Commun. Korean Math. Soc. 19 (2004), No. 1, pp. 179–196

MULTIOBJECTIVE VARIATIONAL PROGRAMMING UNDER GENERALIZED VECTOR VARIATIONAL TYPE I INVEXITY

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ABSTRACT. Mond-Weir type duals for multiobjective variational problems are formulated. Under generalized vector variational type I invexity assumptions on the functions involved, sufficient optimality conditions, weak and strong duality theorems are proved efficient and properly efficient solutions of the primal and dual problems.

1. Introduction and preliminaries

Multiobjective variational programming problem arises when more than one objective functional is to be optimized over a feasible region. There are three kinds of solutions for such problem, that is, properly efficient solutions, efficient solutions and weakly efficient solutions. An important factor in the development of variational problems was the investigation of a number of mechanical and physical problems.

In 1992, Bector and Husain [1] first applied duality method of ordinary multiobjective optimization problem to multiobjective variational problem, and obtained duality results for properly efficient solution under convexity assumptions on involved functions. Since then, many authors ([2], [7] - [12]) have studied optimality conditions and duality theorems for multiobjective variational problems under generalized convexity assumptions on involved functions.

Very recently, Hanson et al. [4] defined vector type I invexity, along the lines of Hanson [3], and Jeyakumar and Mond [5] extending the

Received December 10, 2002.
2000 Mathematics Subject Classification: 90C29, 90C46.
Key words and phrases: efficient solutions, properly efficient solutions, vector variational type I invex problem, generalized vector variational type I invex problem, optimality, duality.
This work was supported by Korea Research Foundation Grant. (KRF-2002-037-C00006).

pseudo, quasi, quasi-pseudo, pseudo-quasi vector type I invexity of Kaul et al. [6], and obtained several sufficient optimality conditions and duality results for ordinary multiobjective optimization problem under the just mentioned vector type invexity assumptions.

The purpose of this paper is to extend the optimality and duality results Hanson et al. [4] to multiobjective variational problems. We introduce variational versions of several concepts of vector type I invexity considered by Kaul et al. [6], and establish sufficient optimality conditions and duality results for efficient or properly efficient solutions of a multiobjective variational problem under the just mentioned vector variational type I invexity assumptions.

Throughout this paper, we will use the following notations.

Let $I = [a, b]$ be a real interval: let $f := (f^1, \cdots, f^p) : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p$, $g := (g^1, \cdots, g^m) : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$, where $x : I \to \mathbb{R}^n$ is differentiable with derivative $\dot{x}$, denote the partial derivatives of f by

$$f_x^i = \left[\frac{\partial f^i}{\partial x_1}, \cdots, \frac{\partial f^i}{\partial x_n}\right], \quad f_{\dot{x}}^i = \left[\frac{\partial f^i}{\partial \dot{x}_1}, \cdots, \frac{\partial f^i}{\partial \dot{x}_n}\right], \quad i = 1, \cdots, p.$$

Let $C(I, \mathbb{R}^m)$ denotes the space of continuous functions $\phi : I \to \mathbb{R}^m$, with the uniform norm; $C_+(I, \mathbb{R}^m)$ is the cone of nonnegative functions in $C(I, \mathbb{R}^m)$. Denote by X the space of piecewise smooth functions $x : I \to \mathbb{R}^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \iff x(t) = \alpha + \int_a^t u(s)ds,$$

where α is a given boundary value: thus $D = d/dt$ except at discontinuities.

Our problem is the multiobjective variational problem (VP) defined as follows;

(VP)

$$\text{Minimize} \quad \int_a^b f(t, x, \dot{x})dt = \left(\int_a^b f^1(t, x, \dot{x})dt, \cdots, \int_a^b f^p(t, x, \dot{x})dt\right)$$

subject to $x(a) = \alpha, \ x(b) = \beta,$

$$g^j(t, x, \dot{x}) \leqq 0, \ t \in I, \ \forall j = 1, \cdots, m.$$

Let X_0 be the set of feasible solutions for (VP), that is,

$$X_0 := \{x \in X \mid g(t, x, \dot{x}) \leqq 0, \ \forall t \in I\}.$$

The following definitions will be needed in the sequel.

DEFINITION 1.1. (1) A point $x^* \in X_0$ is said to be an efficient solution of (VP) if there is no other feasible point $x \in X_0$ such that

$$\int_a^b f^i(t,x,\dot{x})dt \leqq \int_a^b f^i(t,x^*,\dot{x}^*)dt \quad \text{for all} \quad i \in \{1,\cdots,p\}$$

and

$$\int_a^b f^j(t,x,\dot{x})dt < \int_a^b f^j(t,x^*,\dot{x}^*)dt \quad \text{for some} \quad j \in \{1,\cdots,p\}.$$

(2) A point $x^* \in X_0$ is said to be a properly efficient solution of (VP) if it is an efficient for (VP) and if there exists a scalar $M > 0$ such that for all $i \in \{1,\cdots,p\}$,

$$\begin{aligned}&\int_a^b f^i(t,x^*,\dot{x}^*)dt - \int_a^b f^i(t,x,\dot{x})dt \\ &\leqq M\left(\int_a^b f^j(t,x,\dot{x})dt - \int_a^b f^j(t,x^*,\dot{x}^*)dt\right),\end{aligned}$$

for some j such that

$$\int_a^b f^j(t,x,\dot{x})dt > \int_a^b f^j(t,x^*,\dot{x}^*)dt$$

whenever $x \in X_0$ and

$$\int_a^b f^i(t,x,\dot{x})dt < \int_a^b f^i(t,x^*,\dot{x}^*)dt.$$

Now we give variational versions of vector type I problems of several concepts of generalized vector type I invexity considered by Kaul et al. [6].

DEFINITION 1.2. The problem (VP) is vector variational type I invex at u and $\dot{u}$ with respect to η if $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that $\forall i \in \{1,\cdots,p\},\ j \in \{1,\cdots,m\}$, we have

$$\begin{aligned}&\int_a^b f^i(t,x,\dot{x})dt - \int_a^b f^i(t,u,\dot{u})dt \\ &\geqq \int_a^b \eta(t,x,\dot{x},u,\dot{u})^T \left[f_x^i(t,u,\dot{u}) - \frac{d}{dt}f_{\dot{x}}^i(t,u,\dot{u})\right] dt\end{aligned} \tag{1}$$

and

$$-\int_a^b g^j(t,u,\dot{u})dt \geqq \int_a^b \eta(t,x,\dot{x},u,\dot{u})^T \left[g_x^j(t,u,\dot{u}) - \frac{d}{dt}g_{\dot{x}}^j(t,u,\dot{u})\right] dt$$

for all $x, u \in X_0$ with $(\dot{x}, \dot{u})$ piecewise smooth on I.

If strict inequality holds in (1) (whenever $x \neq u$) we say that (VP) is of semi strictly vector variational type I invex at u and $\dot{u}$ with respect to η.

DEFINITION 1.3. (1) The problem (VP) is pseudo vector variational type I invex at u and $\dot{u}$ with respect to η if $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, then there exists $\tau \in \mathbb{R}^p_+$ and piecewise smooth $y : I \to \mathbb{R}^m_+$ such that $\forall i \in \{1, \cdots, p\}$, $j \in \{1, \cdots, m\}$, we have

$$\tag{2} \begin{aligned} &\sum_{i=1}^{p} \tau_i \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[f_x^i(t, u, \dot{u}) - \frac{d}{dt} f_{\dot{x}}^i(t, u, \dot{u}) \right] dt \geqq 0 \\ &\Longrightarrow \sum_{i=1}^{p} \tau_i \left[\int_a^b f^i(t, x, \dot{x}) dt - \int_a^b f^i(t, u, \dot{u}) dt \right] \geqq 0 \end{aligned}$$

and

$$\tag{3} \begin{aligned} &\sum_{j=1}^{m} \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[y_j(t) g_x^j(t, u, \dot{u}) - \frac{d}{dt} y_j(t) g_{\dot{x}}^j(t, u, \dot{u}) \right] dt \geqq 0 \\ &\Longrightarrow \sum_{j=1}^{m} \int_a^b y_j(t) g^j(t, u, \dot{u}) dt \leqq 0. \end{aligned}$$

If the second inequalities in (2) and (3) are both strict we say that (VP) is strictly pseudo vector variational type I invex at u and $\dot{u}$ with respect to η.

(2) The problem (VP) is quasi pseudo vector variational type I invex at u and $\dot{u}$ with respect to η if $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, then there exists $\tau \in \mathbb{R}^p_+$ and piecewise smooth $y : I \to \mathbb{R}^m_+$ such that $\forall i \in \{1, \cdots, p\}$, $j \in \{1, \cdots, m\}$, we have

$$\begin{aligned} &\sum_{i=1}^{p} \tau_i \left[\int_a^b f^i(t, x, \dot{x}) dt - \int_a^b f^i(t, u, \dot{u}) dt \right] \leqq 0 \\ &\Longrightarrow \sum_{i=1}^{p} \tau_i \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[f_x^i(t, u, \dot{u}) - \frac{d}{dt} f_{\dot{x}}^i(t, u, \dot{u}) \right] dt \leqq 0 \end{aligned}$$

and

$$\tag{4} \begin{aligned} &\sum_{j=1}^{m} \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[y_j(t) g_x^j(t, u, \dot{u}) - \frac{d}{dt} y_j(t) g_{\dot{x}}^j(t, u, \dot{u}) \right] dt \geqq 0 \\ &\Longrightarrow \sum_{j=1}^{m} \int_a^b y_j(t) g^j(t, u, \dot{u}) dt \leqq 0. \end{aligned}$$

If the second inequality holds in (4) is strict, we say that (VP) is quasi strictly pseudo vector variational type I invex at u and $\dot{u}$ with respect to η.

(3) The problem (VP) is pseudo quasi vector variational type I invex at u and $\dot{u}$ with respect to η if $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, then there exists $\tau \in \mathbb{R}^p_+$ and piecewise smooth $y : I \to \mathbb{R}^m_+$ such that $\forall i \in \{1, \cdots, p\}$, $j \in \{1, \cdots, m\}$, we have

$$\begin{aligned}(5) \qquad &\sum_{i=1}^{p} \tau_i \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[f_x^i(t, u, \dot{u}) - \frac{d}{dt} f_{\dot{x}}^i(t, u, \dot{u}) \right] dt \geqq 0 \\ &\Longrightarrow \sum_{i=1}^{p} \tau_i \left[\int_a^b f^i(t, x, \dot{x}) dt - \int_a^b f^i(t, u, \dot{u}) dt \right] \geqq 0\end{aligned}$$

and

$$\sum_{j=1}^{m} \int_a^b y_j(t) g^j(t, u, \dot{u}) dt \geqq 0$$

$$\Longrightarrow \sum_{j=1}^{m} \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[y_j(t) g_x^j(t, u, \dot{u}) - \frac{d}{dt} y_j(t) g_{\dot{x}}^j(t, u, \dot{u}) \right] dt \leqq 0.$$

If the second inequality holds in (5) is strict, we say that (VP) is strictly pseudo quasi vector variational type I invex at u and $\dot{u}$ with respect to η.

Now we give an example for Definition 1.2.

EXAMPLE 1.1. Consider the following multiobjective variational problem:

$$\begin{aligned}\text{(VP)} \quad \text{Minimize} \quad & \left(\int_a^b f^1(t, x, \dot{x}) dt, \int_a^b f^2(t, x, \dot{x}) dt \right) \\ & := \left(\int_0^1 (x_1^3(t) + x_2(t)) dt, \int_0^1 (x_1(t) + x_2^3(t)) dt \right) \\ \text{subject to} \quad & g^1(t, x, \dot{x}) := 1 - x_1(t) \leqq 0, \\ & g^2(t, x, \dot{x}) := 1 - x_2(t) \leqq 0.\end{aligned}$$

Clearly, $\int_a^b f^1(t, x, \dot{x}) dt$ and $\int_a^b f^2(t, x, \dot{x}) dt$ are not convex at u and $\dot{u}$.

When $9u_1^2 u_2^2 - 1 = 0$, we let $\eta_1(t, x, \dot{x}, u, \dot{u}) = \min\{\eta_1^*(t, x, \dot{x}, u, \dot{u}), -u_1\}$, $\eta_2(t, x, \dot{x}, u, \dot{u}) = \min\{\eta_2^*(t, x, \dot{x}, u, \dot{u}), -u_2\}$ and $\eta(t, x, \dot{x}, u, \dot{u}) = \{\eta_1(t, x, \dot{x}, u, \dot{u}), \eta_2(t, x, \dot{x}, u, \dot{u})\}$, where $\eta_1^*(t, x, \dot{x}, u, \dot{u}) = \frac{1}{3u_1^2} \min\{x_1^3 + x_2 - u_1^3 - u_2, 3u_1^2(x_1 + x_2^3 - u_1 - u_2^3)\}$ and $\eta_2^* = 0$.

When $9u_1^2u_2^2 - 1 \neq 0$, we let

$$\begin{aligned}&\eta_1(t,x,\dot{x},u,\dot{u})\\ =\ &\min\{\frac{3x_1^3u_2^2 - x_1 + 3x_2u_2^2 - x_2^3 - 3u_1^3u_2^2 + u_1 - 2u_2^3}{9u_1^2u_2^2 - 1}, x_1 - u_1\},\end{aligned}$$

$$\begin{aligned}&\eta_2(t,x,\dot{x},u,\dot{u})\\ =\ &\min\{\frac{3x_2^3u_1^2 + 3x_1u_1^2 - 3u_1^2u_2^3 - 2u_1^3 - x_2 - x_1^3 + u_2}{9u_1^2u_2^2 - 1}, x_2 - u_2\},\end{aligned}$$

and

$$\eta(t,x,\dot{x},u,\dot{u}) = \{\eta_1(t,x,\dot{x},u,\dot{u}), \eta_2(t,x,\dot{x},u,\dot{u})\}.$$

Then we can easily check the following:

$$\begin{aligned}&\int_0^1 f^i(t,x,\dot{x})dt - \int_0^1 f^i(t,u,\dot{u})dt\\ \geqq\ &\int_0^1 \eta(t,x,\dot{x},u,\dot{u})^T\left[f_x^i(t,u,\dot{u}) - \frac{d}{dt}f_{\dot{x}}^i(t,u,\dot{u})\right]dt\end{aligned}$$

for $i = 1, 2$,

$$-\int_0^1 g^j(t,u,\dot{u})dt \geqq \int_0^1 \eta(t,x,\dot{x},u,\dot{u})^T\left[g_x^j(t,u,\dot{u}) - \frac{d}{dt}g_{\dot{x}}^j(t,u,\dot{u})\right]dt$$

for $j = 1, 2$. Then (VP) is vector variational type I invex at u and $\dot{u}$ with respect to η.

2. Sufficient optimality conditions

In this section, we present sufficient Kuhn-Tucker type optimality conditions for the problem (VP).

THEOREM 2.1. *Suppose that*
(i) $\bar{x} \in X_0$;

(ii) *there exist* $\tau^0 \in \mathbb{R}^p_+$ *and piecewise smooth* $y^0 : I \to \mathbb{R}^m_+$ *such that*

$$(a)\quad \sum_{i=1}^{p} \tau_i^0 \left[f_x^i(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right] + \sum_{j=1}^{m} \left[y_j^0(t) g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} y_j^0(t) g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}}) \right] = 0 \text{ a.e. on } I,$$

$$(b)\quad \sum_{j=1}^{m} y_j^0(t) g^j(t,\bar{x},\dot{\bar{x}}) = 0;$$

(iii) *the problem* (VP) *is quasi strictly pseudo vector variational type I invex with respect to* τ^0, $y^0(\cdot)$ *and* η.

Then $\bar{x}$ *is an efficient solution for* (VP).

PROOF. Suppose $\bar{x}$ is not an efficient solution of (VP). Then there exists $x \in X_0$ such that $\int_a^b f(t,x,\dot{x})dt \leqq \int_a^b f(t,\bar{x},\dot{\bar{x}})dt$ and $\int_a^b f(t,x,\dot{x})dt \neq \int_a^b f(t,\bar{x},\dot{\bar{x}})dt$.

This implies that

$$\sum_{i=1}^{p} \tau_i^0 \left[\int_a^b f^i(t,x,\dot{x})dt - \int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt \right] \leqq 0.$$

From the above inequality and the hypothesis (iii), it follows that

$$(6)\quad \sum_{i=1}^{p} \tau_i^0 \int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T \left[f_x^i(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right] dt \leqq 0.$$

By the inequality (6) and hypothesis (ii)(a) we have

$$\sum_{j=1}^{m} \int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T \left[y_j^0(t) g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} y_j^0(t) g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}}) \right] dt \geqq 0.$$

From the above inequality and hypothesis (iii), it follows that

$$(7)\quad \sum_{j=1}^{m} \int_a^b y_j^0(t) g^j(t,\bar{x},\dot{\bar{x}})dt < 0.$$

Now by hypothesis (i) and (ii)(b), it follows that

$$\int_a^b y_j^0(t) g^j(t,\bar{x},\dot{\bar{x}})dt = 0,$$

for every j, which further implies that

$$\sum_{j=1}^{m}\int_a^b y_j^0(t)g^j(t,\bar{x},\dot{\bar{x}})dt = 0.$$

The last equation contradicts the inequality (7) and hence the conclusion follows. □

THEOREM 2.2. *Suppose that*

(i) $\bar{x} \in X_0$;

(ii) *there exist* $\tau^0 \in int\mathbb{R}_+^p$ *and piecewise smooth* $y^0 : I \to \mathbb{R}_+^m$ *continuous on* I, *such that*

$$(a)\quad \sum_{i=1}^{p}\tau_i^0\left[f_x^i(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt}f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})\right]$$
$$+\sum_{j=1}^{m}\left[y_j^0(t)g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt}y_j^0(t)g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}})\right] = 0, \text{ a.e. on } I,$$

$$(b)\quad \sum_{j=1}^{m}y_j^0(t)g^j(t,\bar{x},\dot{\bar{x}}) = 0;$$

(iii) *the problem* (VP) *is pseudo quasi vector variational type I invex with respect to* τ^0, $y^0(\cdot)$ *and* η.

Then $\bar{x}$ *is a properly efficient solution for* (VP).

PROOF. Suppose that $\bar{x}$ is not an efficient solution of (VP). Then there exists $x \in X_0$ such that

$$\int_a^b f(t,x,\dot{x})dt \leqq \int_a^b f(t,\bar{x},\dot{\bar{x}})dt$$

and

$$\int_a^b f(t,x,\dot{x})dt \neq \int_a^b f(t,\bar{x},\dot{\bar{x}})dt.$$

This implies that

$$(8)\qquad \sum_{i=1}^{p}\tau_i^0\left[\int_a^b f^i(t,x,\dot{x})dt - \int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt\right] < 0.$$

By the hypotheses (i) and (ii)(b), we have

$$\sum_{j=1}^{m}\int_a^b y_j^0(t)g^j(t,\bar{x},\dot{\bar{x}})dt = 0.$$

From the above equality and the hypothesis (iii), it follows that

$$(9)\quad \sum_{j=1}^{m}\int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T\left[y_j^0(t)g_x^j(t,\bar{x},\dot{\bar{x}})-\frac{d}{dt}y_j^0(t)g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}})\right]dt \leqq 0.$$

Now by (9) and the hypothesis (ii)(a), we have

$$(10)\qquad \sum_{i=1}^{p}\tau_i^0\int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T\left[f_x^i(t,\bar{x},\dot{\bar{x}})-\frac{d}{dt}f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})\right]dt \geqq 0.$$

Finally, by (10) and the hypothesis (iii), we have for all $x \in X_0$,

$$(11)\qquad \sum_{i=1}^{p}\tau_i^0\left[\int_a^b f^i(t,x,\dot{x})dt-\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt\right] \geqq 0.$$

Since (11) contradict (8), we have the conclusion that $\bar{x}$ is an efficient solution of (VP).

We assume that $p \geqq 2$. Next let

$$M=(p-1)\max_{i,j}\frac{\tau_j^0}{\tau_i^0},\ i\neq j:\ 1\leqq i,j\leqq p.$$

Suppose $\bar{x}$ is not a properly efficient for (VP). Then there exists $x \in X_0$ such that for some i with $\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt > \int_a^b f^i(t,x,\dot{x})dt$,

$$(12)\qquad \begin{aligned}&\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt-\int_a^b f^i(t,x,\dot{x})dt\\ &> M\left[\int_a^b f^j(t,x,\dot{x})dt-\int_a^b f^j(t,\bar{x},\dot{\bar{x}})dt\right]\\ &\qquad \forall j \text{ such that } \int_a^b f^j(t,x,\dot{x})dt > \int_a^b f^j(t,\bar{x},\dot{\bar{x}})dt.\end{aligned}$$

From (12) it follows that

$$\begin{aligned}&\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt-\int_a^b f^i(t,x,\dot{x})dt\\ &> (p-1)\frac{\tau_j^0}{\tau_i^0}\left[\int_a^b f^j(t,x,\dot{x})dt-\int_a^b f^j(t,\bar{x},\dot{\bar{x}})dt\right],\forall j\neq i.\end{aligned}$$

This implies that

$$\frac{\tau_i^0}{p-1}\left[\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt - \int_a^b f^i(t,x,\dot{x})dt\right]$$

$$> \tau_j^0\left[\int_a^b f^j(t,x,\dot{x})dt - \int_a^b f^j(t,\bar{x},\dot{\bar{x}})dt\right]. \tag{13}$$

Summing (13) with respect to $j(\neq i)$, we have that

$$\tau_i^0\left[\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt - \int_a^b f^i(t,x,\dot{x})dt\right]$$

$$> \sum_{j\neq i}\tau_j^0\left[\int_a^b f^j(t,x,\dot{x})dt - \int_a^b f^j(t,\bar{x},\dot{\bar{x}})dt\right].$$

Since $\sum_j \tau_j^0\left[\int_a^b f^j(t,x,\dot{x})dt - \int_a^b f^j(t,\bar{x},\dot{\bar{x}})dt\right] < 0$, contradicts (11) and hence $\bar{x}$ is a properly efficient solution for (VP). □

THEOREM 2.3. *Suppose that*

(i) $\bar{x} \in X_0$;

(ii) *there exist* $\tau^0 \in \mathbb{R}_+^p$ *and piecewise smooth* $y^0 : I \to \mathbb{R}_+^m$ *such that*

$$(a)\quad \sum_{i=1}^{p}\tau_i^0\left[f_x^i(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt}f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}})\right]$$

$$+\sum_{j=1}^{m}\left[y_j^0(t)g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt}y_j^0(t)g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}})\right] = 0 \text{ a.e. on } I,$$

$$(b)\quad \sum_{j=1}^{m} y_j^0(t)g^j(t,\bar{x},\dot{\bar{x}}) = 0;$$

(iii) *the problem* (VP) *is semi strictly quasi vector variational type I invex with respect to* τ^0, $y^0(\cdot)$ *and* η.

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PROOF. Suppose that $\bar{x}$ is not an efficient solution of (VP). Then there exists $x \in X_0$ such that

$$\int_a^b f(t,x,\dot{x})dt \leqq \int_a^b f(t,\bar{x},\dot{\bar{x}})dt$$

and

$$\int_a^b f(t,x,\dot{x})dt \neq \int_a^b f(t,\bar{x},\dot{\bar{x}})dt.$$

This implies that

$$(14) \qquad \sum_{i=1}^{p} \tau_i^0 \left[\int_a^b f^i(t,x,\dot{x})dt - \int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt \right] \leqq 0.$$

From inequality (14) and the hypothesis (iii), it follows that

$$(15) \qquad \sum_{i=1}^{p} \tau_i^0 \int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T \left[f_x^i(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right] dt < 0.$$

By the inequality (15) and the hypotheses (ii)(b), (iii) imply that
(16)

$$\sum_{j=1}^{m} \int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T \left[y_j^0(t) g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} y_j^0(t) g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}}) \right] dt \leqq 0.$$

Adding (15) and (16) we see that the hypothesis (ii)(a) is contradicted. Hence the conclusion follows. □

THEOREM 2.4. *Suppose that*

(i) $\bar{x} \in X_0$;

(ii) *there exist* $\tau^0 \in int\mathbb{R}_+^p$ *and piecewise smooth* $y^0 : I \to \mathbb{R}_+^m$ *such that*

$$(a) \quad \sum_{i=1}^{p} \tau_i^0 \left[f_x^i(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right] + \sum_{j=1}^{m} \left[y_j^0(t) g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} y_j^0(t) g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}}) \right] = 0 \text{ a.e. on } I,$$

$$(b) \quad \sum_{j=1}^{m} y_j^0(t) g^j(t,\bar{x},\dot{\bar{x}}) = 0;$$

(iii) *the problem* (VP) *is strictly pseudo vector variational type I invex with respect to* τ^0, $y^0(\cdot)$ *and* η.

Then $\bar{x}$ *is a properly efficient solution of* (VP).

PROOF. By hypothesis (ii)(b) it follows that

$$\sum_{j=1}^{m} \int_a^b y_j^0(t) g^j(t,\bar{x},\dot{\bar{x}})dt = 0,$$

which implies by the hypothesis (iii) that

$$\sum_{j=1}^{m} \int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T \left[y_j^0(t) g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} y_j^0(t) g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}}) \right] dt < 0,$$

which in turn implies by the hypothesis (ii)(a) that

$$(17) \qquad \sum_{i=1}^{p} \tau_i^0 \int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T \left[f_x^i(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right] dt > 0.$$

Now from (17) and hypothesis (iii), we have

$$(18) \qquad \sum_{i=1}^{p} \tau_i^0 \left[\int_a^b f^i(t,x,\dot{x})dt - \int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt \right] > 0.$$

Next if $\bar{x}$ is not an efficient solution of (VP), then there exists $x \in X_0$ such that

$$\int_a^b f(t,x,\dot{x})dt \leqq \int_a^b f(t,\bar{x},\dot{\bar{x}})dt$$

and

$$\int_a^b f(t,x,\dot{x})dt \neq \int_a^b f(t,\bar{x},\dot{\bar{x}})dt$$

which implies that

$$(19) \qquad \sum_{i=1}^{p} \tau_i^0 \left[\int_a^b f^i(t,x,\dot{x})dt - \int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt \right] \leqq 0.$$

Since (18) and (19) contradict each other, the conclusion follows.

To establish the proper efficiency of $\bar{x}$ of (VP), we follow the same argument as in the proof of Theorem 2.2 except in the end we appeal to the inequality (18) for a contradiction. □

3. Duality theory

In this section, we prove weak and strong duality theorems for (VP). We first introduce a necessary optimality condition for (VP), which will be used for proving strong duality theorems.

THEOREM 3.1. [1] **(Necessity)** *Suppose that*

(i) $\bar{x}$ *is a properly efficient for* (VP);

(ii) *there exists* $x^* \in X_0$ *with* $g^i(t,x^*,\dot{x}^*) < 0$ *where* $i \in I(\bar{x}) := \{i \mid g^i(t,\bar{x},\dot{\bar{x}}) = 0\}$ *such that*

$$-\int_a^b g^j(t,\bar{x},\dot{\bar{x}})dt > \int_a^b \eta(t,x,\dot{x},\bar{x},\dot{\bar{x}})^T \cdot \left[g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}}) \right] dt, \ \forall x \in X_0.$$

Then there exist $\tau^0 \in int\mathbb{R}^p_+$ and $y^0 \in \mathbb{R}^m_+$ with τ^0, y^0 not all zero, satisfying such that

$$\sum_{i=1}^{p} \tau_i^0 \left[f_x^i(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right] + \sum_{j=1}^{m} \left[y_j^0(t) g_x^j(t,\bar{x},\dot{\bar{x}}) - \frac{d}{dt} y_j^0(t) g_{\dot{x}}^j(t,\bar{x},\dot{\bar{x}}) \right] = 0 \text{ a.e. on } I.$$

By using a necessary optimality conditions for (VP), we now define the following multiobjective maximization variational problem as the Mond-Weir type dual (VD) of (VP):

$$\text{(VD)} \quad \text{Maximize} \quad \int_a^b f(t,u,\dot{u})dt = \left(\int_a^b f^1(t,u,\dot{u})dt, \cdots, \int_a^b f^p(t,u,\dot{u})dt \right)$$

$$\text{subject to} \quad \sum_{i=1}^{p} \tau_i \left[f_x^i(t,u,\dot{u}) - \frac{d}{dt} f_{\dot{x}}^i(t,u,\dot{u}) \right] + \sum_{j=1}^{m} \left[y_j(t) g_x^j(t,u,\dot{u}) - \frac{d}{dt} y_j(t) g_{\dot{x}}^j(t,u,\dot{u}) \right] = 0,$$

$$\sum_{j=1}^{m} y_j(t) g^j(t,u,\dot{u}) = 0,$$

$$\tau_i \geqq 0, \ \sum_{i=1}^{p} \tau_i = 1, \ y(t) \geqq 0, \ t \in I.$$

Let Y^0 be the set of feasible solutions of problem (VD):

$$Y^0 = \left\{ (u,\tau,y) : \sum_{i=1}^{p} \tau_i \left[f_x^i(t,u,\dot{u}) - \frac{d}{dt} f_{\dot{x}}^i(t,u,\dot{u}) \right] + \sum_{j=1}^{m} \left[y_j(t) g_x^j(t,u,\dot{u}) - \frac{d}{dt} y_j(t) g_{\dot{x}}^j(t,u,\dot{u}) \right] = 0, \right.$$
$$\left. \sum_{j=1}^{m} y_j(t) g^j(t,u,\dot{u}) = 0, \ \tau_i \geqq 0, \ \sum_{i=1}^{p} \tau_i = 1, \ y(t) \geqq 0, \ t \in I \right\}.$$

THEOREM 3.2. **(Weak Duality)** *Suppose that*

(i) $x \in X_0$;

(ii) $(u, \tau, y) \in Y^0$ *and* $\tau > 0$;

(iii) *the problem* (VP) *is pseudo quasi vector variational type I invex with respect to* τ, $y(\cdot)$ *and* η.

Then

$$\int_a^b f(t, x, \dot{x})dt \leqq \int_a^b f(t, u, \dot{u})dt$$

and

$$\int_a^b f(t, x, \dot{x})dt \neq \int_a^b f(t, u, \dot{u})dt.$$

PROOF. By the hypothesis (ii) we have

$$\sum_{j=1}^{m} \int_a^b y_j(t) g^j(t, u, \dot{u})dt = 0. \tag{20}$$

By the hypothesis (iii) and (20) it follows that

$$\sum_{j=1}^{m} \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[y_j(t) g^j_x(t, u, \dot{u}) - \frac{d}{dt} y_j(t) g^j_{\dot{x}}(t, u, \dot{u}) \right] dt \leqq 0. \tag{21}$$

Using the inequality (21) and the hypothesis (ii) we have

$$\sum_{i=1}^{p} \tau_i \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[f^i_x(t, u, \dot{u}) - \frac{d}{dt} f^i_{\dot{x}}(t, u, \dot{u}) \right] dt \geqq 0. \tag{22}$$

Hypothesis (iii) and (22), give

$$\sum_{i=1}^{p} \tau_i \left[\int_a^b f^i(t, x, \dot{x})dt - \int_a^b f^i(t, u, \dot{u})dt \right] \geqq 0. \tag{23}$$

Now suppose to the contrary that

$$\int_a^b f(t, x, \dot{x})dt \leq \int_a^b f(t, u, \dot{u})dt.$$

Then since $\tau > 0$, we have

$$\sum_{i=1}^{p} \tau_i \left[\int_a^b f^i(t, x, \dot{x})dt - \int_a^b f^i(t, u, \dot{u})dt \right] < 0,$$

which contradicts (23). Hence the conclusion follows. □

THEOREM 3.3. **(Weak Duality)** *Suppose that*

(i) $x \in X_0$;

(ii) $(u, \tau, y) \in Y^0$ *and* $\tau > 0$;

(iii) *the problem* (VP) *is semi strictly vector variational type I invex with respect to* τ, $y(\cdot)$ *and* η.

Then

$$\int_a^b f(t, x, \dot{x})dt \leqq \int_a^b f(t, u, \dot{u})dt$$

and

$$\int_a^b f(t, x, \dot{x})dt \neq \int_a^b f(t, u, \dot{u})dt.$$

PROOF. By the hypothesis (ii) we have

$$\sum_{j=1}^{m} \int_a^b y_j(t) g^j(t, u, \dot{u})dt = 0. \tag{24}$$

By (24) and the hypothesis (iii) it follows that

$$\sum_{j=1}^{m} \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[y_j(t) g_x^j(t, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt} y_j(t) g_{\dot{x}}^j(t, \bar{u}, \dot{\bar{u}}) \right] dt \leqq 0. \tag{25}$$

Using the inequality (25) and the hypothesis (ii) we have

$$\sum_{i=1}^{p} \tau_i \int_a^b \eta(t, x, \dot{x}, u, \dot{u})^T \left[f_x^i(t, \bar{u}, \dot{\bar{u}}) - \frac{d}{dt} f_{\dot{x}}^i(t, \bar{u}, \dot{\bar{u}}) \right] dt \geqq 0. \tag{26}$$

By (26) and the hypothesis (iii) we have

$$\sum_{i=1}^{p} \tau_i \left[\int_a^b f^i(t, x, \dot{x})dt - \int_a^b f^i(t, u, \dot{u})dt \right] > 0. \tag{27}$$

Now suppose to the contrary that

$$\int_a^b f(t, x, \dot{x})dt \leq \int_a^b f(t, u, \dot{u})dt.$$

Then since $\tau > 0$, we have

$$\sum_{i=1}^{p} \tau_i \left[\int_a^b f^i(t, x, \dot{x})dt - \int_a^b f^i(t, u, \dot{u})dt \right] < 0,$$

which contradicts (27). Hence the conclusion follows. □

THEOREM 3.4. **(Strong Duality)** *Suppose that*
(i) $\bar{x}$ *is a properly efficient solution of problem* (VP);
(ii) *the hypothesis* (ii) *of Theorem 3.1 is satisfied.*

Then there exist $\tau^0 \in int\mathbb{R}^p_+$, *and* $y^0 \in \mathbb{R}^m_+$ *such that* $(\bar{x}, \tau^0, y^0) \in Y^0$ *and the objective of* (VP) *and* (VD) *have the same values at* $\bar{x}$ *and* $(\bar{x}, \tau^0, y^0)$, *respectively. Moreover, if the problem* (VP) *is pseudo quasi vector variational type I invex at all feasible solution of* (VD) *then* $(\bar{x}, \tau^0, y^0) \in Y^0$ *is an efficient solution of* (VD).

PROOF. Let $M = \{1, \cdots, m\}$. By Theorem 3.1, there exist $\tau^0 \in int\mathbb{R}^p$ and $y^0 \in \mathbb{R}^I$, $y^0(t) \geqq 0$ such that

$$\sum_{i=1}^{p} \tau_i^0 \left[f_x^i(t, \bar{x}, \dot{\bar{x}}) - \frac{d}{dt} f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}}) \right] + \sum_{j \in I} \left[y_j^0(t) g_x^j(t, \bar{x}, \dot{\bar{x}}) - \frac{d}{dt} y_j^0(t) g_{\dot{x}}^j(t, \bar{x}, \dot{\bar{x}}) \right] = 0, \ \forall t \in I.$$

Since $\int_a^b g^i(t, \bar{x}, \dot{\bar{x}})dt = 0$ for all $i \in I$,

$$\int_a^b y_i^0(t) g^i(t, \bar{x}, \dot{\bar{x}})dt = 0 \quad \text{for all} \quad i \in I.$$

Taking $y_i^0 = 0$ for all $i \in M \setminus I$, we have

$$\int_a^b y_i^0(t) g^i(t, \bar{x}, \dot{\bar{x}})dt = 0 \quad \text{for all} \quad i \in M.$$

It also follows that

$$\sum_{i=1}^{p} \tau_i^0 \left[f_x^i(t, \bar{x}, \dot{\bar{x}}) - \frac{d}{dt} f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}}) \right] + \sum_{j=1}^{m} \left[y_j^0(t) g_x^j(t, \bar{x}, \dot{\bar{x}}) - \frac{d}{dt} y_j^0(t) g_{\dot{x}}^j(t, \bar{x}, \dot{\bar{x}}) \right] = 0.$$

Therefore $(\bar{x}, \tau^0, y^0) \in Y^0$. Trivially, the objective function values of (VP) and (VD) are equal.

Next suppose that $(\bar{x}, \tau^0, y^0)$ is not an efficient solution of (VD). Then there exists a point $(y^*, \widetilde{\tau}, \widetilde{y}) \in Y^0$ such that $\int_a^b f(t, \bar{x}, \dot{\bar{x}})dt \leqq \int_a^b f(t, y^*, \dot{y}^*)dt$, and $\int_a^b f(t, \bar{x}, \dot{\bar{x}})dt \neq \int_a^b f(t, y^*, \dot{y}^*)dt$, which violates the weak duality. Hence $(\bar{x}, \tau^0, y^0)$ is indeed an efficient solution of (VP).

The proof of the following theorem is very similar to the proof of Theorem 3.4, except that we appeal to the weak duality Theorem 3.3 instead of Theorem 3.2. □

THEOREM 3.5. **(Strong Duality)** *Suppose that*

(i) $\bar{x}$ *is a properly efficient solution of problem* (VP);

(ii) *the hypothesis* (ii) *of Theorem 3.1 is satisfied.*

Then there exist $\tau^0 \in int\mathbb{R}^p_+$, *and* $y^0 \in \mathbb{R}^m_+$ *such that* $(\bar{x}, \tau^0, y^0) \in Y^0$ *and the objective of* (VP) *and* (VD) *have the same values at* $\bar{x}$ *and* $(\bar{x}, \tau^0, y^0)$, *respectively. Moreover, if the problem* (VP) *is semi strictly vector variational type I invex at all feasible solution of* (VD) *then* $(\bar{x}, \tau^0, y^0) \in Y^0$ *is an efficient solution of* (VD).

References

[1] C. R. Bector and I. Husain, *Duality for multiobjective variational problems*, J. Math. Anal. Appl. **166** (1992), 214–229.

[2] B. D. Craven, *On continuous programming with generalized convexity*, Asia-Pacific J. Oper. Res. **10** (1993), 219–232.

[3] M. A. Hanson, *On sufficiency of Kuhn-Tucker conditions*, J. Math. Anal. Appl. **80** (1981), 545–550.

[4] M. A. Hanson, R. Pini and C. Singh, *Multiobjective Programming under Generalized Type I Invexity*, J. Math. Anal. Appl. **261** (2001), 562–577.

[5] V. Jeyakumar and B. Mond, *On generalized convex mathematical programming*, J. Austral. Math. Soc. Ser. B **34** (1992), 43–53.

[6] R. N. Kaul, S. K. Suneja and M. K. Srivastava, *Optimality criteria and duality in multiple objective optimization involving generalized invexity*, J. Optim. Theory Appl. **80** (1994), 465–482.

[7] D. S. Kim and W. J. Lee, *Symmetric duality for multiobjective variational problems with invexity*, J. Math. Anal. Appl. **218** (1998), 34–48.

[8] S. K. Mishra, *Generalized proper efficiency and duality for a class of nondifferentiable multiobjective variational problems with V-invexity*, J. Math. Anal. Appl. **202** (1996), 53–71.

[9] B. Mond, S. Chandra and I. Husain, *Duality for variational problems with invexity*, J. Math. Anal. Appl. **134** (1988), 322–328.

[10] R. N. Mukherjee and S. K. Mishra, *Sufficient optimality criteria and duality for multiobjective variational problems with V-Invexity*, Indian J. Pure Appl. Math. **25** (1994), 801–813.

[11] G. J. Zalmai, *Optimality conditions and duality for a class of continuous-time programming problems with nonlinear operator equality and inequality constraints*, J. Math. Anal. Appl. **153** (1990), 309–330.

[12] ———, *Proper efficiency conditions and duality models for constrained multiobjective variational problems containing arbitrary norms*, J. Math. Anal. Appl. **196** (1995), 411–427.

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