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THE GENERALISED INTEGRATION BY PARTS FORMULA FOR APPELL SEQUENCES AND RELATED RESULTS

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Abstract. A generalised integration by parts formula for sequences of absolutely continuous functions that satisfy the $w-$Appell condition and different estimates for the remainder are provided. Applications for particular instances of such sequences are pointed out as well.

1. Introduction

In [6], Matić et. al introduced the concept of *harmonic sequences of polynomials* by assuming that the polynomial $\{P_n\}_{n\in\mathbb{N}}$ satisfies the condition

$$P_0 = 1, \quad P_n'(t) = P_{n-1}(t) \quad \text{for all } t \in \mathbb{R} \text{ and } n \in \mathbb{N}. \tag{1.1}$$

With this assumption, they proved the following generalised Taylor's formula:

Theorem 1. *Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that, for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x \in I$*

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \times \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)\right] + R_n(f; a, x), \tag{1.2}$$

where

$$R_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)\, dt \tag{1.3}$$

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and $\{P_n\}_{n\in\mathbb{N}}$ *is a harmonic sequence of polynomials.*

As examples of such polynomials, they mentioned the following

$$P_n(t) := \frac{1}{n!}(t-x)^n, \quad t \in \mathbb{R};$$

or

$$P_n(t) := \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n, \quad t \in \mathbb{R};$$

or

$$P_n(t) := \frac{(x-a)^n}{n!} B_n\left(\frac{t-a}{x-a}\right), \ n \geq 1, \ P_0(t) = 1,$$

where $B_n(\cdot)$ are the *Bernoulli's polynomials,* or

$$P_n(t) := \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right), \ n \geq 1, \ P_0(t) = 1,$$

where $E_n(\cdot)$ are the *Euler's polynomials.*

Amongst others, they proved the following general estimation result for the remainder $R_n(f;a,x)$.

COROLLARY 1. *Under the assumptions of Theorem 1 and if* $x \geq a$, *then*

$$(1.4) \qquad |R_n(f;a,x)| \leq \begin{cases} \max\limits_{t\in[a,x]} |P_n(t)| \int_a^x \left|f^{(n+1)}(s)\right| ds; \\ \\ \max\limits_{t\in[a,x]} \left|f^{(n+1)}(t)\right| \int_a^x |P_n(s)| \, ds; \\ \\ \left(\int_a^x |P_n(s)|^q \, ds\right)^{\frac{1}{q}} \left(\int_a^x \left|f^{(n+1)}(s)\right|^p ds\right)^{\frac{1}{p}}, \\ \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \ p > 1. \end{cases}$$

Now, if one would choose in (1.2) $f(x) = \int_a^b g(t)\,dt$ and put $x = b$, we could then state the following generalised integration by parts formula

$$(1.5) \quad \int_a^b g(t)\,dt = \sum_{k=1}^n (-1)^{k+1}\left[P_k(b)\,g^{(k-1)}(b) - P_k(a)\,g^{(k-1)}(a)\right] + S_n(f;a,b),$$

where

$$(1.6) \qquad S_n(f;a,b) = (-1)^n \int_a^b P_n(t)\,g(t)\,dt.$$

Using the classical notation for the Lebesgue norms,

$$\begin{aligned} \|h\|_{\infty} &:= ess \sup_{t\in[a,b]} |h(t)|, \\ \|h\|_{p} &:= \left(\int_a^b |h(t)|^p\,dt\right)^{\frac{1}{p}}, \; p \geq 1 \end{aligned}$$

the remainder (1.6) may be bounded in the following manner

$$|S_n(g;a,b)| \leq \begin{cases} \|P_n\|_{\infty} \left\|g^{(n)}\right\|_1, \\ \|P_n\|_1 \left\|g^{(n)}\right\|_{\infty}, \\ \|P_n\|_q \left\|g^{(n)}\right\|_p, \text{ where } \frac{1}{p}+\frac{1}{q}=1,\ p>1. \end{cases} \tag{1.7}$$

For other results based on the integration by parts formula (1.5), see [1]-[5] and [7]-[9].

2. The generalised integration by parts formula

We shall start with the following definition.

DEFINITION 1. Let $w : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. The sequence of absolutely continuous functions $\{w_k\}_{k=\overline{0,n}}$ $(n \geq 1)$ defined on $[a,b]$ are said to be of *$w-$Appell type* if

$$w_0 = w \text{ a.e. on } [a,b]; \tag{2.1}$$

$$w_k' = w_{k-1} \text{ a.e. on } [a,b], \text{ for all } k = 1,\ldots,n. \tag{2.2}$$

REMARK 1. It is obvious that any sequence of harmonic polynomials is a sequence of $w-$Appell type with $w = 1$.

REMARK 2. Having given an absolutely continuous function $w : [a,b] \to \mathbb{R}$ we may construct a sequence of $w-$Appell type in the following canonical fashion:

$$\begin{aligned} w_1(t) &= \int_a^t w(s)\,ds + c_1,\ t \in [a,b],\ c_1 \in \mathbb{R}; \\ w_2(t) &= \int_a^t w_1(s)\,ds + c_2,\ t \in [a,b],\ c_2 \in \mathbb{R}; \\ &\cdots \\ w_n(t) &= \int_a^t w_{n-1}(s)\,ds + c_n,\ t \in [a,b],\ c_n \in \mathbb{R}. \end{aligned}$$

The following generalised integration by parts formula associated with the sequence $\{w_k\}_{k=\overline{0,n}}$ naturally holds.

LEMMA 1. *Let $w : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $\{w_k\}_{k=\overline{0,n}}$ a sequence of $w-$Appell type. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$is absolutely continuous on $[a,b]$ and $w_n g^{(n)} \in L_1[a,b]$, then we have the equality*

$$\int_a^b w(t)\, g(t)\, dt = A_n(w,g;a,b) + R_n(w,g;a,b), \tag{2.3}$$

where

$$A_n(w,g;a,b) = \sum_{k=1}^{n} (-1)^{k-1} \left[w_k(b)\, g^{(k-1)}(b) - w_k(a)\, g^{(k-1)}(a) \right] \tag{2.4}$$

and

$$R_n(w,g;a,b) = (-1)^n \int_a^b w_n(t)\, g^{(n)}(t)\, dt. \tag{2.5}$$

PROOF. We prove (2.3) by mathematical induction.
For $n = 1$, we have

$$\int_a^b w(t)\, g(t)\, dt = w_1(b)\, g(b) - w_1(a)\, g(a) - \int_a^b w_1(t)\, g^{(1)}(t)\, dt$$

which follows on applying the integration by parts formula taking into account that $w_1' = w$ a.e. on $[a,b]$.

Assume that (2.3) holds for $m \in \{1, \ldots, n-1\}$, i.e.,

$$\begin{aligned} & \int_a^b w(t)\, g(t)\, dt \\ = & \sum_{k=1}^{m} (-1)^{k-1} \left[w_k(b)\, g^{(k-1)}(b) - w_k(a)\, g^{(k-1)}(a) \right] \\ & + (-1)^m \int_a^b w_m(t)\, g^{(m)}(t)\, dt. \end{aligned} \tag{2.6}$$

As w_m and $g^{(m)}$ are absolutely continuous and $w'_{m+1} = w_m$ a.e. on $[a,b]$, then, by the integration by parts formula we may write

$$
\begin{aligned}
(2.7)\quad \int_a^b w_m(t)\, g^{(m)}(t)\, dt &= \int_a^b w'_{m+1}(t)\, g^{(m)}(t)\, dt \\
&= w_{m+1}(b)\, g^{(m)}(b) - w_{m+1}(a)\, g^{(m)}(a) \\
&\quad - \int_a^b w_{m+1}(t)\, g^{(m+1)}(t)\, dt.
\end{aligned}
$$

Using (2.6), we get

$$
\begin{aligned}
\int_a^b w(t)\, g(t)\, dt &= \sum_{k=1}^{m} (-1)^{k-1} \left[w_k(b)\, g^{(k-1)}(b) - w_k(a)\, g^{(k-1)}(a) \right] \\
&\quad + (-1)^m \left[w_{m+1}(b)\, g^{(m)}(b) - w_{m+1}(a)\, g^{(m)}(a) \right. \\
&\quad \left. - \int_a^b w_{m+1}(t)\, g^{(m+1)}(t)\, dt \right] \\
&= \sum_{k=1}^{m+1} (-1)^{k-1} \left[w_k(b)\, g^{(k-1)}(b) - w_k(a)\, g^{(k-1)}(a) \right] \\
&\quad + (-1)^{m+1} \int_a^b w_{m+1}(t)\, g^{(m+1)}(t)\, dt
\end{aligned}
$$

showing that (2.3) holds for $n = m + 1$ as well.

The lemma is thus proved. □

REMARK 3. If $w_n = P_n$, $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials, then by (2.3) we recapture the identity (1.6).

The following result concerning estimates of the remainder $R_n(w, g; a, b)$ holds.

THEOREM 2. *With the assumptions of Lemma 1, we have the estimates:*

(2.8)

$$
|R_n(g;a,b)| \le \begin{cases} \|w_n\|_1 \left\|g^{(n)}\right\|_\infty & \text{if } w_n \in L_1[a,b],\ g^{(n)} \in L_\infty[a,b]; \\[2ex] \|w_n\|_q \left\|g^{(n)}\right\|_p & \text{if } w_n \in L_q[a,b],\ g^{(n)} \in L_p[a,b]; \\ & \text{and } \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; \\ \|w_n\|_\infty \left\|g^{(n)}\right\|_1 & \text{if } w_n \in L_\infty[a,b]. \end{cases}
$$

The proof follows by the representation (2.5) and Hölder's inequalities for $\|\cdot\|_p$ −norms ($p \in [1,\infty]$).

In what follows, we point out a number of useful examples.

1. Define $w_0^{(1)}(t) := e^{\alpha t}$, $\alpha \in \mathbb{R}\backslash\{0\}$ and consider the sequence $w_k^{(1)}(t) := \frac{1}{\alpha^k}e^{\alpha t}$, $k = 0, 1, \ldots$. Then

$$\frac{dw_k^{(1)}(t)}{dt} = \frac{1}{\alpha^{k-1}}e^{\alpha t} = w_{k-1}^{(1)}(t), \text{ for } k = 1, 2, \ldots$$

showing that $\left\{w_k^{(1)}\right\}_{k\in\mathbb{N}}$ is an $w_0^{(1)}$−Appell type sequence.

If we use Lemma 1, we may state the identity:

$$\int_a^b e^{\alpha t} g(t)\,dt = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{\alpha^k}\left[e^{\alpha b} g^{(k-1)}(b) - e^{\alpha a} g^{(k-1)}(a)\right] + \frac{(-1)^n}{\alpha^n}\int_a^b e^{\alpha t} g^{(n)}(t)\,dt. \tag{2.9}$$

As

$$\left\|w_n^{(1)}\right\|_1 = \int_a^b \left|\frac{e^{\alpha t}}{\alpha^n}\right| dt = \frac{1}{|\alpha|^n}\int_a^b e^{\alpha t}dt = \frac{1}{|\alpha|^n}\cdot\left|\frac{e^{\alpha b} - e^{\alpha a}}{\alpha}\right|,$$

$$\begin{aligned}\left\|w_n^{(1)}\right\|_q &= \left(\int_a^b \left|\frac{e^{\alpha t}}{\alpha^n}\right|^q dt\right)^{\frac{1}{q}} \\ &= \frac{1}{|\alpha|^n}\left(\int_a^b e^{\alpha q t}dt\right)^{\frac{1}{q}} = \frac{1}{|\alpha|^n}\left|\frac{e^{\alpha q b} - e^{\alpha q a}}{\alpha q}\right|^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\left\|w_n^{(1)}\right\|_\infty &= \frac{1}{|\alpha|^n}\sup_{t\in[a,b]}\left|e^{\alpha t}\right| = \frac{1}{|\alpha|^n}\max\left\{e^{\alpha a}, e^{\alpha b}\right\} \\ &= \frac{1}{|\alpha|^n}\cdot\frac{e^{\alpha a} + e^{\alpha b} + \left|e^{\alpha a} - e^{\alpha b}\right|}{2}.\end{aligned}$$

Then, by Theorem 2 we may state the inequality:

$$\left|\int_a^b e^{\alpha t} g(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{\alpha^k}\left[e^{\alpha b} g^{(k-1)}(b) - e^{\alpha a} g^{(k-1)}(a)\right]\right| \tag{2.10}$$

$$\leq \begin{cases} \dfrac{\left|e^{\alpha b} - e^{\alpha a}\right|}{|\alpha|^{n+1}} \left\|g^{(n)}\right\|_\infty & \text{if } g^{(n)} \in L_\infty[a,b]; \\ \dfrac{1}{|\alpha|^n} \left|\dfrac{e^{\alpha q b} - e^{\alpha q a}}{\alpha q}\right|^{\frac{1}{q}} \left\|g^{(n)}\right\|_p, & \text{if } g^{(n)} \in L_p[a,b]; \\ & \text{and } \frac{1}{p} + \frac{1}{q} = 1,\ p > 1; \\ \dfrac{1}{|\alpha|^n} \cdot \dfrac{e^{\alpha a} + e^{\alpha b} + \left|e^{\alpha a} - e^{\alpha b}\right|}{2} \left\|g^{(n)}\right\|_1, & \end{cases}$$

where $\alpha \in \mathbb{R}\backslash\{0\}$.

2. Define $w_0^{(2)}(t) = t^\alpha$, $\alpha > -1$ and consider

$$w_k^{(2)}(t) := \frac{t^{\alpha+k}}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}, \quad k = 0, 1, \ldots.$$

Then

$$\frac{dw_k^{(1)}(t)}{dt} = \frac{t^{\alpha+k-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)} = w_{k-1}^{(2)}(t), \text{ for } k = 1, 2, \ldots,$$

which shows that $\left\{w_k^{(2)}\right\}_{k=0,1,\ldots}$ is a $w_0^{(2)}$–Appell type sequence.

If we use Lemma 1, we may state the identity:

$$\begin{aligned} \int_a^b t^\alpha g(t)\,dt &= \sum_{k=1}^{n} (-1)^{k-1} \frac{\left[b^{\alpha+k} g^{(k-1)}(b) - a^{\alpha+k} g^{(k-1)}(a)\right]}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)} \\ &\quad + \frac{(-1)^n}{(\alpha+1)\cdots(\alpha+n)} \int_a^b t^{\alpha+n} g^{(n)}(t)\,dt,\ [a,b] \in (0,\infty). \end{aligned} \tag{2.11}$$

As, for $[a,b] \subset (0,\infty)$,

$$\begin{aligned} &\left\|w_n^{(2)}\right\|_1 \\ =\ & \int_a^b \left|\frac{t^{\alpha+n}}{(\alpha+1)\cdots(\alpha+n)}\right| dt = \frac{b^{\alpha+n+1} - a^{\alpha+n+1}}{(\alpha+1)\cdots(\alpha+n)(\alpha+n+1)} \\ =\ & \frac{(b-a)}{(\alpha+1)\cdots(\alpha+n)} L_{\alpha+n}^{\alpha+n}(a,b), \end{aligned}$$

where

$$L_m(a,b) := \begin{cases} b & \text{if } \ a=b \\ \left[\dfrac{b^{m+1}-a^{m+1}}{(m+1)(b-a)}\right]^{\frac{1}{m}} & \text{if } \ a\neq b \end{cases}$$

is the $m-$logarithmic mean, $m \in \mathbb{R}\setminus\{-1,0\}$; and for $q>1$

$$\begin{aligned}\left\|w_n^{(2)}\right\|_q &= \left(\int_a^b \left|\frac{t^{\alpha+n}}{(\alpha+1)\cdots(\alpha+n)}\right|^q dt\right)^{\frac{1}{q}} \\ &= \frac{1}{(\alpha+1)\cdots(\alpha+n)} \\ &\quad\times\left[\frac{b^{(\alpha+n)q+1}-a^{(\alpha+n)q+1}}{((\alpha+n)q+1)(b-a)}\cdot(b-a)\right]^{\frac{1}{q}\cdot\frac{(\alpha+n)q}{(\alpha+n)q}} \\ &= \frac{(b-a)^{\frac{1}{q}}}{(\alpha+1)\cdots(\alpha+n)}L_{(\alpha+n)q}^{\alpha+n}(a,b),\end{aligned}$$

and, finally

$$\begin{aligned}\left\|w_n^{(2)}\right\|_\infty &= \sup_{t\in[a,b]}\left|\frac{t^{\alpha+n}}{(\alpha+1)\cdots(\alpha+n)}\right| \\ &= \frac{1}{(\alpha+1)\cdots(\alpha+n)}b^{\alpha+n}, \quad n\geq 1\end{aligned}$$

then by Theorem 2, we may state the inequality

$$\begin{aligned}(2.12)\quad &\left|\int_a^b t^\alpha g(t)\,dt-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}\right. \\ &\qquad\left.\cdot\left[b^{\alpha+k}g^{(k-1)}(b)-a^{\alpha+k}g^{(k-1)}(a)\right]\right| \\ &\leq\begin{cases}\dfrac{(b-a)\,L_{\alpha+n}^{\alpha+n}(a,b)}{(\alpha+1)\cdots(\alpha+n)}\left\|g^{(n)}\right\|_\infty & \text{if } \quad g^{(n)}\in L_\infty[a,b]; \\ \dfrac{(b-a)^{\frac{1}{q}}\,L_{(\alpha+n)q}^{\alpha+n}(a,b)}{(\alpha+1)\cdots(\alpha+n)}\left\|g^{(n)}\right\|_p & \text{if } \quad g^{(n)}\in L_p[a,b]; \\ & \text{and } \quad \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ \dfrac{b^{\alpha+n}}{(\alpha+1)\cdots(\alpha+n)}\left\|g^{(n)}\right\|_1 & \text{if } \quad g^{(n)}\in L_1[a,b],\end{cases}\end{aligned}$$

for all $\alpha > -1$.

3. Define $w_0^{(3)}(t) = \sin(\alpha t)$, $\alpha \in \mathbb{R}\backslash\{0\}$ and consider the sequence

$$w_k^{(3)}(t) = \frac{(-1)^k}{\alpha^k} \sin\left(\alpha t + k \cdot \frac{\pi}{2}\right).$$

Then

$$\begin{aligned} \frac{dw_k^{(3)}(t)}{dt} &= \frac{(-1)^k}{\alpha^k} \alpha \cos\left[\alpha t + k \cdot \frac{\pi}{2}\right] = \frac{(-1)^k}{\alpha^{k-1}} \sin\left[\frac{\pi}{2} - \left(\alpha t + k\frac{\pi}{2}\right)\right] \\ &= \frac{(-1)^{k-1}}{\alpha^{k-1}} \sin\left(\alpha t + k\frac{\pi}{2} - \frac{\pi}{2}\right) \\ &= \frac{(-1)^{k-1}}{\alpha^{k-1}} \sin\left(\alpha t + (k-1)\frac{\pi}{2}\right) \\ &= w_{k-1}^{(3)}(t), \quad t \in \mathbb{R}, \end{aligned}$$

which shows that $\left\{w_k^{(3)}\right\}_{k=0,1,\ldots}$ is an $w_0^{(3)}$–Appell type sequence.

If we use Lemma 1, we may state the identity

$$\begin{aligned} &\int_a^b \sin(\alpha t) \cdot g(t)\, dt \qquad (2.13) \\ &= \sum_{k=1}^{n} \frac{1}{\alpha^k} \left[\sin\left(\alpha a + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(a) - \sin\left(\alpha b + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(b)\right] \\ &\qquad + \frac{1}{\alpha^n} \int_a^b \sin\left(\alpha t + k\frac{\pi}{2}\right) g^{(n)}(t)\, dt. \end{aligned}$$

We compute

$$\begin{aligned} \left\|w_n^{(3)}\right\|_2 &:= \left[\frac{1}{\alpha^{2n}} \int_a^b \sin^2\left(\alpha t + n\frac{\pi}{2}\right) dt\right]^{\frac{1}{2}} \\ &= \frac{1}{|\alpha|^n} \left[\int_a^b \left[\frac{1 - \cos(2\alpha t + n \cdot \pi)}{2}\right] dt\right]^{\frac{1}{2}} \\ &= \frac{1}{|\alpha|^n} \left[\frac{1}{2}(b-a) - \frac{1}{4\alpha}\left[\sin(2\alpha b + n\pi) - \sin(2\alpha a + n\pi)\right]\right]^{\frac{1}{2}} \\ &= \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\,|\alpha|^n} \left[1 - \frac{\sin(2\alpha b + n\pi) - \sin(2\alpha a + n\pi)}{2\alpha(b-a)}\right]^{\frac{1}{2}}. \end{aligned}$$

Consequently, using Theorem 2 for Hilbertian norms, we may state that

$$
\begin{aligned}
(2.14)\quad & \left| \int_a^b \sin(\alpha t)\, g(t)\, dt \right. \\
& - \sum_{k=1}^{n} \frac{1}{\alpha^k} \left[\sin\left(\alpha a + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(a) \right. \\
& \qquad \left. \left. - \sin\left(\alpha b + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(b) \right] \right| \\
\leq\ & \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\,|\alpha|^n} \left[1 - \frac{\sin(2b\alpha + n\pi) - \sin(2a\alpha + n\pi)}{2\alpha(b-a)} \right]^{\frac{1}{2}} \left\| g^{(n)} \right\|_2,
\end{aligned}
$$

provided $g^{(n)} \in L_2[a,b]$.

4. Define $w_0^{(4)}(t) = \cos(\alpha t)$, $\alpha \in \mathbb{R}\setminus\{0\}$ and consider the sequence $w_k^{(4)}(t)$ $= \frac{(-1)^k}{\alpha^k} \cos\left(\alpha t + k \cdot \frac{\pi}{2}\right)$. Then

$$
\begin{aligned}
\frac{dw_k^{(4)}(t)}{dt} &= \frac{(-1)^k}{\alpha^k} \left[-\alpha \sin\left[\alpha t + k \cdot \frac{\pi}{2}\right] \right] = \frac{(-1)^{k-1}}{\alpha^{k-1}} \sin\left(\alpha t + k\frac{\pi}{2}\right) \\
&= \frac{(-1)^{k-1}}{\alpha^{k-1}} \cos\left[\frac{\pi}{2} - \left(\alpha t + k\frac{\pi}{2}\right)\right] \\
&= \frac{(-1)^{k-1}}{\alpha^{k-1}} \cos\left[\alpha t + (k-1)\frac{\pi}{2}\right] \\
&= w_{k-1}^{(4)}(t),\ t \in \mathbb{R},
\end{aligned}
$$

which shows that $\left\{ w_k^{(4)} \right\}_{k=0,1,\dots}$ is an $w_0^{(4)}$–Appell type sequence.

If we use Lemma 1, we may state the identity

$$
\begin{aligned}
(2.15)\quad & \int_a^b \cos(\alpha t)\, g(t)\, dt \\
=\ & \sum_{k=1}^{n} \frac{1}{\alpha^k} \left[\cos\left(\alpha a + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(a) - \cos\left(\alpha b + k\frac{\pi}{2}\right) \cdot g^{(k-1)}(b) \right] \\
& + \frac{1}{\alpha^n} \int_a^b \cos\left(\alpha t + k\frac{\pi}{2}\right) g^{(n)}(t)\, dt.
\end{aligned}
$$

We compute

$$\begin{aligned}\left\|w_n^{(4)}\right\|_2 &:= \left[\frac{1}{\alpha^{2n}}\int_a^b \cos^2\left(\alpha t + n\frac{\pi}{2}\right)dt\right]^{\frac{1}{2}} \\ &= \frac{1}{|\alpha|^n}\left[\int_a^b\left[\frac{1+\cos(2\alpha t + n\cdot\pi)}{2}\right]dt\right]^{\frac{1}{2}} \\ &= \frac{1}{|\alpha|^n}\left[\frac{1}{2}(b-a)+\frac{1}{4\alpha}\left[\sin(2\alpha b + n\pi) - \sin(2\alpha a + n\pi)\right]\right]^{\frac{1}{2}} \\ &= \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\,|\alpha|^n}\left[1+\frac{\sin(2\alpha b+n\pi)-\sin(2\alpha a+n\pi)}{2\alpha(b-a)}\right]^{\frac{1}{2}}.\end{aligned}$$

Consequently, using Theorem 2 for Hilbertian norms, we may state that

(2.16)

$$\begin{aligned}&\left|\int_a^b \cos(\alpha t)\, g(t)\,dt\right. \\ &\left.-\sum_{k=1}^{n}\frac{1}{\alpha^k}\left[\cos\left(\alpha a + k\frac{\pi}{2}\right)\cdot g^{(k-1)}(a) - \cos\left(\alpha b + k\frac{\pi}{2}\right)\cdot g^{(k-1)}(b)\right]\right| \\ &\le \frac{(b-a)^{\frac{1}{2}}}{\sqrt{2}\,|\alpha|^n}\left[1+\frac{\sin(2b\alpha+n\pi)-\sin(2a\alpha+n\pi)}{2\alpha(b-a)}\right]^{\frac{1}{2}}\left\|g^{(n)}\right\|_2,\end{aligned}$$

provided $g^{(n)} \in L_2[a,b]$.

3. A perturbed version via Korkine's identity

The following identity which can be easily proved by direct computation is known in the literature as Korkine's identity:

$$\begin{aligned}(3.1)\qquad &\frac{1}{b-a}\int_a^b u(t)\,v(t)\,dt - \frac{1}{b-a}\int_a^b u(t)\,dt\cdot\frac{1}{b-a}\int_a^b v(t)\,dt \\ &= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (u(t)-u(s))(v(t)-v(s))\,dtds,\end{aligned}$$

provided that $u, v : [a,b] \to \mathbb{R}$ are measurable and all the involved integrals exist.

The following representation of the weighted integral $\int_a^b w(t)\,g(t)\,dt$ holds.

LEMMA 2. *Let $w : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $\{w_k\}_{k=\overline{0,n+1}}$ be a sequence of $w-$Appell type. If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$ and $w_n g^{(n)} \in L_1[a,b]$, then we have the equality*

$$\int_a^b w(t)\,g(t)\,dt = B_n(w,g;a,b) + S_n(w,g;a,b), \tag{3.2}$$

where

$$B_n(w,g;a,b) = A_n(w,g;a,b) + (-1)^n \left[w_{n+1}(b) - w_{n+1}(a)\right]\left[g^{(n-1)};a,b\right] \tag{3.3}$$

and the remainder $S_n(w,g;a,b)$ can be represented by

$$\begin{aligned} &S_n(w,g;a,b) \\ &= \frac{(-1)^n}{2(b-a)}\int_a^b\int_a^b (w_n(t)-w_n(s))\left(g^{(n)}(t)-g^{(n)}(s)\right)dtds \end{aligned} \tag{3.4}$$

where $\left[g^{(n-1)};a,b\right]$ is the devided difference, i.e., we recall that

$$\left[g^{(n-1)};a,b\right] := \frac{g^{(n-1)}(b)-g^{(n-1)}(a)}{b-a}.$$

PROOF. Using Korkine's identity, we may write

$$\begin{aligned} &\int_a^b w_n(t)\,g^{(n)}(t)\,dt \\ &= \frac{1}{b-a}\int_a^b w_n(t)\,dt \cdot \int_a^b g^{(n)}(t)\,dt \\ &\quad + \frac{1}{2(b-a)}\int_a^b\int_a^b (w_n(t)-w_n(s))\left(g^{(n)}(t)-g^{(n)}(s)\right)dtds \\ &= \left[w_{n+1}(b)-w_{n+1}(a)\right]\cdot\frac{g^{(n-1)}(b)-g^{(n-1)}(a)}{b-a} \\ &\quad + \frac{1}{2(b-a)}\int_a^b\int_a^b (w_n(t)-w_n(s))\left(g^{(n)}(t)-g^{(n)}(s)\right)dtds, \end{aligned}$$

giving the following representation for the remainder $R_n(w,g;a,b)$:

$$\begin{aligned} &R_n(w,g;a,b) \\ &= (-1)^n\left[w_{n+1}(b)-w_{n+1}(a)\right]\cdot\frac{g^{(k-1)}(b)-g^{(k-1)}(a)}{b-a} \\ &\quad + \frac{(-1)^n}{2(b-a)}\int_a^b\int_a^b (w_n(t)-w_n(s))\left(g^{(n)}(t)-g^{(n)}(s)\right)dtds. \end{aligned}$$

Using the identity (2.3) we deduce (3.2). □

For an absolutely continuous function $h : [\alpha, \beta] \to \mathbb{R}$, we denote

$$\|h\|_{[\alpha,\beta],1} := \left| \int_{\alpha}^{\beta} |h(\tau)| \, d\tau \right|.$$

Using this notation, we may state the following result involving the estimation of the remainder $S(w, g; a, b)$ in the perturbed formula (3.2).

THEOREM 3. *With the assumption of Lemma 2, we have*

$$\begin{aligned} & |S(w, g; a, b)| \\ \leq \; & \frac{1}{2(b-a)} \int_a^b \int_a^b \|w_{n+1}\|_{[s,t],1} \left\| g^{(n+1)} \right\|_{[s,t],1} dtds \\ =: \; & M_n \left(w_{n+1}, g^{(n+1)} \right). \end{aligned} \tag{3.5}$$

PROOF. Using the representation (3.4), we may write

$$\begin{aligned} & |S(w, g; a, b)| \\ \leq \; & \frac{1}{2(b-a)} \int_a^b \int_a^b |w_n(t) - w_n(s)| \left| g^{(n)}(t) - g^{(n)}(s) \right| dtds \\ = \; & \frac{1}{2(b-a)} \int_a^b \int_a^b \left| \int_s^t w_{n+1}(\tau) \, d\tau \right| \left| \int_s^t g^{(n+1)}(\sigma) \, d\sigma \right| dtds \\ \leq \; & \frac{1}{2(b-a)} \int_a^b \int_a^b \|w_{n+1}\|_{[s,t],1} \left\| g^{(n+1)} \right\|_{[s,t],1} dtds \end{aligned}$$

and the estimate (3.5) is obtained. □

In practical applications some bounds of $M_n\left(w_{n+1}, g^{(n+1)}\right)$ could be more useful.

Now, it obvious that

$$\begin{aligned} & M_n \left(w_{n+1}, g^{(n+1)} \right) \\ \leq \; & \frac{1}{2(b-a)} ess \sup_{(t,s) \in [a,b]} \left\{ \|w_{n+1}\|_{[s,t],1} \right\} \cdot \int_a^b \int_a^b \left\| g^{(n+1)} \right\|_{[s,t],1} dtds \\ = \; & \frac{1}{2(b-a)} \left\| \|w_{n+1}\|_{[\cdot,\cdot],1} \right\|_{\infty} \left\| \left\| g^{(n+1)} \right\|_{[\cdot,\cdot],1} \right\|_1 . \end{aligned}$$

Also, by Hölder's integral inequality for double integrals, we may write that

$$\begin{aligned} M_n\left(w_{n+1}, g^{(n+1)}\right) &\leq \frac{1}{2(b-a)}\left(\int_a^b\int_a^b \|w_{n+1}\|_{[s,t],1}^q \, dtds\right)^{\frac{1}{q}} \\ &\quad \cdot\left(\int_a^b\int_a^b \left\|g^{(n+1)}\right\|_{[s,t],1}^p dtds\right)^{\frac{1}{p}} \\ &= \frac{1}{2(b-a)}\left\|\|w_{n+1}\|_{[\cdot,\cdot],1}\right\|_q \left\|\left\|g^{(n+1)}\right\|_{[\cdot,\cdot],1}\right\|_p, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

In a similar manner,

$$M_n\left(w_{n+1}, g^{(n+1)}\right) \leq \frac{1}{2(b-a)}\left\|\|w_{n+1}\|_{[\cdot,\cdot],1}\right\|_1 \left\|\left\|g^{(n+1)}\right\|_{[\cdot,\cdot],1}\right\|_\infty .$$

Consequently, we may state the following corollary.

COROLLARY 2. *With the assumptions of Lemma 2, we have the following bounds for the remainder $S_n(w,g;a,b)$:*

$$(3.6)\qquad |S_n(w,g;a,b)| \leq \begin{cases} \frac{1}{2(b-a)}\left\|\|w_{n+1}\|_{[\cdot,\cdot],1}\right\|_\infty \left\|\|g^{(n+1)}\|_{[\cdot,\cdot],1}\right\|_1; \\ \frac{1}{2(b-a)}\left\|\|w_{n+1}\|_{[\cdot,\cdot],1}\right\|_q \left\|\|g^{(n+1)}\|_{[\cdot,\cdot],1}\right\|_p, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2(b-a)}\left\|\|w_{n+1}\|_{[\cdot,\cdot],1}\right\|_1 \left\|\|g^{(n+1)}\|_{[\cdot,\cdot],1}\right\|_\infty, \end{cases}$$

where $\|\cdot\|_s$ $(s\in[1,\infty])$ are the usual Lebesgue norms on

$$L_s([a,b]\times[a,b])^2.$$

Now, if we use the natural notations

$$\|h\|_{[t,s],\beta} := \left|\int_t^s |h(\tau)|^\beta d\tau\right|^{\frac{1}{\beta}}, \ \beta>1$$

and

$$\|h\|_{[t,s],\infty} := ess\sup\{|h(\tau)|,\ \tau\in[t,s]\ \ ([s,t])\},$$

then by Hölder's integral inequality, we have

$$(3.7)\qquad \begin{gathered} \|w_{n+1}\|_{[s,t],1} \leq |s-t|^{\frac{1}{\alpha}} \|w_{n+1}\|_{[s,t],\beta}, \\ \beta>1,\ \frac{1}{\beta}+\frac{1}{\alpha}=1 \ \text{ or } \ \alpha=1,\ \beta=\infty \end{gathered}$$

and

$$\text{(3.8)} \qquad \left\|g^{(n+1)}\right\|_{[s,t],1} \le |s-t|^{\frac{1}{\gamma}} \left\|g^{(n+1)}\right\|_{[s,t],\delta}, \ \gamma > 1, \ \frac{1}{\gamma} + \frac{1}{\delta} = 1 \ \text{ or } \ \gamma = 1, \ \delta = \infty.$$

Multiplying (3.7) and (3.8) and integrating over $(t,s) \in [a,b]^2$, we deduce

$$\text{(3.9)} \qquad M_n\left(w_{n+1}, g^{(n+1)}\right) \le \frac{1}{2(b-a)} \int_a^b \int_a^b |s-t|^{\frac{1}{\alpha}+\frac{1}{\gamma}} \|w_{n+1}\|_{[s,t],\beta} \left\|g^{(n+1)}\right\|_{[s,t],\delta} dtds,$$

where $\alpha, \beta > 1$, $\frac{1}{\beta} + \frac{1}{\alpha} = 1$ or $\alpha = 1$, $\beta = \infty$; $\gamma, \delta > 1$, $\frac{1}{\gamma} + \frac{1}{\delta} = 1$ or $\gamma = 1$ and $\delta = \infty$, which is an inequality interesting in itself.

The terms $M_n\left(w_{n+1}, g^{(n+1)}\right)$ can be bounded in a simpler form as follows.

COROLLARY 3. *With the assumptions of Lemma 2 and if α, β, γ, δ are as above and $r_i > 1$, $\left(i = \overline{1,3}\right)$, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, then we have the inequality:*

$$\text{(3.10)} \qquad M_n\left(w_{n+1}, g^{(n+1)}\right) \le K_{r_1} \left\| \|w_{n+1}\|_{[\cdot,\cdot],\beta} \right\|_{r_2} \left\| \left\|g^{(n+1)}\right\|_{[\cdot,\cdot],\delta} \right\|_{r_3},$$

where

$$K_{r_1} = \frac{2^{\frac{1}{r_1}-1} (b-a)^{\frac{1}{\alpha}+\frac{1}{\delta}+\frac{2}{r_1}-1}}{\left[\left(\frac{1}{\alpha}+\frac{1}{\delta}\right) r_1 + 1\right]^{\frac{1}{r_1}} \left[\left(\frac{1}{\alpha}+\frac{1}{\delta}\right) r_1 + 2\right]^{\frac{1}{r_1}}}.$$

PROOF. Using Hölder's integral inequality for three terms with $r_i > 1$, $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1$, we may write:

$$\text{(3.11)} \qquad \int_a^b \int_a^b |s-t|^{\frac{1}{\alpha}+\frac{1}{\gamma}} \|w_{n+1}\|_{[s,t],\beta} \left\|g^{(n+1)}\right\|_{[s,t],\delta} dtds$$
$$\le \left(\int_a^b \int_a^b |s-t|^{\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right) r_1} dtds\right)^{\frac{1}{r_1}} \times \left(\int_a^b \int_a^b \|w_{n+1}\|_{[s,t],\beta}^{r_2} dtds\right)^{\frac{1}{r_2}} \times \left(\int_a^b \int_a^b \left\|g^{(n+1)}\right\|_{[s,t],\delta}^{r_3} dtds\right)^{\frac{1}{r_3}}$$

$$= \left(\int_a^b \int_a^b |s-t|^{\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1} dtds\right)^{\frac{1}{r_1}} \left\| \|w_{n+1}\|_{[\cdot,\cdot],\beta} \right\|_{r_2} \left\| \left\|g^{(n+1)}\right\|_{[\cdot,\cdot],\delta} \right\|_{r_3}.$$

We remark that, for a positive p, we have

$$\begin{aligned} & \int_a^b \int_a^b |x-y|^p \, dxdy \\ = & \int_a^b \left(\int_a^b |y-x|^p \, dy\right) dx \\ = & \int_a^b \left(\int_a^x (x-y)^p \, dy + \int_x^b (y-x)^p \, dy\right) dx \\ = & \int_a^b \left[\frac{(x-a)^{p+1} + (b-x)^{p+1}}{p+1}\right] dx = \frac{2(b-a)^{p+2}}{(p+1)(p+2)} \end{aligned}$$

and then

$$\begin{aligned} & \left(\int_a^b \int_a^b |s-t|^{\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1} dtds\right)^{\frac{1}{r_1}} \\ = & \left[\frac{2(b-a)^{\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1+2}}{\left[\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1+1\right]\left[\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)r_1+2\right]}\right]^{\frac{1}{r_1}} \\ = & \frac{2^{\frac{1}{r_1}}(b-a)^{\frac{1}{\alpha}+\frac{1}{\delta}+\frac{2}{r_1}}}{\left[\left(\frac{1}{\alpha}+\frac{1}{\delta}\right)r_1+1\right]^{\frac{1}{r_1}}\left[\left(\frac{1}{\alpha}+\frac{1}{\delta}\right)r_1+2\right]^{\frac{1}{r_1}}}. \end{aligned}$$

Using (3.9) and (3.11) we obtain (3.10). □

REMARK 4. If one would use the examples **1-4** considered in Section 2, some particular inequalities may be stated. We omit the details.

4. Some bounds via a Grüss type inequality

In [4], Cheng and Sun have proved the following integral inequality of Grüss type for univariate real functions:

LEMMA 3. *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable functions such that*

$$\varphi \le g(x) \le \Phi \text{ for a.e. } x \in [a,b]. \tag{4.1}$$

Then

$$\left|\frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \frac{1}{2}(\Phi-\varphi)\frac{1}{b-a}\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|dt. \tag{4.2}$$

For a generalization of this result in the abstract settings of Lebesgue integrals and weighted means, see [3].

The following result also holds.

THEOREM 4. *Let $w:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $\{w_k\}_{k=\overline{0,n+1}}$ be a sequence of w–Appell type. If $g:[a,b]\to\mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$, $w_n g^{(n)}\in L_1[a,b]$, and there exists the constants γ_n,Γ_n so that*

$$-\infty<\gamma_n\le g^{(n)}\le\Gamma_n<\infty \text{ a.e. on } [a,b],$$

then we have the equality

$$\int_a^b w(t)\,g(t)\,dt=B_n(w,g;a,b)+S_n(w,g;a,b),$$

where

$$B_n(w,g;a,b) = A_n(w,g;a,b)+(-1)^n\left[w_{n+1}(b)-w_{n+1}(a)\right]\left[g^{(n-1)};a,b\right], \tag{4.3}$$

$A_n(w,g;a,b)$ is given in equation (2.4) and the remainder $S_n(w,g;a,b)$ satisfies the estimate

$$|S_n(w,g;a,b)|\le\frac{1}{2}(\Gamma_n-\gamma_n)\int_a^b\left|w_n(t)-[w_{n+1};a,b]\right|dt \tag{4.4}$$

where $[w_{n+1};a,b]$ is the devided difference.

PROOF. From (3.4) we have

$$S_n(w,g;a,b) = (b-a)(-1)^n\left|\frac{1}{b-a}\int_a^b w_n(t)\,g^{(n)}(t)\,dt -\frac{1}{b-a}\int_a^b w_n(t)\,dt\cdot\frac{1}{b-a}\int_a^b g^{(n)}(t)\,dt\right|.$$

Thus, by the use of Lemma 3 we have

$$\begin{aligned} & |S_n(w,g;a,b)| \\ \leq \; & \frac{1}{2}(\Gamma_n - \gamma_n)\int_a^b \left| w_n(t) - \frac{1}{b-a}\int_a^b w_n(s)\,ds \right| dt \end{aligned}$$

and the theorem is proved. □

REMARK 5. If one would use the examples **1-4** considered in Section 2, some particular inequalities may be stated. We omit the details.

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