + c'_n)c''_n + L^3(d_n + c_n)(d'_n + c'_n)c''_n \\
&\quad + L^2(d_n + c_n)c'_n + L^2 d_n(d'_n + c'_n)c''_n + Ld_n c'_n \\
&\quad + Lc_n + Lc'_n]B
\end{aligned} \tag{2.13}
$$

for $n \geq 0$. Substituting (2.11) and (2.13) into (2.8), we obtain that

$$
\begin{aligned}
&\|x_{n+1} - q\| \\
&\leq [1 - A(x_{n+1}, q)d_n]\|x_n - q\| + [3d_n^2 + 2Ld_n^2 \\
&\quad + 4L^3 d_n^2(d'_n + c'_n) + Ld_n d'_n + 2L^3 d_n(d'_n + c'_n) + Ld_n^2 \\
&\quad + L^2 d_n(d_n + c_n) + 2L^4 d_n(d_n + c_n)(d'_n + c'_n) \\
&\quad + Lc_n(1 + 2d_n) + 2L^3 c_n(1 + 2d_n)(d'_n + c'_n)]\|x_n - q\| \\
&\quad + [2L^2 d_n^2 c''_n(d'_n + c'_n) + 2Ld_n^2 c'_n + L^2 d_n c''_n(d'_n + c'_n) \\
&\quad + L^3 d_n c''_n(d_n + c_n)(d'_n + c'_n) + L^2 d_n c'_n(d_n + c_n) \\
&\quad + L^2 d_n^2 c''_n(d'_n + c'_n) + Ld_n^2 c'_n + Ld_n c_n + Ld_n c'_n \\
&\quad + L^2 c_n c''_n(1 + 2d_n)(d'_n + c'_n) + Lc_n c'_n(1 + 2d_n) \\
&\quad + c_n(1 + 2d_n)]B \\
&\leq [1 + M_1(d_n^2 + d_n d'_n + c_n)]\|x_n - q\| - A(x_{n+1}, q)d_n\|x_n - q\| \\
&\quad + M_2 B(d_n^2 + c'_n d_n + c_n + d_n d'_n)
\end{aligned} \tag{2.14}
$$

for $n \geq 0$ and some positive constants M_1 and M_2. Put

$$
\begin{aligned}
&r_n = \|x_n - q\|, \quad s_n = d_n, \quad k_n = M_1(d_n^2 + d'_n d_n + c_n), \\
&t_n = M_2 B(d_n^2 + c'_n d_n + c_n + d_n d'_n) \quad \text{for } n \geq 0.
\end{aligned}
$$

It follows from (2.3), (2.4), (2.14) and Lemma 1.1 that $r_n \to 0$ as $n \to \infty$. This completes the proof. $\square$

THEOREM 2.2. *Suppose that X is a real Banach space, K is a nonempty convex subset of X and $T : K \to K$ is a uniformly continuous ϕ-hemicontractive operator. For any $x_0, u_0, v_0, w_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be defined by (2.1), where $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a''_n\}$, $\{b''_n\}$ and $\{c''_n\}$ are real sequences in $[0,1]$ satisfying (2.2) and the following conditions*

$$b_n + c_n \in (0,1), \quad n \geq 0; \tag{2.15}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c'_n = \lim_{n\to\infty} \frac{c_n}{b_n + c_n} = 0; \tag{2.16}$$

$$\sum_{n=0}^{\infty} b_n = +\infty. \tag{2.17}$$

If $R(T)$ is bounded, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

PROOF. As in the proof of Theorem 2.1, we conclude that $F(T)$ is a singleton. Put $F(T) = \{q\}$. Thus for $x \in K$,

$$\begin{aligned}\langle (I-T)x - (I-T)q, j(x-q)\rangle &\geq \phi(\|x-q\|)\|x-q\| \\ &\geq A(x,q)\|x-q\|^2,\end{aligned}$$

where $A(x,q) = \frac{\phi(\|x-q\|)}{1+\phi(\|x-q\|)+\|x-q\|} \in [0,1)$. It follows from Lemma 1.1 of Kato [11] that

$$\|x-q\| \leq \|x-q+r[(I-T-A(x,q))x - (I-T-A(x,q))q]\|$$

for $x \in K$ and $r > 0$. Set $d_n = b_n + c_n$, $d'_n = b'_n + c'_n$ and $d''_n = b''_n + c''_n$ for $n \geq 0$ and

$$\begin{aligned}D = \sup\{&\|Tx_n - q\|, \|Ty_n - q\|, \|Tz_n - q\|, \\ &\|u_n - q\|, \|v_n - q\|, \|w_n - q\| : n \geq 0\}.\end{aligned} \tag{2.18}$$

Since $R(T)$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded, it follows that $D < \infty$. According to (2.1) and (2.18), we deduce that

$$\sup\{\|x_n - q\|, \|y_n - q\| : n \geq 0\} \leq D. \tag{2.19}$$

In view of (2.1) we infer that

$$\begin{aligned}(2.20)\qquad &(1-d_n)x_n \\ &= x_{n+1} - d_nTy_n - c_n(u_n - Ty_n) \\ &= [1-(1-A(x_{n+1},q))d_n]x_{n+1} + d_n(I-T-A(x_{n+1},q))x_{n+1} \\ &\quad + d_n(Tx_{n+1} - Ty_n) - c_n(u_n - Ty_n)\end{aligned}$$

and

$$(2.21)\quad (1-d_n)q = [1-(1-A(x_{n+1},q))d_n]q + d_n(I-T-A(x_{n+1},q))q$$

for $n \geq 0$. It follows from (2.2), (2.5), (2.20) and (2.21) that

$$\begin{aligned}&(1-d_n)\|x_n - q\| \\ &\geq [1-(1-A(x_{n+1},q))d_n]\|x_{n+1} - q + \frac{d_n}{1-(1-A(x_{n+1},q))d_n} \\ &\quad \times [(I-T-A(x_{n+1},q))x_{n+1} - (I-T-A(x_{n+1},q))q]\| \\ &\quad - d_n\|Tx_{n+1} - Ty_n\| - c_n\|u_n - Ty_n\| \\ &\geq [1-(1-A(x_{n+1},q))d_n]\|x_{n+1} - q\| - d_n\|Tx_{n+1} - Ty_n\| - 2Dc_n,\end{aligned}$$

which implies that

$$\begin{aligned}(2.22)\qquad &\|x_{n+1} - q\| \\ &\leq \frac{1-d_n}{1-(1-A(x_{n+1},q))d_n}\|x_n - q\| \\ &\quad + \frac{d_n}{1-(1-(Ax_{n+1},q))d_n}\|Tx_{n+1} - Ty_n\| \\ &\quad + \frac{2Dc_n}{1-(1-A(x_{n+1},q))d_n} \\ &\leq (1-A(x_{n+1},q)d_n)\|x_n - q\| + Md_n\|Tx_{n+1} - Ty_n\| + Mc_n\end{aligned}$$

for $n \geq 0$, where M is some constant. According to (2.1), (2.2), (2.15) and (2.16), we get that

$$\begin{aligned}&\|x_{n+1} - y_n\| \\ &\leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \\ &\leq b_n\|Ty_n - x_n\| + c_n\|u_n - x_n\| + b_n'\|Tz_n - x_n\| + c_n'\|v_n - x_n\| \\ &\leq 2D(d_n' + d_n) \\ &\to 0\end{aligned}$$

as $n \to \infty$. The uniform continuity of T ensures that

$$\|Tx_{n+1} - Ty_n\| \to 0 \quad \text{as } n \to \infty. \tag{2.23}$$

Set $\inf\{A(x_{n+1}, q) : n \geq 0\} = r$. We assert that $r = 0$. If not, then $r > 0$. In view of (2.22), we have that

$$\|x_{n+1} - q\| \leq (1 - rd_n)\|x_n - q\| + Md_n\|Tx_{n+1} - Ty_n\| + Mc_n \tag{2.24}$$

for all $n \geq 0$. Let $\alpha_n = \|x_n - q\|$, $\omega_n = rd_n$, $\beta_n = Mr^{-1}\|Tx_{n+1} - Ty_n\| + Mr^{-1}c_n d_n^{-1}$ and $\gamma_n = 0$ for $n \geq 0$. By (2.16), (2.17), (2.23), (2.24) and Lemma 1.2 we conclude that $\|x_n - q\| \to 0$ as $n \to \infty$, which means that $r = 0$, which is a contradiction. Therefore $r = 0$ and there exists a subsequence $\{\|x_{n_i+1} - q\|\}_{i=1}^{\infty}$ of $\{\|x_{n+1} - q\|\}_{n=0}^{\infty}$ satisfying

$$\|x_{n_i+1} - q\| \to 0 \quad \text{as } i \to \infty. \tag{2.25}$$

Employing (2.16), (2.23) and (2.25) we conclude that given $\varepsilon > 0$, there exists a positive integer m satisfying

$$\|x_{n_m+1} - q\| < \varepsilon \tag{2.26}$$

and

$$M\|Tx_{n+1} - Ty_n\| + M\frac{c_n}{d_n} < \min\left\{\frac{1}{2}\varepsilon, \frac{\phi(\varepsilon)\varepsilon}{1 + \phi(\frac{3}{2}\varepsilon) + \frac{3}{2}\varepsilon}\right\} \tag{2.27}$$

for $n \geq N$. Now we show that

$$\|x_{n_m+j} - q\| \leq \varepsilon \quad \text{for } j \geq 1. \tag{2.28}$$

Clearly (2.26) means that (2.28) holds for $j = 1$. Assume that (2.28) holds for $j = k$. If $\|x_{n_m+k+1} - q\| > \varepsilon$, we obtain that by (2.22) and (2.27)

$$\begin{aligned} &\|x_{n_m+k+1} - q\| \\ &\leq \|x_{n_m+k} - q\| + Md_{n_m+k}\|Tx_{n_m+k+1} - Ty_{n_m+k}\| + Mc_{n_m+k} \\ &\leq \varepsilon + \min\left\{\frac{1}{2}\varepsilon, \frac{\phi(\varepsilon)\varepsilon}{1 + \phi(\frac{3}{2}\varepsilon) + \frac{3}{2}\varepsilon}\right\}d_{n_m+k} \\ &\leq \frac{3}{2}\varepsilon. \end{aligned} \tag{2.29}$$

Note that $\phi(\|x_{n_m+k+1} - q\|) > \phi(\varepsilon)$. It follows from (2.29) that

$$A(x_{n_m+k+1}, q) \geq \frac{\phi(\varepsilon)}{1 + \phi(\frac{3}{2}\varepsilon) + \frac{3}{2}\varepsilon}. \tag{2.30}$$

By virtue of (2.22), (2.27) and (2.30), we obtain the following estimates:

$$\begin{aligned}
&\|x_{n_m+k+1} - q\| \\
&\leq \left(1 - \frac{\phi(\varepsilon)}{1 + \phi(\frac{3}{2}\varepsilon) + \frac{3}{2}\varepsilon} d_{n_m+k}\right) \|x_{n_m+k} - q\| \\
&\quad + M d_{n_m+k} \|T x_{x_m+k+1} - T y_{n_m+k}\| + M c_{n_m+k} \\
&\leq \left(1 - \frac{\phi(\varepsilon)}{1 + \phi(\frac{3}{2}\varepsilon) + \frac{3}{2}\varepsilon} d_{n_m+k}\right) \varepsilon \\
&\quad + \min\left\{\frac{1}{2}\varepsilon, \frac{\phi(\varepsilon)\varepsilon}{1 + \phi(\frac{3}{2}\varepsilon) + \frac{3}{2}\varepsilon}\right\} d_{n_m+k} \\
&\leq \varepsilon.
\end{aligned}$$

That is,

$$\varepsilon < \|x_{n_m+k+1} - q\| \leq \varepsilon,$$

which is a contradiction. Hence $\|x_{n_m+k+1} - q\| \leq \varepsilon$. By induction, (2.28) holds for $j \geq 1$. It follows from (2.28) that $x_n \to q$ as $n \to \infty$. This completes the proof. □

REMARK 2.1. Theorems 2.1 and 2.2 extend and improve Theorem 3.3 of Chang [1], Theorem 3.2 of Chang et al. [2], Theorem of Chidume [3], Theorem 2 of Chidume [4], Theorem 2 of Chidume [5], Theorem 4 of Chidume [6], Theorem 13 of Chidume [7], Theorems 2 and 4 of Chidume and Osilike [9], Theorems 2 and 4 of Deng [10], Theorem 3.3 of Liu and Kang [13], Theorem 3.1 of Liu and Kang [14], Theorem 2 of Osilike [15], Theorem 2 of Osilike [16], Theorems 3.2 and 4.2 of Tan and Xu [17] in the following ways:

(a) that X be either real uniformly smooth Banach space in [1], [2], [5], [6] or real q-uniformly smooth Banach space in [10], [15], [17], or real smooth Banach space in [7], is replaced by the more general real Banach space;

(b) that T be strongly psedo-contractive operator in [1]-[7], [9], [10] is replaced by the more general class of ϕ-hemicontractive operators;

(c) the Mann iteration sequence in [3], [4], [9], [17], the Ishikawa iteration sequence in [1], [2], [5]-[7], [9], [10], [15]-[17] and the Ishikawa iteration sequence with errors in [13], [14] are replaced by the more general three-step iteration sequence with errors.

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