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MULTIPLE L_p ANALYTIC GENERALIZED FOURIER-FEYNMAN TRANSFORM ON THE BANACH ALGEBRA

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ABSTRACT. In this paper, we use a generalized Brownian motion process to define a generalized Feynman integral and a generalized Fourier-Feynman transform. We also define the concepts of the multiple L_p analytic generalized Fourier-Feynman transform and the generalized convolution product of functionals on function space $C_{a,b}[0,T]$. We then verify the existence of the multiple L_p analytic generalized Fourier-Feynman transform for functionals on function space that belong to a Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$. Finally we establish some relationships between the multiple L_p analytic generalized Fourier-Feynman transform and the generalized convolution product for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$.

1. Introduction

The concept of L_1 analytic Fourier-Feynman transform (FFT) was introduced by Brue in [2]. In [3], Cameron and Storvick introduced the concept of an L_2 analytic FFT on Wiener space. In [13], Johnson and Skoug developed an L_p analytic FFT theory for $1 \leq p \leq 2$ which extended the results in [2, 3] and gave various relationships between the L_1 and L_2 theories. In [9]-[11], Huffman, Park and Skoug developed an L_p analytic FFT theory on certain classes of functionals defined on Wiener space and they defined a convolution product(CP) of two functionals on Wiener space and then found several interesting properties for the FFT

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and the CP on Wiener space. In [1], Ahn investigated the L_1 analytic FFT theory on the Fresnel class of an abstract Wiener space. In [5], Chang, Song, and Yoo studied the analytic FFT and the first variation on abstract Wiener space and the Fresnel class $\mathcal{F}(B)$.

The Wiener process is free of drift and is stationary in time while the stochastic process used in this paper is nonstationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [15]. In [6], Chang and Chung studied the conditional function space integral and in [7], Chang and Skoug studied the L_p analytic generalized Fourier-Feynman transform(GFFT) and first variation on function space $C_{a,b}[0,T]$. Recently, in [8], Chang, Choi and Skoug obtained the integration by parts formulas for the generalized Feynman integral and the L_1 and L_2 analytic GFFT on function space.

In Section 2 of this paper, we introduce the basic concepts and the notations for our research. In Section 3, we study the L_p analytic GFFT and the generalized CP(GCP). In Section 4, we investigate the essential properties for the multiple L_p analytic GFFT and the GCP on a function space $C_{a,b}[0,T]$. Finally, we establish some relationships between the multiple L_p analytic GFFT and the GCP for functionals in $\mathcal{S}(L^2_{a,b}[0,T])$.

2. Definitions and preliminaries

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega), \cdots, Y(t_n,\omega))$ is normally distributed with the density function

$$
\begin{aligned}
K(\vec{t},\vec{\eta}) &= \Big((2\pi)^n \prod_{j=1}^{n} (b(t_j) - b(t_{j-1}))\Big)^{-1/2} \\
&\quad \cdot \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\bigg\}
\end{aligned} \tag{2.1}
$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \cdots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$, $a'(t) \in L^2[0,T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(s) > 0$ for each $s \in [0,T]$.

As explained in [18, pp.18–20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By Theorem 14.2 [18, p.187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable [14] provided ρB is $\mathcal{B}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.).

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$: i.e.,

(2.2)
$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)]. \tag{2.3}$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthonormal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j,\phi_k)_{a,b}=\begin{cases}0, & j\neq k\\ 1, & j=k\end{cases},$$

and for each $v\in L^2_{a,b}[0,T]$, let

$$v_n(t)=\sum_{j=1}^{n}(v,\phi_j)_{a,b}\phi_j(t) \tag{2.4}$$

for $n=1,2,\cdots$. Then for each $v\in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund(PWZ) stochastic integral $\langle v,x\rangle$ is defined by the formula

$$\langle v,x\rangle=\lim_{n\to\infty}\int_0^T v_n(t)dx(t) \tag{2.5}$$

for all $x\in C_{a,b}[0,T]$ for which the limit exists ; one can show that for each $v\in L^2_{a,b}[0,T]$, the PWZ integral $\langle v,x\rangle$ exists for μ-a.e. $x\in C_{a,b}[0,T]$.

We denote the function space integral of a $\mathcal{B}(C_{a,b}[0,T])$-measurable functional F by

$$E[F]=\int_{C_{a,b}[0,T]}F(x)d\mu(x), \tag{2.6}$$

whenever the integral exists.

We are now ready to state the definition of the generalized analytic Feynman integral.

DEFINITION 2.1. Let $\mathbb{C}$ denote the complex numbers. Let $\mathbb{C}_+=\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda>0\}$ and $\tilde{\mathbb{C}}_+=\{\lambda\in\mathbb{C}:\lambda\neq 0 \text{ and } \mathrm{Re}\lambda\geq 0\}$. Let $F:C_{a,b}[0,T]\longrightarrow\mathbb{C}$ be such that for each $\lambda>0$, the function space integral

$$J(\lambda)=\int_{C_{a,b}[0,T]}F(\lambda^{-\frac{1}{2}}x)d\mu(x)$$

exists for all $\lambda>0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda)=J(\lambda)$ for all $\lambda>0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda\in\mathbb{C}_+$ we write

$$E^{\mathrm{an}_\lambda}[F]\equiv E_x^{\mathrm{an}_\lambda}[F(x)]=J^*(\lambda). \tag{2.7}$$

Let $q \neq 0$ be a real number and let F be a functional such that $E^{\text{an}_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F over $C_{a,b}[0,T]$ with parameter q and we write

$$E^{\text{anf}_q}[F] \equiv E_x^{\text{anf}_q}[F(x)] = \lim_{\lambda \to -iq} E^{\text{an}_\lambda}[F] \tag{2.8}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Next we state the definitions of the analytic GFFT and the GCP.

DEFINITION 2.2. For $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, let

$$T_\lambda(F)(y) = E_x^{\text{an}_\lambda}[F(y+x)]. \tag{2.9}$$

In the standard Fourier theory, the integrals involved are often interpreted in the mean; a similar concept is useful in the FFT theory [13]. Let $p \in (1,2]$ and let p and p' be related by $1/p + 1/p' = 1$. Let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$\lim_{n\to\infty} E[|H_n(\rho y) - H(\rho y)|^{p'}] = 0.$$

Then we write

$$H \approx \text{l.i.m.}_{n\to\infty} H_n$$

and we call H the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ.

We are ready to state the definition of the L_p analytic GFFT.

DEFINITION 2.3. Let q be a nonzero real number and let F be a measurable functional on $C_{a,b}[0,T]$. For $p \in (1,2]$, we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(p)}(F)(y) = \text{l.i.m.}_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.10}$$

if it exists. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.11}$$

if it exists.

We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$.

DEFINITION 2.4. Let F and G be measurable functionals on $C_{a,b}[0,T]$. For $\lambda \in \tilde{\mathbb{C}}_+$, we define their GCP $(F*G)_\lambda$ (if it exists) by
(2.12)
$$(F*G)_\lambda(y) = \begin{cases} E_x^{\text{an}_\lambda}[F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})], & \lambda \in \mathbb{C}_+ \\ E_x^{\text{anf}_q}[F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})], & \lambda = -iq,\ q \in \mathbb{R},\ q \neq 0. \end{cases}$$

REMARK 2.1. (1) When $\lambda = -iq$, we denote $(F*G)_\lambda$ by $(F*G)_q$.
(2) For any real $q \neq 0$, we briefly describe F_q^* and ${}^*F_q$ of a functional F on $C_{a,b}[0,T]$ as follows :

$$F_q^* = (F*1)_q \quad \text{and} \quad {}^*F_q = (1*F)_q. \tag{2.13}$$

The following generalized analytic Feynman integral formula is used several times in this paper.

$$E_x[\exp\{i\lambda^{-\frac{1}{2}}\langle v, x\rangle\}] = \exp\bigg\{-\frac{1}{2\lambda}(v^2, b') + i\lambda^{-\frac{1}{2}}(v, a')\bigg\} \tag{2.14}$$

for all $\lambda \in \tilde{\mathbb{C}}_+$ and $v \in L^2_{a,b}[0,T]$, where

$$(v, a') = \int_0^T v(t)a'(t)dt = \int_0^T v(t)da(t) \tag{2.15}$$

and

$$(v^2, b') = \int_0^T v^2(t)b'(t)dt = \int_0^T v^2(t)db(t). \tag{2.16}$$

3. Transforms and convolutions

First we give the definition of the Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ referred to in Section 1 above.

DEFINITION 3.1. Let $M(L^2_{a,b}[0,T])$ be the space of complex-valued, countably additive (and hence finite) Borel measures on $L^2_{a,b}[0,T]$. The Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ consists of those functionals F on $C_{a,b}[0,T]$ expressible in the form

$$F(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle u, x\rangle\}df(u) \tag{3.1}$$

for s-a.e. $x \in C_{a,b}[0,T]$, where the associated measure f is an element of $M(L^2_{a,b}[0,T])$.

REMARK 3.1. (i) When $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $\mathcal{S}(L^2_{a,b}[0,T])$ reduces to the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [4]. For further work on $\mathcal{S}$, see the references referred to in Section 20.1 of [12].

(ii) $M(L^2_{a,b}[0,T])$ is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

(iii) One can show that the correspondence $f \mapsto F$ is injective, carries convolution into pointwise multiplication and that $\mathcal{S}(L^2_{a,b}[0,T])$ is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{L^2_{a,b}[0,T]} |df(u)|.$$

In [4], Cameron and Storvick carry out these arguments in detail for the Banach algebra $\mathcal{S}$.

REMARK 3.2. If $a(t) \equiv 0$ on $[0,T]$, then for all $F \in \mathcal{S}(L^2_{a,b}[0,T])$ with associated measure f, the generalized analytic Feynman integral $E^{\text{anf}_q}[F]$ will always exist for all real $q \neq 0$ and be given by the formula

$$E^{\text{anf}_q}[F] = \int_{L^2_{a,b}[0,T]} \exp\left\{ -\frac{i(u^2,b')}{2q} \right\} df(u). \tag{3.2}$$

However, for $a(t)$ and $b(t)$ as in Section 2, and proceeding formally using equations (3.1) and (2.14), we see that $E^{\text{anf}_q}[F]$ will be given by the formula

$$E^{\text{anf}_q}[F] = \int_{L^2_{a,b}[0,T]} \exp\left\{ -\frac{i(u^2,b')}{2q} + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u,a') \right\} df(u) \tag{3.3}$$

if it exists. But the integral on the right hand-side of (3.3) might not exist if the real part of

$$-\frac{i(u^2,b')}{2q} + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u,a')$$

is positive. However

$$\left| \exp\left\{ -\frac{i(u^2,b')}{2q} + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u,a') \right\} \right| = \begin{cases} \exp\{-(2q)^{-1/2}(u,a')\}, & q > 0 \\ \exp\{(-2q)^{-1/2}(u,a')\}, & q < 0 \end{cases},$$

and so the generalized analytic Feynman integral $E^{\mathrm{anf}_q}[F]$ will certainly exist provided the associated measure f satisfies the condition

$$\int_{L^2_{a,b}[0,T]} \exp\left\{\frac{1}{\sqrt{|2q|}}\int_0^T |u(s)|d|a|(s)\right\}|df(u)| < \infty. \tag{3.4}$$

In our next theorem, we obtain the L_p analytic GFFT $T_q^{(p)}(F)$ of a functional F in $\mathcal{S}(L^2_{a,b}[0,T])$.

THEOREM 3.1. *Let q_0 be a nonzero real number and let F be an element of $\mathcal{S}(L^2_{a,b}[0,T])$ whose associated measure f satisfies the condition (3.4) above with q replaced with q_0. Then for all $p \in [1,2]$ and all real q with $|q| \geq |q_0|$, the L_p analytic GFFT of F, $T_q^{(p)}(F)$ exists and is given by the formula*

(3.5)

$$T_q^{(p)}(F)(y) = \int_{L^2_{a,b}[0,T]} \exp\left\{i\langle u,y\rangle - \frac{i}{2q}(u^2,b') + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u,a')\right\}df(u)$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore $T_q^{(p)}(F)$ is an element of $\mathcal{S}(L^2_{a,b}[0,T])$ with associated measure ϕ defined by

$$\phi(E) = \int_E \exp\left\{-\frac{i}{2q}(u^2,b') + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u,a')\right\}df(u) \tag{3.6}$$

for $E \in \mathcal{B}(L^2_{a,b}[0,T])$.

PROOF. By (2.9), the Fubini theorem, and (2.14), we have that for all $\lambda > 0$,

(3.7)

$$\begin{aligned} T_\lambda(F)(y) &= E_x[F(y+\lambda^{-1/2}x)] \\ &= \int_{L^2_{a,b}[0,T]} E_x[\exp\{i\langle u,y\rangle + i\lambda^{-1/2}\langle u,x\rangle\}]df(u) \\ &= \int_{L^2_{a,b}[0,T]} \exp\left\{i\langle u,y\rangle - \frac{1}{2\lambda}(u^2,b') + \frac{i}{\sqrt{\lambda}}(u,a')\right\}df(u) \end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$. But the last expression above is analytic throughout $\mathbb{C}_+$ and is continuous on $\tilde{\mathbb{C}}_+$. Thus the equation (3.5) is established. Let ϕ be defined by (3.6) for each $E \in \mathcal{B}(L^2_{a,b}[0,T])$. By using (3.4) above, we obtain that

$$(3.8) \qquad ||\phi|| \le \int_{L^2_{a,b}[0,T]} \exp\left\{ \frac{1}{\sqrt{|2q_0|}} \int_0^T |u(s)|d|a|(s) \right\} |df(u)| < \infty.$$

Hence we have the desired result. □

In our next theorem, we obtain the GCP of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$.

THEOREM 3.2. *Let q_0 be a nonzero real number and let F and G be elements of $\mathcal{S}(L^2_{a,b}[0,T])$ whose associated measures f and g satisfy the condition*

$$(3.9) \quad \int_{L^2_{a,b}[0,T]} \exp\left\{ \frac{1}{\sqrt{|4q_0|}} \int_0^T |u(s)|d|a(s)| \right\} \big[|df(u)| + |dg(u)|\big] < \infty.$$

Then their GCP $(F*G)_q$ exists for all $p \in [1,2]$ and all real q with $|q| \ge |q_0|$ and is given by the formula

(3.10)

$$(F*G)_q(y) = \int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i}{4q}((u-v)^2, b') + i\left(\frac{i}{2q}\right)^{\frac{1}{2}} (u-v, a') \right\} df(u)dg(v)$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore $(F*G)_q$ is an element of $\mathcal{S}(L^2_{a,b}[0,T])$.

PROOF. By using (2.12), the Fubini theorem, and (2.14), we have that for all $\lambda > 0$,

$$(3.11) \quad \begin{aligned} &(F*G)_\lambda(y) \\ &= E_x\left[F\left(\frac{y+\lambda^{-1/2}x}{\sqrt{2}}\right) G\left(\frac{y-\lambda^{-1/2}x}{\sqrt{2}}\right)\right] \\ &= \int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} E_x\left[\exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle + \frac{i}{\sqrt{2\lambda}}\langle u-v, x\rangle \right\}\right] df(u)dg(v) \\ &= \int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{1}{4\lambda}((u-v)^2, b') + \frac{i}{\sqrt{2\lambda}}(u-v, a') \right\} df(u)dg(v) \end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$. But the last expression above is analytic throughout $\mathbb{C}_+$, and is continuous on $\tilde{\mathbb{C}}_+$. Thus we have the equation (3.10) above.

Let a set function $h : \mathcal{B}(L^2_{a,b}[0,T] \times L^2_{a,b}[0,T]) \to \mathbb{C}$ be defined by

(3.12)
$$h(E) = \int_E \exp\bigg\{ -\frac{i}{4q}((u-v)^2, b') + i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u-v, a')\bigg\} df(u)dg(v)$$

for each $E \in \mathcal{B}(L^2_{a,b}[0,T] \times L^2_{a,b}[0,T])$. Then h is a complex Borel measure on $\mathcal{B}(L^2_{a,b}[0,T] \times L^2_{a,b}[0,T])$. Now we define a function $\varphi : L^2_{a,b}[0,T] \times L^2_{a,b}[0,T] \to L^2_{a,b}[0,T]$ by

$$\varphi(u,v) = \frac{1}{\sqrt{2}}(u+v). \tag{3.13}$$

Then φ is continuous and so it is Borel measurable. Let $\tilde{h} = h \circ \varphi^{-1}$. By the condition (3.9) above, we have that for real q with $|q| \ge |q_0|$

(3.14)
$$\begin{aligned}
||\tilde{h}|| &= \int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} |dh(u,v)| \\
&\le \int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} \bigg| \exp\bigg\{ -\frac{i}{4q}((u-v)^2, b') \\
&\qquad + i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u-v, a')\bigg\}\bigg| |df(u)||dg(v)| \\
&\le \int_{L^2_{a,b}[0,T]} \exp\bigg\{ \frac{1}{\sqrt{|4q_0|}} \int_0^T |u(s)| d|a|(s) \bigg\} |df(u)| \\
&\qquad \cdot \int_{L^2_{a,b}[0,T]} \exp\bigg\{ \frac{1}{\sqrt{|4q_0|}} \int_0^T |v(s)| d|a|(s) \bigg\} |dg(v)| < \infty.
\end{aligned}$$

Hence $\tilde{h} = h \circ \varphi^{-1}$ belongs to $M(L^2_{a,b}[0,T])$ and

$$(F * G)_q(y) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle r, y\rangle\} d\tilde{h}(r) \tag{3.15}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Hence $(F * G)_q$ exists and is given by (3.10) for all real q with $|q| \ge |q_0|$ and it belongs to $\mathcal{S}(L^2_{a,b}[0,T])$. □

REMARK 3.3. Let F, f, and q_0 be as in Theorem 3.2. Then for all real q with $|q| \geq |q_0|$, F_q^* and ${}^*F_q$ exist. Furthermore, F_q^* and ${}^*F_q$ are in $\mathcal{S}(L^2_{a,b}[0,T])$.

THEOREM 3.3. *Let F, G, f, g, and q_0 be as in Theorem 3.2. Then for all $p \in [1,2]$ and all real q with $|q| \geq |q_0|$,*

$$T_q^{(p)}((F*G)_q)(y) = T_q^{(p)}(F_q^*)(y)T_q^{(p)}({}^*G_q)(y) \tag{3.16}$$

for s-a.e. $y \in C_{a,b}[0,T]$, where F_q^* and ${}^*G_q$ are given by (2.13). Also, both of the expressions in (3.16) *are given by the expression*

$$\begin{aligned}\int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]} \exp\Big\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i}{2q}(u^2+v^2, b') \\ + 2i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u,a')\Big\} df(u)dg(v).\end{aligned} \tag{3.17}$$

PROOF. By using (2.9), (2.12), the Fubini theorem, and (2.14), we have that for all $\lambda > 0$,

$$T_\lambda((F*G)_\lambda)(y) = T_\lambda(F_\lambda^*)(y)T_\lambda({}^*G_\lambda)(y) \tag{3.18}$$

for s-a.e. $y \in C_{a,b}[0,T]$. But both of the expressions on the right-hand side of equation (3.18) are analytic functions of λ throughout $\mathbb{C}_+$, and are continuous functions of λ on $\tilde{\mathbb{C}}_+$ for all $y \in C_{a,b}[0,T]$. By using (3.9), $T_q^{(p)}((F*G)_q)$ exists for all real q with $|q| \geq |q_0|$ and is given by (3.16) for all desired values of p and q. □

THEOREM 3.4. *Let F, G, f, g, and q_0 be as in Theorem 3.3. Then*

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{-q}} T_q^{(p)}((F*G)_q)(y)d\mu(y) \\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{-q}} T_q^{(p)}(F_q^*)(y)T_q^{(p)}({}^*G_q)(y)d\mu(y) \\ &= \int_{C_{a,b}[0,T]}^{\mathrm{anf}_{q}} (F_{-q}^*)_q^*(y)(G_{-q}^*)_q^*(-y)d\mu(y)\end{aligned} \tag{3.19}$$

for all $p \in [1,2]$ and all real q with $|q| \geq |q_0|$.

PROOF. Fix p and q. Then for $\lambda > 0$, using (3.17) and the Fubini theorem we have

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} T_q^{(p)}((F*G)_q)(y/\sqrt{\lambda})d\mu(y)\\
&=\int_{C_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]}\exp\bigg\{\frac{1}{\sqrt{2\lambda}}\langle u+v,y\rangle\\
&\qquad -\frac{i}{2q}(u^2+v^2,b')+2i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u,a')\bigg\}df(u)dg(v)d\mu(y) \qquad (3.20)\\
&=\int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]}\exp\bigg\{-\frac{1}{4\lambda}((u+v)^2,b')+\frac{i}{\sqrt{2\lambda}}(u+v,a')\\
&\qquad -\frac{i}{2q}(u^2+v^2,b')+2i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u,a')\bigg\}df(u)dg(v).
\end{aligned}$$

But the last expression is an analytic function of λ throughout $\tilde{\mathbb{C}}_+$ and is continuous throughout $\tilde{\mathbb{C}}_+$, and so letting $\lambda = -i(-q) = iq$, we obtain that

(3.21)

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{\mathrm{anf}_{-q}} T_q^{(p)}((F*G)_q)(y)d\mu(y)\\
&=\int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]}\exp\bigg\{\frac{i}{4q}((u+v)^2,b')+i\Big(\frac{-i}{2q}\Big)^{\frac{1}{2}}(u+v,a')\\
&\qquad -\frac{i}{2q}(u^2+v^2,b')+2i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u,a')\bigg\}df(u)dg(v)\\
&=\int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]}\exp\bigg\{-\frac{i}{4q}((u-v)^2,b')+i\Big(\frac{-i}{2q}\Big)^{\frac{1}{2}}(u+v,a')\\
&\qquad +2i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u,a')\bigg\}df(u)dg(v).
\end{aligned}$$

Clearly, the condition (3.9) will imply the existence of (3.21). On the other hand, using (2.13), (3.10), the Fubini theorem, and (2.14), we obtain that for $\lambda > 0$,

$$
\begin{aligned}
&(F^*_{-q})^*_\lambda(y)\\
&= \int_{C_{a,b}[0,T]} F^*_{-q}\Big(\frac{y+\lambda^{-\frac{1}{2}}x}{\sqrt{2}}\Big)d\mu(x)\\
&= \int_{L^2_{a,b}[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u,y\rangle - \frac{1}{4\lambda}(u^2,b') + \frac{i}{\sqrt{2\lambda}}(u,a')\\
&\qquad + \frac{i}{4q}(u^2,b') + i\Big(\frac{-i}{2q}\Big)^{\frac{1}{2}}(u,a')\Big\}df(u)
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
&(G^*_{-q})^*_\lambda(-y)\\
&= \int_{C_{a,b}[0,T]} G^*_{-q}\Big(\frac{-y+\lambda^{-\frac{1}{2}}x}{\sqrt{2}}\Big)d\mu(x)\\
&= \int_{L^2_{a,b}[0,T]} \exp\Big\{-\frac{i}{\sqrt{2}}\langle v,y\rangle - \frac{1}{4\lambda}(v^2,b') + \frac{i}{\sqrt{2\lambda}}(v,a')\\
&\qquad + \frac{i}{4q}(v^2,b') + i\Big(\frac{-i}{2q}\Big)^{\frac{1}{2}}(v,a')\Big\}dg(v)
\end{aligned}
\tag{3.23}
$$

s-a.e. $y \in C_{a,b}[0,T]$. By using (3.22) and (3.23), we have that for $\lambda > 0$

$$
\begin{aligned}
&\int_{C_{a,b}[0,T]} (F^*_{-q})^*_\lambda(y/\sqrt{\lambda})(G^*_{-q})^*_\lambda(-y/\sqrt{\lambda})d\mu(y)\\
&= \int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]} \exp\Big\{-\frac{1}{4\lambda}((u-v)^2,b') + \frac{i}{\sqrt{2\lambda}}(u-v,a')\\
&\qquad - \frac{1}{4\lambda}(u^2+v^2,b') + \frac{i}{\sqrt{2\lambda}}(u+v,a')\\
&\qquad + \frac{i}{4q}(u^2+v^2,b') + i\Big(\frac{-i}{2q}\Big)^{\frac{1}{2}}(u+v,a')\Big\}df(u)dg(v).
\end{aligned}
\tag{3.24}
$$

But the last expression above is an anlytic function of λ throughout $\tilde{\mathbb{C}}_+$ and is continuous throughout on $\tilde{\mathbb{C}}_+$ and so letting $\lambda \to -iq$ we obtain

that
(3.25)

$$\int_{C_{a,b}[0,T]}^{\text{anf}_q} (F^*_{-q})^*_q(y)(G^*_{-q})^*_q(-y)d\mu(y)$$
$$= \int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]} \exp\bigg\{ -\frac{i}{4q}((u-v)^2, b') + 2i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u,a') + i\Big(\frac{-i}{2q}\Big)^{\frac{1}{2}}(u+v,a')\bigg\}df(u)dg(v).$$

Now (3.21) and (3.25) together yield (3.19). □

REMARK 3.4. In Theorem 3.4 above, if $a(t) \equiv 0$, then for all $q \neq 0$,
(3.26)

$$T_q^{(p)}(F_q^*)(y) = T_q^{(p)}(F)(y/\sqrt{2}) \text{ and } T_q^{(p)}({}^*G_q)(y) = T_q^{(p)}(G)(y/\sqrt{2})$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore

$$(F^*_{-q})^*_q(y) = F(y/\sqrt{2}) \text{ and } (G^*_{-q})^*_q(-y) = G(-y/\sqrt{2}).$$

Hence we have the following Parseval's identity

$$\int_{C_{a,b}[0,T]}^{\text{anf}_{-q}} T_q^{(p)}((F*G)_q)(y)d\mu(y)$$
$$= \int_{C_{a,b}[0,T]}^{\text{anf}_{-q}} T_q^{(p)}(F)(y/\sqrt{2})T_q^{(p)}(G)(y/\sqrt{2})d\mu(y)$$
$$= \int_{C_{a,b}[0,T]}^{\text{anf}_{q}} F(y/\sqrt{2})G(-y/\sqrt{2})d\mu(y).$$

4. Multiple L_p analytic GFFT and the GCP

In this section we will give a definition of the multiple L_p analytic GFFT of a functional on $C_{a,b}[0,T]$ and then establish some relationships between the multiple L_p analytic GFFT and the GCP of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$.

First, we state the definition of the multiple L_p analytic GFFT of a functional F on $C_{a,b}[0,T]$.

DEFINITION 4.1. Let F be a measurable functional defined on $C_{a,b}[0,T]$ and define a transform $(T_\gamma)^{(n)}(F)$ $(\gamma>0)$ of F by

$$(T_\gamma)^{(n)}(F) = \underbrace{(T_\gamma \circ \cdots \circ T_\gamma)}_{n-\text{times}}(F), \tag{4.1}$$

that is, $(T_\gamma)^{(n)}$ means the n-times composition of T_γ, where T_γ is given by (2.9) in Definition 2.2 and n is a nonnegative integer. When λ is in $\mathbb{C}_+$, the transform $(T_\lambda)^{(n)}(F)$ means the analytic extension of $(T_\gamma)^{(n)}(F)$ $(\gamma>0)$ as the function of $\lambda \in \mathbb{C}_+$.

Let $(T_\lambda)^{(n)}(F)$ be an analytic extension of $(T_\gamma)^{(n)}(F)$ as a function of $\lambda \in \mathbb{C}_+$. In case that $1<p\le 2$, for each $q\in\mathbb{R}-\{0\}$, we define the multiple L_p analytic GFFT $(T_q^{(p)})^{(n)}(F)$ of F by

$$(T_q^{(p)})^{(n)}(F) = \text{l.i.m.}_{\lambda\to -iq}(T_\lambda)^{(n)}(F), \tag{4.2}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

In case that $p=1$, for each $q\in\mathbb{R}-\{0\}$, we define the multiple L_1 analytic GFFT $(T_q^{(1)})^{(n)}(F)$ of F by

$$(T_q^{(1)})^{(n)}(F) = \lim_{\lambda\to -iq}(T_\lambda)^{(n)}(F), \tag{4.3}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Note that $(T_\lambda)^{(0)}(F)\equiv F\equiv (T_q^{(p)})^{(0)}(F)$, $(T_\lambda)^{(1)}(F)\equiv T_\lambda(F)$, and $(T_q^{(p)})^{(1)}(F)\equiv T_q^{(p)}(F)$.

We have already shown that for $F\in\mathcal{S}(L^2_{a,b}[0,T])$ with condition (3.4), the L_p GFFT $T_q^{(p)}(F)$ belongs to the Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$. Hence by using the mathematical induction and proceeding as in the proof of Theorem 3.1, we can obtain the following theorem.

THEOREM 4.1. *Let q_0 be a nonzero real number and let n be a nonnegative integer. Let $F\in\mathcal{S}(L^2_{a,b}[0,T])$ be given by* (3.1) *whose associated measure f satisfies the condition*

$$\int_{L^2_{a,b}[0,T]} \exp\left\{\frac{n}{\sqrt{|2q_0|}}\int_0^T |u(s)|d|a|(s)\right\}|df(u)|<\infty. \tag{4.4}$$

Then for all $p \in [1,2]$ *and all real* q *with* $|q| \geq |q_0|$, *the multiple* L_p *analytic GFFT* $(T_q^{(p)})^{(n)}(F)$ *exists and is given by*

$$(4.5)\quad \begin{aligned}&(T_q^{(p)})^{(n)}(F)(y)\\ &= \int_{L^2_{a,b}[0,T]} \exp\bigg\{ i\langle u,y\rangle - \frac{in}{2q}(u^2,b') + in\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(u,a')\bigg\} df(u)\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$. *Furthermore,* $(T_q^{(p)})^{(n)}(F)$ *is an element of* $\mathcal{S}(L^2_{a,b}[0,T])$ *with associated measure*

$$\phi_n(E) = \int_E \exp\bigg\{ - \frac{in}{2q}(u^2,b') + in\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(u,a')\bigg\} df(u)$$

for $E \in \mathcal{B}(L^2_{a,b}[0,T])$.

Note that (4.5) is reduced to (3.5), if we take $n=1$ in (4.5).

Next, we obtain the GCP of the multiple L_p analytic GFFT's of functionals in $\mathcal{S}(L^2_{a,b}[0,T])$.

THEOREM 4.2. *Let* q_0 *be a nonzero real number and let* n *be a non-negative integer. Let* F *and* G *be elements of* $\mathcal{S}(L^2_{a,b}[0,T])$ *whose associated measures* f *and* g *satisfy the condition*

$$(4.6)\quad \int_{L^2_{a,b}[0,T]} \exp\bigg\{\frac{n}{\sqrt{|2q_0|}}\int_0^T |u(s)|d|a|(s)\bigg\}\big[|df(u)| + |dg(v)|\big] < \infty.$$

Then for all $p \in [1,2]$, *all real* q *with* $|q| \geq |q_0|$ *and a nonnegative integer* m, *the GCP* $((T_q^{(p)})^{(n)}(F) * (T_q^{(p)})^{(m)}(G))_q(y)$ *exists and is given by (4.7) below. Furthermore* $((T_q^{(p)})^{(n)}(F) * (T_q^{(p)})^{(m)}(G))_q$ *is an element of* $\mathcal{S}(L^2_{a,b}[0,T])$.

PROOF. By using (4.5) and (3.10) we observe that for all $p \in [1,2]$ and all q with $|q| \geq |q_0|$

$$(4.7)\quad \begin{aligned}&((T_q^{(p)})^{(n)}(F) * (T_q^{(p)})^{(m)}(G))_q(y)\\ &= \int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{in}{2q}(u^2,b') - \frac{im}{2q}(v^2,b')\\ &\qquad + in\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(u,a') + im\Big(\frac{i}{q}\Big)^{\frac{1}{2}}(v,a')\\ &\qquad - \frac{i}{4q}((u-v)^2,b') + i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u-v,a')\bigg\} df(u)dg(v)\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$. Furthermore, proceeding as in the proof of Theorem 3.2 above and using (4.6), we see that $((T_q^{(p)})^{(n)}(F)*(T_q^{(p)})^{(m)}(G))_q$ is an element of $\mathcal{S}(L_{a,b}^2[0,T])$. □

Note that (4.7) is reduced to (3.10), if we take $m = n = 0$ in (3.10).

In our next theorem, we obtain the multiple L_p analytic GFFT of the convolution product for two functionals in $\mathcal{S}(L_{a,b}^2[0,T])$.

THEOREM 4.3. *Let F, G, f, g and q_0 be as in Theorem 4.2. Then for all $p \in [1,2]$ and all real q the following equation with $|q| \geq |q_0|$,*

$$\begin{aligned}(T_q^{(p)})^{(n)}&((F*G)_q)(y)\\ = \int_{L_{a,b}^2[0,T]}\int_{L_{a,b}^2[0,T]} &\exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle\\ &-\frac{i}{4q}((u-v)^2,b')+i\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u-v,a')\\ &-\frac{in}{4q}((u+v)^2,b')+in\Big(\frac{i}{2q}\Big)^{\frac{1}{2}}(u+v,a')\bigg\}df(u)dg(v)\end{aligned} \tag{4.8}$$

holds for s-a.e. $y \in C_{a,b}[0,T]$, where n is a nonnegative integer. Furthermore, $(T_q^{(p)})^{(n)}((F*G)_q)(y)$ is an element of $\mathcal{S}(L_{a,b}^2[0,T])$.

PROOF. By using equations (3.10) and (4.5), we can easily obtain the equation (4.8) above. Moreover, the condition (4.6) will imply the existence of the equation (4.8). □

Finally, we show that the L_p analytic GFFT of the GCP of the multiple L_p analytic GFFT's is a product of the multiple L_p analytic GFFT's of the transforms for functionals in $\mathcal{S}(L_{a,b}^2[0,T])$.

THEOREM 4.4. *Let F, G, f, g, q_0, n and m be as in Theorem 4.2. Then for all $p \in [1,2]$ and all real q the following equation with $|q| \geq |q_0|$,*

$$\begin{aligned}&T_q^{(p)}(((T_q^{(p)})^{(n)}(F)*(T_q^{(p)})^{(m)}(G))_q)(y)\\ &= (T_{q/2}^{(p)})^{(n)}(T_q^{(p)}(F_q^*))(y)(T_{q/2}^{(p)})^{(m)}(T_q^{(p)}(^*G_q))(y)\end{aligned} \tag{4.9}$$

holds for s-a.e. $y \in C_{a,b}[0,T]$, *where* F_q^* *and* ${}^*G_q$ *are as in* (2.13). *Also, both expressions in* (4.9) *are given by the expression*

$$\int_{L^2_{a,b}[0,T]}\int_{L^2_{a,b}[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i(n+1)}{2q}(u^2,b') - \frac{i(m+1)}{2q}(v^2,b') + in\Big(\frac{i}{q}\Big)^{\frac12}(u,a') + im\Big(\frac{i}{q}\Big)^{\frac12}(v,a') + i\sqrt{2}\Big(\frac{i}{q}\Big)^{\frac12}(u,a')\bigg\}df(u)dg(v).$$

Furthermore, the transform $T_q^{(p)}(((T_q^{(p)})^{(n)}(F) * (T_q^{(p)})^{(m)}(G))_q)$ *is an element of* $\mathcal{S}(L^2_{a,b}[0,T])$.

PROOF. By using (4.5), (3.10) and (3.5), we can obtain the equation (4.9) above. □

REMARK 4.1. In Theorem 4.4 above, if $a(t) \equiv 0$, then

$$(T_{q/2}^{(p)})^{(n)}(T_q^{(p)}(F)(\cdot/\sqrt{2}))(y) = (T_q^{(p)})^{(n+1)}(F)(y/\sqrt{2}) \tag{4.10}$$

and

$$(T_{q/2}^{(p)})^{(m)}(T_q^{(p)}(G)(\cdot/\sqrt{2}))(y) = (T_q^{(p)})^{(m+1)}(G)(y/\sqrt{2}). \tag{4.11}$$

Hence by using (3.26), (4.10) and (4.11) we obtain that

$$\begin{aligned}&T_q^{(p)}(((T_q^{(p)})^{(n)}(F) * (T_q^{(p)})^{(m)}(G))_q)(y)\\ &= (T_q^{(p)})^{(n+1)}(F)(y/\sqrt{2})(T_q^{(p)})^{(m+1)}(G)(y/\sqrt{2})\end{aligned}$$

for s-a.e. $y \in C_{a,b}[0,T]$.

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