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ON CONTACT SLANT SUBMANIFOLD OF $L \times_f F$

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ABSTRACT. It is well known that the warped product $L \times_f F$ of a line L and a Kaehler manifold F is an almost contact Riemannian manifold which is characterized by some tensor equations appeared in (1.7) and (1.8). In this paper we determine contact slant submanifolds tangent to the structure vector field of $L \times_f F$.

1. A special almost contact metric structure on the warped product space $L \times_f F$

Let F be a Kaehler manifold and c a nonzero constant. Let $f(t) = ce^t$ be a function on a line L. Then the warped product space $L \times_f F$ admits an almost contact metric structure as follows (for more details, see [7]). Denote by (J, G) the Kaehler structure of F and let $(t, x_1, \cdots, x_{2n})$ be a local coordinate of $L \times_f F$ where t and $(x_1, \cdots, x_{2n})$ are the local coordinates of L and F, respectively. We define a Riemannian metric tensor g, a vector field ξ and a 1-form η as follows.

$$g_{(t,x)} = \begin{pmatrix} 1 & 0 \\ 0 & f^2(t)G_{(x)} \end{pmatrix}, \tag{1.1}$$

$$\xi = d/dt, \quad \eta(X) = g(X, \xi). \tag{1.2}$$

We also define a $(1,1)$-tensor field ϕ by

$$\phi_{(t,x)} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\phi}_{(t,x)} \end{pmatrix}, \tag{1.3}$$

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where

$$\tilde{\phi}_{(t,x)} = (exp(t\xi))_* J_x (exp(-t\xi))_*.$$

Then we can easily verify that the aggregate (ϕ, ξ, η, g) defines an almost contact metric structure on $L \times_f F$ in the sense that satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1 \tag{1.4}$$

$$\phi^2 X = -X + \eta(X)\xi, \qquad g(\xi, X) = \eta(X) \tag{1.5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1.6}$$

for any vector fields X and Y on $L \times_f F$. Moreover, in this case (ϕ, ξ, η, g) satisfies

$$(\widetilde{\nabla}_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{1.7}$$

$$\widetilde{\nabla}_X \xi = X - \eta(X)\xi, \tag{1.8}$$

where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. In the rest of the paper an almost contact metric structure satisfying (1.7) and (1.8) will be called *Kenmotsu structure*. We notice that Kenmotsu structure is normal but not Sasakian in the sense of [1, 9] and especially is not compact because of (1.8). Moreover, in order that Kenmotsu structure has constant ϕ-holomorphic sectional curvature c it is necessary and sufficient that its curvature tensor $\widetilde{R}$ satisfies

$$\begin{aligned}\widetilde{R}(X,Y)Z =& \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &+ g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\}\end{aligned}$$

for any vector fields X, Y, Z ([7]).

2. Fundamental properties

In this section, we recall some basic formulas and definitions about slant submanifolds in both Complex and Contact Geometry, which we shall use later. For details and background on complex and contact manifolds, we refer to the standard references ([1, 10]).

A submanifold N of an almost Hermitian manifold $(\widetilde{N}, g, J)$ is said to be *slant* ([4]) if for each nonzero vector X tangent to N at p, the angle $\theta(X)$, $0 \leq \theta(X) \leq \frac{\pi}{2}$, between JX and T_pX is a constant, called the *slant angle* of the submanifold. In particular, holomorphic and totally real submanifolds appear as slant submanifolds with slant angle 0 and $\frac{\pi}{2}$, respectively. A slant submanifold is called *proper slant* if it is neither holomorphic nor totally real. In the case where N is a Riemannian surface and $\widetilde{N}$ is a Kaehler manifold, S.S.Chern and J.G.Wolfson introduced the notation of *Kaehler angle*, defined to be the angle between $J\partial/\partial x$ and $\partial/\partial y$, where $z = x + \sqrt{-1}y$ is a local complex coordinate on N ([5]). It is clear that if N is a surface with constant angle α, then it is a slant submanifold with slant angle θ satisfying $\theta = \alpha$ (resp.$\theta = \pi - \alpha$) when $\alpha \in [0, \frac{\pi}{2}]$ (resp. $\alpha \in (\frac{\pi}{2}, \pi]$).

Now, for any tangent vector field X we put $JX = PX + FX$, where PX (resp. FX)denotes the tangential (resp. normal) component of JX. Then, $\theta-$slant submanifolds are characterized by the formula :

$$P^2 = -\cos^2\theta \; Id$$

([4]). A special type of proper slant submanifold is that of *Kaehlerian slant* submanifold, i.e., a proper slant submanifold satisfying $\nabla P = 0$, where ∇ denotes the induced connection on N. It is easy to show that a Kaehlerian slant submanifold is a Kaehlerian manifold with respect to the induced metric and with the almost complex structure given by $(\sec\theta)P$.

In a similar way, given a submanifold M tangent to the structure vector field ξ of an almost contact metric manifold $(\widetilde{M}, \phi, \xi, \eta, g)$, it is said to be slant ([2, 3]) if the angle $\theta(X)$ between ϕX and T_pM is a constant, which is independent of the choice of $p \in M$ and $X \in T_p(M)\setminus$ Span(ξ_p). In particular, for $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$) we obtain the invariant (resp. anti−invariant)submanifolds. Now, if we denote by PX (resp. FX) the tangential (resp. normal) component of ϕX, there is an equation which characterizes $\theta-$slant submanifolds :

$$P^2 = -\cos^2\theta(Id - \eta \otimes \xi). \tag{2.1}$$

Especially if a $\theta-$slant submanifold satisfies

$$(\nabla_Y P)X = \cos^2\theta\{-\eta(X)PY - g(Y, PX)\xi\}, \tag{2.2}$$

for any tangent vector fields X and Y, then the submanifold is called a *Kenmotsu slant submanifold*, where ∇ denotes the induced connection on the submanifold.

3. Main result

Let M be a submanifold of a Kenmotsu manifold $\widetilde{M}$ in which the structure vector field ξ is tangent to M. Denoting by ∇ and $\nabla^{\perp}$ the induced connections on M and the normal bundle $T^{\perp}M$ of M respectively, we have the equations of Gauss and Weingarten

$$\widetilde{\nabla}_Y X = \nabla_Y X + h(Y, x), \tag{3.1}$$

$$\widetilde{\nabla}_Y N = -A_N Y + \nabla^{\perp}_Y N \tag{3.2}$$

for tangent vector fields X, Y and normal vector field N to M, where h and A_N denote the second fundamental form and the shape operator in the direction of N which are related by

$$g(h(X,Y), N) = g(A_N X, Y). \tag{3.3}$$

It is clear that (1.8) and (3.3) yield

$$A_N \xi = 0 \tag{3.4}$$

for any normal vector field N to M since the structure vector field ξ is tangent to M. If the Kenmotsu manifold $\widetilde{M}$ has constant ϕ-holomorphic sectional curvature c, then the equation of Gauss for M is given by

$$\begin{aligned} & g(R(X,Y)Z,W) \\ = & \frac{c-3}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ + & \frac{c+1}{4}\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ + & g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ + & g(X,\phi Z)g(\phi Y,W) - g(Y,\phi Z)g(\phi X,W) \\ + & 2g(X,\phi Y)g(\phi Z,W)\} + g(h(Y,Z),h(X,W)) \\ - & g(h(X,Z),h(Y,W)) \end{aligned} \tag{3.5}$$

for tangent vector fields X, Y, Z, W to M.

For a tangent vector field X and normal vector field N to M, we put

$$\phi X = PX + FX, \quad \phi N = tN + t^{\perp}N. \tag{3.6}$$

Then we can easily see that P and $t^{\perp}$ are skew-symmetric endomorphisms acting on T_pM and $T_p^{\perp}M$, respectively. From (3.1), (3.2) and (3.6), it follows easily that

$$(\widetilde{\nabla}_Y \phi)X + \phi(\widetilde{\nabla}_Y X) = \nabla_Y(PX) + h(Y, PX).$$

From this

$$\begin{aligned} &- \eta(X)\phi Y - g(Y, \phi X)\xi + \phi(\nabla_Y X + h(Y, X)) \\ &= (\nabla_Y P)X + P\nabla_Y X + h(Y, PX) - A_{FX}Y + D_Y(FX), \end{aligned}$$

or equivalently,

$$\begin{aligned} &- \eta(X)\{PY + FY\} - g(Y, \phi X)\xi + P\nabla_Y X \\ &+ F\nabla_Y X + th(Y, X) + t^{\perp}h(Y, X) \\ = &(\nabla_Y P)X + P\nabla_Y X + h(Y, PX) - A_{FX}Y + (\nabla_Y F)X + F\nabla_Y A. \end{aligned}$$

Thus we have the following formulas

$$(\nabla_Y P)X = -\eta(X)PY - g(Y, PX)\xi + th(Y, X) + A_{FX}Y, \tag{3.7}$$

$$(\nabla_Y F)X = -\eta(X)FY + t^{\perp}h(Y, X) - h(Y, PX). \tag{3.8}$$

From now on we assume that M is a Kenmotsu slant submanifold tangent to the structure vector field ξ . Then, from (2.2) and (3.7), we have

$$(1 - \cos^2\theta)\{\eta(X)PY + g(Y, PX)\xi\} = th(Y, X) + A_{FX}Y. \tag{3.9}$$

Putting $X = \xi$ in (3.9) and using $P\xi = F\xi = 0$ and (3.4), we obtain

$$(1 - \cos^2\theta)PY = 0, \tag{3.10}$$

for any tangent vector field Y. Applying P to (3.10) and making use of (2.1), we can easily see that $\cos^2\theta(1 - cos^2\theta) = 0$, provided dim $M \geq 2$.

Thus we have

THEOREM 1. *There exist no proper Kenmotsu slant submanifolds of the warped product* $L \times_f F$.

THEOREM 2. *A Kenmotsu slant submanifolds of the warped product* $L \times_f F$ *is anti-invariant or locally a warped product* $(-\epsilon, \epsilon) \times_f V$, *where* V *is a Kaehler manifold.*

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