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COMPARISON STUDY OF BIVARIATE LAPLACE DISTRIBUTIONS WITH THE SAME MARGINAL DISTRIBUTION

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ABSTRACT

Bivariate Laplace distributions for which both marginal distributions and Laplace are discussed. Three kinds of bivariate Laplace distributions which are extended bivariate exponential distributions of Gumbel (1960) are introduced in this paper. These symmetrical distributions are compared with asymmetrical distributions of Kotz *et al.* (2000). Their probability density functions, cumulative distribution functions are derived. Conditional skewnesses and kurtoses are also defined. Their correlation coefficients are calculated and compared with others. We proposed bivariate random vector generating methods whose distributions are bivariate Laplace. With sample means and medians obtained from generated random vectors, variance and covariance matrices of means and medians are calculated and discussed with those of bivariate normal distribution.

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1. INTRODUCTION

Given marginal distributions are not decisive to their bivariate distribution. Fréchet (1951) shows that with given marginal distributions, there are many bivariate distributions corresponding to their marginals. This is just about existence of bivariate distribution not for the structure.

Kozubowski and Podgorski (2000) suggest asymmetric multivariate distributions whose marginals are Laplace. With them, generalized multivariate Laplace

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distributions which are worked by Johnson (1987), Anderson (1992), Fernandez *et al.* (1995), and Ernst (1998) are explored in this paper.

We will extend Gumbel (1960)'s work to three types of the bivariate Laplace (double exponential) distribution whose marginals are Laplace. These symmetric distributions obtained in this paper are different from the bivariate Laplace distributions which are defined from the multivariate one of Kozubowski and Podgorski (2000) and the generalized multivariate one of Fernandez *et al.* (1995) and Ernst (1998). Three types of distributions are described and compared with the contour shapes of others.

General multivariate Laplace distributions are reviewed in Section 2. In Section 3, three types of symmetric bivariate Laplace distribution are discussed. The probability density functions (*pdf*) and cumulative distribution functions (*cdf*) are derived with plots of the functions. We find conditional moments include the conditional skewness and kurtosis as well as the correlation coefficient. In Section 4, a method to generate a bivariate random vector whose distribution is the bivariate Laplace is proposed. The variance and covariance matrices of means and medians obtained from their generated random vectors are calculated and compared with those of the bivariate normal distributions in Section 5. And in Section 6, characteristics of three types of bivariate Laplace distribution suggested in Section 3 are mentioned and compared with other bivariate Laplace distributions obtained from literature reviews.

2. MULTIVARIATE LAPLACE DISTRIBUTION

2.1. *Symmetric multivariate Laplace distributions*

A symmetric Laplace distribution has already been extended to the multivariate case that McGraw and Wagner (1968) listed a bivariate Laplace distribution as a special case of elliptically contoured law, while Johnson and Kotz (1970) provided its density function. These are marginal distribution function and joint density function such as

$$f_X(x) = \frac{1}{\sqrt{2}\sigma}\exp\Big(-\frac{\sqrt{2}\,|x|}{\sigma}\Big),$$

$$f_{X,Y}(x,y) = \frac{1}{\sigma\sqrt{\pi(1-\rho^2)}}J_0\Big(\frac{\sqrt{2}\,q}{\sigma}\Big),$$

where $\sigma > 0$, $q = \sqrt{(x_1^2 - 2\rho x_1 x_2 + x_2^2)/(1-\rho^2)}$, and $J_\lambda(\cdot)$ is Bessel function of the second kind with index λ.

Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)'$ be a random vector whose mean vector and variance-covariance matrix are defined $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)' \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} = (\sigma_{ij})$, a $p \times p$ symmetric positive definite matrix, respectively. If the random vector $\mathbf{X}$ has a density function, then $\mathbf{X}$ has an elliptical contoured distribution if and only if its density function is of the form

$$f(\mathbf{x}) = k_p|\boldsymbol{\Sigma}|^{\frac{1}{2}} g\Big((\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\Big),$$

where g is a non-negative real function and k_p is a positive proportionality constant. We denote the distribution of $\mathbf{X}$ as $EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}\ ; g(\cdot))$. Specially let $g(t) = \exp(-t^{\lambda/2})$ with $\lambda > 0$. The densities of $\mathbf{X}$ and R $(= \sqrt{(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}\ )$ are expressed such as

$$f(\mathbf{x}) = k_{p,\lambda}|\boldsymbol{\Sigma}|^{\frac{1}{2}} \exp\big[- \{(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}^{\frac{\lambda}{2}}\big], \tag{2.1}$$

$$h(r) = \frac{2\pi^{p/2}}{\Gamma(p/2)} k_{p,\lambda} r^{p-1} \exp(-r^{\lambda}),\ r > 0.$$

The density of a generalized gamma random variable which is introduced by Stacy (1962) and described by Johnson and Kotz (1970) is specified as

$$f(\mathbf{x}) = \frac{\lambda\Gamma(p/2)}{2\pi^{p/2}\Gamma(p/\lambda)}|\boldsymbol{\Sigma}|^{\frac{1}{2}} \exp[-\{(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}^{\frac{\lambda}{2}}],$$

where $k_{p,\lambda}$ in equation (2.1) is substituted with $\lambda\Gamma(p/2)/\big(2\pi^{p/2}\Gamma(p/\lambda)\big)$. It is called a multivariate generalized Laplace and denoted by $\mathbf{X} \sim MGL_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \lambda)$ (Kuwana and Kariya, 1991).

This family of univariate distributions includes the Laplace ($\lambda = 1$), the normal ($\lambda = 2$), and the uniform $[\mu - \sigma, \mu + \sigma]$ ($\lambda \to \infty$) distributions (Ernst, 1998). Figure 2.1 shows the shape of $MGL_2(\mathbf{0}, \boldsymbol{\Sigma}, 1)$, where

$$\boldsymbol{\Sigma} = \begin{bmatrix} 2 & -0.5 \cdot \sqrt{2} \cdot 2 \\ -0.5 \cdot \sqrt{2} \cdot 2 & 4 \end{bmatrix}.$$

An elliptical contour curve with $f(\mathbf{x}) = 0.1$ is demonstrated in Figure 2.2.

2.2. Asymmetric multivariate Laplace distribution

The following characteristic function $\phi(t)$ and the density function $f(x)$ of an asymmetric univariate Laplace distribution (AL) were introduced by Hinkley and Revankar (1977) and used for modeling of stock price. which proposed by Madan

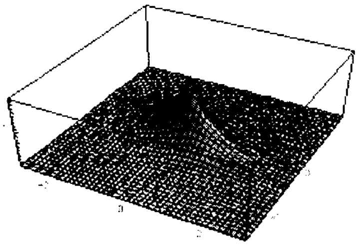

FIGURE 2.1 *Bivariate Laplace of Ernst*

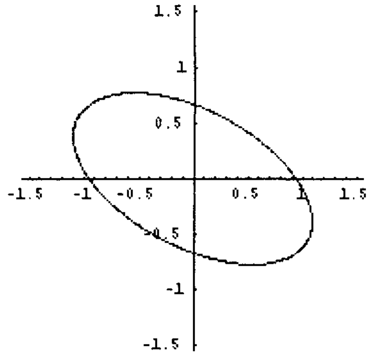

FIGURE 2.2 *An elliptical contour with* $f(x,y) = 0.1$

et al. (1998). This distribution could extended to the asymmetric multivariate Laplace distribution.

$$\phi(t) = (1 + \sigma^2 t^2 - i\mu t)^{-1},$$

$$f(x) = \begin{cases} \dfrac{1}{\sigma}\dfrac{k}{1+k^2}\exp\Big(-\dfrac{k}{\sigma}x\Big), & \text{if } x \geq 0, \\ \dfrac{1}{\sigma}\dfrac{k}{1+k^2}\exp\Big(-\dfrac{1}{\sigma k}x\Big), & \text{if } x < 0, \end{cases}$$

where $k = 2\sigma/(\mu + \sqrt{4\sigma^2 + \mu^2})$.

Kozubowski and Podgorski (2000) extended the multivariate symmetric Laplace distributions discussed by Anderson (1992) to asymmetric multivariate distributions whose characteristic function of the family of asymmetric multivariate Laplace distributions is summarized as the following.

DEFINITION 2.1 (Asymmetric multivariate Laplace distribution). *A random vector* $\mathbf{Y}$ *in* $\mathbb{R}^d$ *has a multivariate asymmetric Laplace distribution* (AL), *if its characteristic function is given by*

$$\phi(\boldsymbol{t}) = \Big(1 + \frac{1}{2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t} - i\boldsymbol{m}'\boldsymbol{t}\Big)^{-1}, \quad \boldsymbol{t} \in \mathbb{R}^d,$$

where $\boldsymbol{t} \in \mathbb{R}^d$, $\boldsymbol{m} \in \mathbb{R}^d$, $\boldsymbol{m} \neq \mathbf{0}$, *and* $\boldsymbol{\Sigma}$ *is a* $d \times d$ *non-negative definite symmetric matrix. It is denoted by* $\mathbf{Y} \sim AL_d(\boldsymbol{m}, \boldsymbol{\Sigma})$.

If the matrix $\boldsymbol{\Sigma}$ is a positive-definite, the distribution is then truly d-dimensional and possessed a probability density function. If $\boldsymbol{m} = \mathbf{0}$, $AL_d(\mathbf{0}, \boldsymbol{\Sigma})$ turns to be a symmetric multivariate Laplace distribution.

They note that the term 'multivariate Laplace law' is somewhat ambiguous since it has been used for at least the following classes of multivariate distributions (Kotz *et al.*, 2000).

Multivariate Laplace law.

1. A multivariate distribution generated by a vector of *iid* univariate Laplace variables. See Marshall and Olkin (1993), Kalashniknov (1997), and Fernandez *et al.* (1995) *etc.*

2. A bivariate distribution with Laplace marginal introduced by Ulrich and Chen (1987).

3. An elliptically contoured distribution given by the characteristic function

$$\phi(\boldsymbol{t}) = \left(1 + \frac{1}{2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}\right)^{-1}, \quad \boldsymbol{t} \in \mathbb{R}^d.$$

   See McGraw and Wagner (1968), Pillai (1985), Johnson (1987), Anderson (1992), and Kotz *et al.* (2000).

4. A special case ($\lambda = 1$) of the multivariate exponential power distribution with the density

$$f(\mathbf{x}) = C\exp\Big[-\{(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}^{\frac{\lambda}{2}}\Big], \quad \mathbf{x} \in \mathbb{R}^d,$$

   where C is a constant. See Fernandez *et al.* (1995), Ernst (1998), and Haro-López and Smith (1999).

5. A multivariate distribution with the density

$$f(\mathbf{x}) = CK_0\Big(\frac{\|\mathbf{x}\|}{\sigma}\Big), \quad \mathbf{x} \in \mathbb{R}^d,$$

   where K_0 is modified Bessel function of the third kind and order zero (Fang *et al.*, 1990). In case $d = 2$ (and only in this case), the characteristic function of this distribution is the following.

$$\phi(\boldsymbol{t}) = \left(1 + \frac{1}{2}\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t}\right)^{-1}, \quad \boldsymbol{t} \in \mathbb{R}^2.$$

Kozubowski and Podgorski (2000) introduced the asymmetric multivariate Laplace density of AL_d. Let $G(\cdot)$ and $F(\cdot)$ be cumulative distribution functions of $AL_d(\boldsymbol{m}, \boldsymbol{\Sigma})$ and $N_d(\mathbf{0}, \boldsymbol{\Sigma})$ random vectors, respectively, and let $g(\cdot)$ and $f(\cdot)$ be the corresponding probability distribution functions.

$$G(\mathbf{y}) = \int_0^\infty F(z^{-\frac{1}{2}}y - z^{\frac{1}{2}}\boldsymbol{m}e^{-z})\,dz,$$
$$g(\mathbf{y}) = \int_0^\infty f(z^{-\frac{1}{2}}y - z^{\frac{1}{2}}\boldsymbol{m}e^{-z})\,dz.$$

We can also express an AL_d density in terms of the modified Bessel function of the third kind. From above $g(\mathbf{y})$ can be expressed as following (see Kotz *et al.*, 2000; Kozubowski and Podgorski, 2000):

$$g(\mathbf{y}) = \frac{2\exp(\mathbf{y}'\boldsymbol{\Sigma}^{-1}\boldsymbol{m})}{(2\pi)^{\frac{d}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\left(\frac{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}{2+\boldsymbol{m}'\boldsymbol{\Sigma}^{-1}\boldsymbol{m}}\right)^{\frac{\nu}{2}} K_\nu\left(\sqrt{(2+\boldsymbol{m}'\boldsymbol{\Sigma}^{-1}\boldsymbol{m})(\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y})}\right),$$

where $\nu = (2-d)/2$ and $K_\lambda(u)$ is the modified Bessel function of the third kind with index λ. Let us illustrate following three densities.

EXAMPLE 1 (*Symmetric multivariate Laplace* $(\boldsymbol{m} = \mathbf{0})$). The probability distribution function of a symmetric case in AL_2 $(\boldsymbol{m} = \mathbf{0})$ is the following.

$$g(\mathbf{y}) = 2(2\pi)^{-\frac{d}{2}}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}\left(\frac{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}{2}\right)^{\frac{\nu}{2}} K_\nu\left(\sqrt{2\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}\right)$$

It is the same as the multivariate Laplace distribution of Anderson (1992).

EXAMPLE 2. If $d = 1$ $(\nu = 1/2)$, then the density function is the same as the asymmetric univariate Laplace form.

$$g(\mathbf{y}) = \frac{1}{\gamma}\exp\left[-\frac{|y|}{\sigma^2}\{\gamma - \mu\cdot\text{sign}(y)\}\right],$$

where $\sigma^2 = \Sigma$, $\mu = m$, and $\gamma = \sqrt{\mu^2 + 2\sigma^2}$.

EXAMPLE 3. For the case $d = 2$ and $\nu = 0$,

$$g(\mathbf{y}) = \pi^{-1}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}\exp(\mathbf{y}'\boldsymbol{\Sigma}^{-1}\boldsymbol{m})K_0\left(\sqrt{(2+\boldsymbol{m}'\boldsymbol{\Sigma}^{-1}\boldsymbol{m})(\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y})}\right).$$

Denoting

$$\boldsymbol{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

we obtain a five parameter family dependent on m_1, m_2, σ_1^2, σ_2^2, and ρ with the densities of the form

$$g(x,y) = \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{(m_1\sigma_2/\sigma_1 - m_2\rho)x + (m_2\sigma_1/\sigma_2 - m_1\rho)y}{\sigma_1\sigma_2(1-\rho^2)}\right\}$$
$$\times K_0\left(C(m_1,m_2,\sigma_1,\sigma_2,\rho)\sqrt{x^2\frac{\sigma_2}{\sigma_1} - 2\rho xy + y^2\frac{\sigma_1}{\sigma_2}}\right)$$

where

$$C(m_1,m_2,\sigma_1,\sigma_2,\rho) = \frac{\sqrt{2\sigma_1\sigma_2(1-\rho^2) + m_1^2\sigma_2/\sigma_1 - 2m_1m_2\rho + m_1^2\sigma_1/\sigma_2}}{\sigma_1\sigma_2(1-\rho^2)}.$$

The parameter m_1 and m_2 give information of the skewness of the density. For two dimensional case (AL_2), some shapes of the following densities and their contours are demonstrated in order to compare with those of densities which will be introduced in next section.

1. $g(x,y;m_1,m_2,\sigma_1,\sigma_2,\rho) = g(x,y;0,0,\sqrt{2},\sqrt{2},0)$

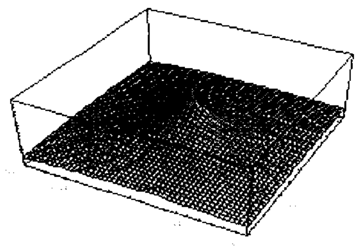

FIGURE 2.3 *AL_2 of $\rho = 0$*

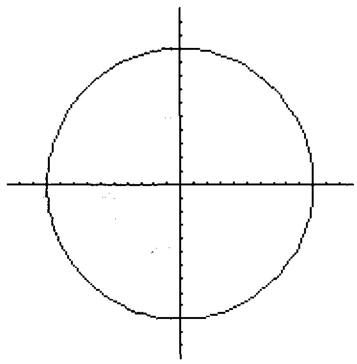

FIGURE 2.4 *AL_2 contour with $\rho = 0$*

2. $g(x,y;m_1,m_2,\sigma_1,\sigma_2,\rho) = g(x,y;0,0,\sqrt{2},\sqrt{2},-0.25)$

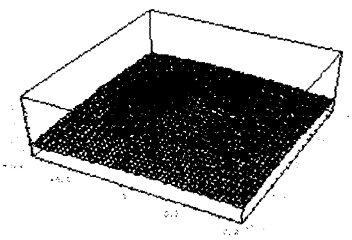

FIGURE 2.5 *AL_2 of $\rho = -0.25$*

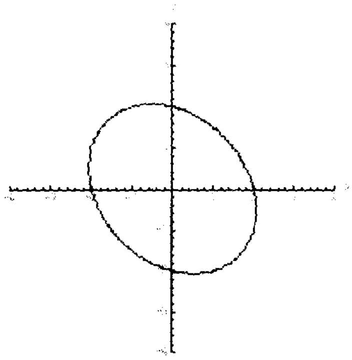

FIGURE 2.6 *AL_2 contour with $\rho = -0.25$*

3. $g(x,y;m_1,m_2,\sigma_1,\sigma_2,\rho) = g(x,y;-2,1,\sqrt{2},\sqrt{2},0.5)$

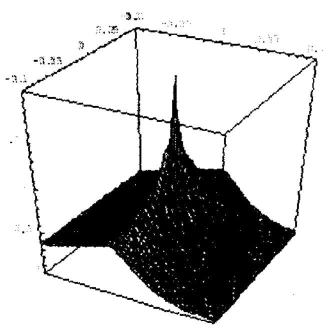

FIGURE 2.7 *Asymmetric AL_2 Laplace*

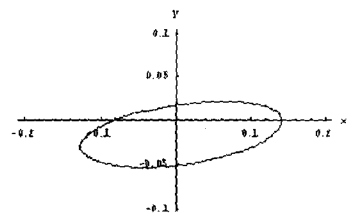

FIGURE 2.8 *Asymmetric. contour of AL_2*

3. ANOTHER SYMMETRIC BIVARIATE LAPLACE DISTRIBUTIONS

3.1. First type of bivariate Laplace distribution

The *pdf* and *cdf* of the first type of the bivariate exponential distribution are as the following (see Gumbel, 1960; Johnson and Kotz, 1970).

For $x \geq 0$, $y \geq 0$ and $0 \leq \theta \leq 1$,

$$\begin{aligned} F_{X,Y}(x,y) &= 1 - e^{-x} - e^{-y} + e^{-x-y-\theta xy}, \\ f_{X,Y}(x,y) &= e^{-x-y-\theta xy}\{(1+\theta x)(1+\theta y) - \theta\}. \end{aligned}$$

The marginal distributions of X and Y are each standard exponential. Let us take absolute value functions on random variables X and Y in the above *pdf.* Then we could define the following *pdf* of the bivariate Laplace distribution.

DEFINITION 3.1. *The pdf of the first type of the bivariate Laplace distribution is defined as*

$$\begin{aligned} f_{X,Y}(x,y) = &\ \frac{1}{4}\exp\big(-|x-\mu_X| - |y-\mu_Y| - \theta|x-\mu_X||y-\mu_Y|\big) \\ &\times\{(1+\theta|x-\mu_X|)(1+\theta|y-\mu_Y|) - \theta\}, \end{aligned}$$

where $-\infty \leq x \leq \infty$, $-\infty \leq y \leq \infty$, $0 \leq \theta \leq 1$, $\mu_X = E(X)$, *and* $\mu_Y = E(Y)$.

The *pdf* of Definition 3.1 satisfies common properties of continuous density function. And one could get the following *cdf* of the first type of the bivariate Laplace distribution by using Definition 3.1.

THEOREM 3.1. *The cdf of the first type of the bivariate Laplace distribution is defined as*

$$F_{X,Y}(x,y) = \begin{cases} \frac{1}{4}\left\{4 - 2e^{-(x-\mu_X)} - 2e^{-(y-\mu_Y)} + e^{-(x-\mu_X)-(y-\mu_Y)-\theta(x-\mu_X)(y-\mu_Y)}\right\}, & x \geq \mu_X, y \geq \mu_Y, \\ \frac{1}{4}\left\{2e^{(y-\mu_Y)} - e^{-(x-\mu_X)+(y-\mu_Y)+\theta(x-\mu_X)(y-\mu_Y)}\right\}, & x \geq \mu_X, y < \mu_Y, \\ \frac{1}{4}\left\{2e^{(x-\mu_X)} - e^{(x-\mu_X)-(y-\mu_Y)+\theta(x-\mu_X)(y-\mu_Y)}\right\}, & x < \mu_X, y \geq \mu_Y, \\ \frac{1}{4}\left\{e^{(x-\mu_X)+(y-\mu_Y)-\theta(x-\mu_X)(y-\mu_Y)}\right\}, & x < \mu_X, y < \mu_Y, \end{cases}$$

where $0 \leq \theta \leq 1$.

The proof can be obtained by double integrations with ease. The shapes of *pdf* and *cdf* with several values of θ and $\mu_X = 0$, $\mu_Y = 0$ are on Figure 3.1 $\sim$ Figure 3.5. And Figure 3.6 is a contour plot with $\theta = 1$.

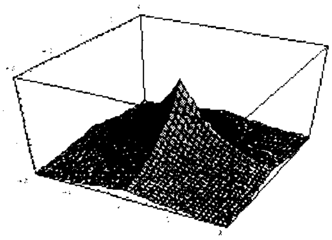

FIGURE 3.1 *pdf with* $\theta = 0$

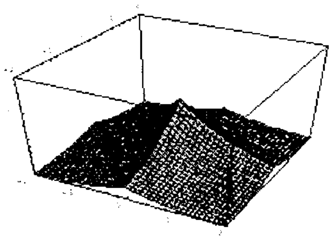

FIGURE 3.2 *pdf with* $\theta = 0.25$

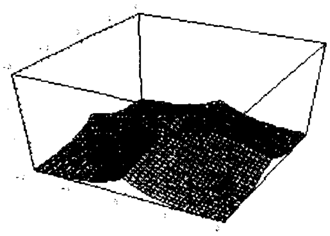

FIGURE 3.3 *pdf with* $\theta = 0.75$

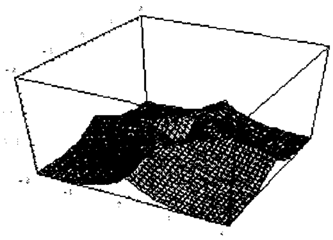

FIGURE 3.4 *pdf with* $\theta = 1$

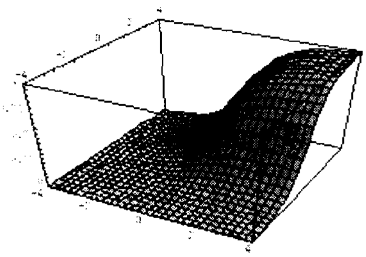

FIGURE 3.5 *cdf with* $\theta = 0$

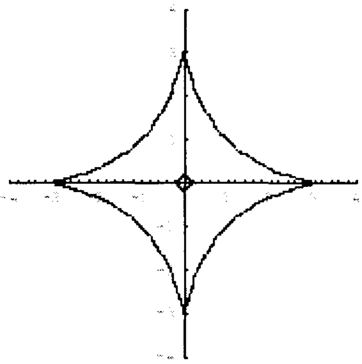

FIGURE 3.6 *Contour with* $\theta = 1$

Note that the correlation coefficient ρ of the first type of the bivariate exponential distribution belongs to $-0.40 \leq \rho \leq 0$ (see Gumbel, 1960; Johnson and Kotz, 1970). The correlation coefficient of the first type of the bivariate Laplace distribution could be derived as the following.

THEOREM 3.2. *The correlation coefficient ρ of the first type of the bivariate Laplace distribution is $\rho = 0$, which is free of θ.*

PROOF. Without any losses of generality, suppose $\mu_X = \mu_Y = 0$.

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E[XY] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{xy}{4} e^{-|x|-|y|-\theta|x|-|y|}\{(1+\theta|x|)(1+\theta|y|)-\theta\}\,dxdy \\ &= 0,\end{aligned}$$

since the *pdf* is symmetric with respect to X and Y axes. □

The conditional k^{th} moments given $Y = y$ are obtained as the following.

THEOREM 3.3. *The conditional k^{th} moments of X of the first type of the bivariate Laplace distribution are derived such that*

$$E\big[(X-\mu_X)^k|y\big] = 0, \text{ where } k \text{ is odd},$$

$$E\big[(X-\mu_X)^k|y\big] = \begin{cases} \dfrac{k!\{1+\theta(y-\mu_Y)+k\theta\}}{\{1+\theta(y-\mu_Y)\}^{k+1}}, & y \geq \mu_Y, \\ \dfrac{k!\{1-\theta(y-\mu_Y)+k\theta\}}{\{1-\theta(y-\mu_Y)\}^{k+1}}, & y < \mu_Y, \text{ where } k \text{ is even}.\end{cases}$$

The proof can be obtained by integrations with ease. With results of Theorem 3.3, one can get the conditional kurtosis and skewness as the following.

COROLLARY 3.1. *The conditional kurtosis and skewness of the first type of the bivariate Laplace distribution are*

$$\frac{E\big[(X-\mu_X)^3|y\big]}{E\big[(X-\mu_X)^2|y\big]^{\frac{3}{2}}} = 0,$$

$$\frac{E\big[(X-\mu_X)^4|y\big]}{E\big[(X-\mu_X)^2|y\big]^{2}} - 3$$

$$= \begin{cases} \dfrac{6\{1+\theta(y-\mu_Y)+4\theta\}\{1+\theta(y-\mu_Y)\}}{\{1+\theta(y-\mu_Y)+2\theta\}^2} - 3, & \text{where } y \geq \mu_Y, \\ \dfrac{6\{1-\theta(y-\mu_Y)+4\theta\}\{1-\theta(y-\mu_Y)\}}{\{1-\theta(y-\mu_Y)+2\theta\}^2} - 3, & \text{where } y < \mu_Y.\end{cases}$$

3.2. Second type of bivariate Laplace distribution

For $x \geq 0$, $y \geq 0$, and $-1 \leq \alpha \leq 1$, the *cdf* and *pdf* of the second type of the bivariate exponential distribution are proposed by Morgenstern (1956) such that,

$$\begin{aligned} F_{X,Y}(x,y) &= F_X(x)F_Y(y)\left[1+\alpha\{1-F_X(x)\}\{1-F_Y(y)\}\right] \\ f_{X,Y}(x,y) &= f_X(x)F_Y(y)\left[1+\alpha\{2F_X(x)-1\}\{2F_Y(y)-1\}\right]. \end{aligned}$$

With similar arguments, the *cdf* and *pdf* of the second type of the bivariate Laplace distribution could be derived.

DEFINITION 3.2. *The cdf and pdf of the second type of the bivariate Laplace distribution are defined as*

$$F_{X,Y}(x,y) = \begin{cases} \left\{1-\frac{1}{2}e^{-(x-\mu_X)}\right\}\left\{1-\frac{1}{2}e^{-(y-\mu_Y)}\right\}\left\{1+\frac{1}{4}\alpha e^{-(x-\mu_X)-(y-\mu_Y)}\right\}, & x \geq \mu_X,\ y \geq \mu_Y, \\ \frac{1}{2}e^{(y-\mu_Y)}\left\{1-\frac{1}{2}e^{-(x-\mu_X)}\right\}\left[1+\frac{1}{2}\alpha\left\{e^{-(x-\mu_X)}-\frac{1}{2}e^{-(x-\mu_X)+(y-\mu_Y)}\right\}\right], & x \geq \mu_X,\ y < \mu_Y, \\ \frac{1}{2}e^{(x-\mu_X)}\left\{1-\frac{1}{2}e^{-(y-\mu_Y)}\right\}\left[1+\frac{1}{2}\alpha\left\{e^{-(y-\mu_Y)}-\frac{1}{2}e^{(x-\mu_X)-(y-\mu_Y)}\right\}\right], & x < \mu_X,\ y \geq \mu_Y, \\ \frac{1}{4}e^{(x-\mu_X)+(y-\mu_Y)}\left[1+\alpha\left\{1-\frac{1}{2}e^{(x-\mu_X)}\right\}\left\{1-\frac{1}{2}e^{(y-\mu_Y)}\right\}\right], & x < \mu_X,\ y < \mu_Y, \end{cases}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}e^{-(x-\mu_X)-(y-\mu_Y)}\left[1+\alpha\left\{1-e^{-(x-\mu_X)}\right\}\left\{1-e^{-(y-\mu_Y)}\right\}\right], & x \geq \mu_X,\ y \geq \mu_Y, \\ \frac{1}{4}e^{-(x-\mu_X)+(y-\mu_Y)}\left[1+\alpha\{1-e^{-(x-\mu_X)}\}\{-1+e^{(x-\mu_Y)}\}\right], & x \geq \mu_X,\ y < \mu_Y, \\ \frac{1}{4}e^{(x-\mu_X)-(y-\mu_Y)}\left[1+\alpha\{-1+e^{-(x-\mu_X)}\}\{1-e^{-(y-\mu_Y)}\}\right], & x < \mu_X,\ y \geq \mu_Y, \\ \frac{1}{4}e^{(x-\mu_X)+(y-\mu_Y)}\left[1+\alpha\left\{1-e^{(x-\mu_X)}\right\}\left\{1-e^{(y-\mu_Y)}\right\}\right], & x < \mu_X,\ y < \mu_Y, \end{cases}$$

where $-1 \leq \alpha \leq 1$.

Shapes of the *pdf* and *cdf* of the second type of the bivariate Laplace distribution are shown in Figure 3.7 $\sim$ Figure 3.9. And Figure 3.10 is an example of a contour plot.

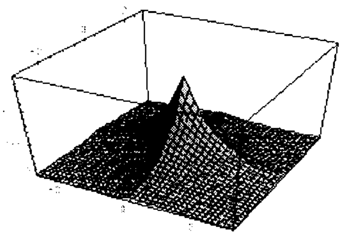

FIGURE 3.7 *pdf with* $\alpha = -1$

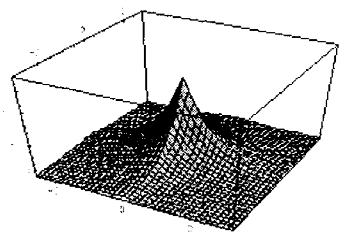

FIGURE 3.8 *pdf with* $\alpha = 1$

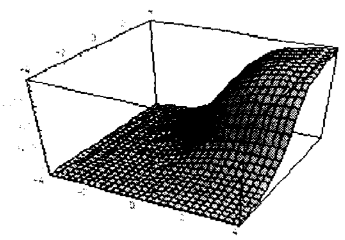

FIGURE 3.9 *cdf with* $\alpha = -1$

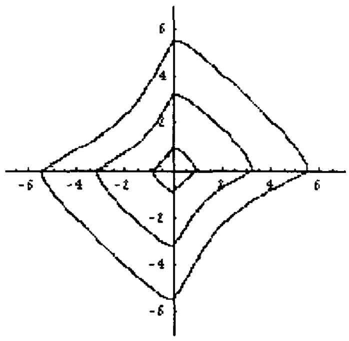

FIGURE 3.10 *Several contours*

The correlation coefficient of the second type of the bivariate exponential distribution in Gumbel's study is obtained as $\rho = \alpha/4$. Note that since $|\alpha| \leq 1$, ρ can not exceed -0.25 or be less than 0.25. Now Theorem 3.4 describes the correlation coefficient of the second type of the bivariate Laplace distribution.

THEOREM 3.4. *The correlation coefficient ρ of the second type of the bivariate Laplace distribution belongs to the interval*

$$-\frac{9}{32} \leq \rho \leq \frac{9}{32}.$$

PROOF. Without any losses of generality, assume $\mu_X = \mu_Y = 0$.

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E[XY] \\ &= \frac{1}{16}\Big\{4+\frac{1}{4}\alpha(3)(3)\Big\} + \frac{1}{16}\Big\{-4-\frac{1}{4}\alpha(3)(-3)\Big\} \\ &\quad +\frac{1}{16}\Big\{-4-\frac{1}{4}\alpha(-3)(3)\Big\} + \frac{1}{16}\Big\{4+\frac{1}{4}\alpha(3)(3)\Big\} \\ &= \frac{9\alpha}{16}\end{aligned}$$

and ${\sigma_X}^2 = {\sigma_Y}^2 = 2$. Therefore $\rho = (9/32)\alpha$. □

The conditional k^{th} moments of X given $Y = y$ are also obtained.

THEOREM 3.5. *The conditional k^{th} moments of the second type of the bivariate Laplace distribution are derived such that*

1. *for an even number k,*

$$E\big[(X-\mu_X)^k|y\big] = k!, \qquad -\infty < y < \infty;$$

2. *for an odd number k,*

$$E\big[(X-\mu_X)^k|y\big] = \begin{cases} k!\alpha\Big(1-\dfrac{1}{2^{k+1}}\Big)\Big\{1-e^{-(y-\mu_Y)}\Big\}, & y \ge \mu_Y, \\ k!\alpha\Big(\dfrac{1}{2^{k+1}}-1\Big)\Big\{1-e^{(y-\mu_Y)}\Big\}, & y < \mu_Y. \end{cases}$$

PROOF.

1. When k is even,

$$E\big[(X-\mu_X)^k|y\big]$$

$$= \begin{cases} \dfrac{1}{2}k!\Big\{1+\alpha-\alpha e^{-(y-\mu_Y)}-\alpha\dfrac{1}{2^k}+\alpha e^{-(y-\mu_Y)}\dfrac{1}{2^k}\Big\}, & x \ge \mu_X,\ y \ge \mu_Y, \\ \dfrac{1}{2}k!\Big\{1-\alpha+\alpha e^{(y-\mu_Y)}+\alpha\dfrac{1}{2^k}-\alpha e^{(y-\mu_Y)}\dfrac{1}{2^k}\Big\}, & x \ge \mu_X,\ y < \mu_Y, \\ \dfrac{1}{2}k!\Big\{1-\alpha+\alpha e^{-(y-\mu_Y)}+\alpha\dfrac{1}{2^k}-\alpha e^{-(y-\mu_Y)}\dfrac{1}{2^k}\Big\}, & x < \mu_X,\ y \ge \mu_Y, \\ \dfrac{1}{2}k!\Big\{1+\alpha-\alpha e^{(y-\mu_Y)}-\alpha\dfrac{1}{2^k}+\alpha e^{(y-\mu_Y)}\dfrac{1}{2^k}\Big\}, & x < \mu_X,\ y < \mu_Y. \end{cases}$$

When $y \ge \mu_Y$,

$$E\big[(X-\mu_X)^k|y\big] = 2\frac{1}{2}k! = k!.$$

and when $y < \mu_Y$,

$$E\big[(X-\mu_X)^k|y\big] = 2\frac{1}{2}k! = k!.$$

Hence for an even number k, $E\big[(X-\mu_X)^k\big] = k!$, $-\infty < y < \infty$.

2. When k is odd,

$$E[(X-\mu_X)^k|y]$$
$$= \begin{cases} \frac{1}{2}k!\left\{1+\alpha-\alpha e^{-(y-\mu_Y)}-\frac{\alpha}{2^k}+\alpha e^{-(y-\mu_Y)}\frac{1}{2^k}\right\}, & x \geq \mu_X,\ y \geq \mu_Y, \\ \frac{1}{2}k!\left\{1-\alpha+\alpha e^{(y-\mu_Y)}+\frac{\alpha}{2^k}-\alpha e^{(y-\mu_Y)}\frac{1}{2^k}\right\}, & x \geq \mu_X,\ y < \mu_Y, \\ \frac{1}{2}k!\left\{-1+\alpha-\alpha e^{-(y-\mu_Y)}-\frac{\alpha}{2^k}+\alpha e^{-(y-\mu_Y)}\frac{1}{2^k}\right\}, & x < \mu_X,\ y \geq \mu_Y, \\ \frac{1}{2}k!\left\{-1-\alpha+\alpha e^{(y-\mu_Y)}+\frac{\alpha}{2^k}-\alpha e^{(y-\mu_Y)}\frac{1}{2^k}\right\}, & x < \mu_X,\ y < \mu_Y. \end{cases}$$

When $y \geq \mu_Y$,

$$E\big[(X-\mu_X)^k|y\big] = k!\alpha\Big(1-\frac{1}{2^{k+1}}\Big)\Big(1-e^{-(y-\mu_Y)}\Big)$$

and when $y < \mu_Y$,

$$E\big[(X-\mu_X)^k|y\big] = k!\alpha\Big(\frac{1}{2^{k+1}}-1\Big)\Big(1-e^{(y-\mu_Y)}\Big).$$

□

The conditional kurtosis and skewness could be summarized by using Theorem 3.5.

COROLLARY 3.2. *The conditional kurtosis and skewness of the second type of the bivariate Laplace distribution are*

$$\frac{E\big[(X-\mu_X)^3|y\big]}{E\big[(X-\mu_X)^2|y\big]^{\frac{3}{2}}} = \begin{cases} \frac{45\sqrt{2}}{32}\alpha\big(1-e^{-(y-\mu_Y)}\big), & y \geq \mu_Y, \\ -\frac{45\sqrt{2}}{32}\alpha\big(1-e^{(y-\mu_Y)}\big), & y < \mu_Y, \end{cases}$$

$$\frac{E\big[(X-\mu_X)^4|y\big]}{E\big[(X-\mu_X)^2|y\big]^2} - 3 = 3.$$

Since the coefficient of kurtosis is 3, the density is more peaked around its center than the normal density.

3.3. Third type of bivariate Laplace distribution

The *pdf* and *cdf* of the third type of the bivariate exponential distribution are

$$\begin{aligned} f_{X,Y}(x,y) &= P(x,y)(x^m+y^m)^{1/(m-2)}\,x^{m-1}\big\{(x^m+y^m)^{1/m}+m-1\big\}, \\ F_{X,Y}(x,y) &= 1-e^{-x}-e^{-y}+P(x,y), \end{aligned}$$

where $x, y > 0$, $1 \le m \le 2$ and $P(x,y) = e^{-(x^m+y^m)^{1/m}}$ (see Gumbel, 1960; Johnson and Kotz, 1970).

The *pdf* of the third type the bivariate Laplace distribution could be derived by taking absolute value functions on x and y.

DEFINITION 3.3. *The third type of the bivariate Laplace distribution has the following pdf:*

$$\begin{aligned} f_{X,Y}(x,y) = {} & P\big(|x-\mu_X|,|y-\mu_Y|\big)\big(|x-\mu_X|^m+|y-\mu_Y|^m\big)^{1/(m-2)} \\ & \times|x-\mu_X|^{m-1}|y-\mu_Y|^{m-1}\Big\{\big(|x-\mu_X|^m+|y-\mu_Y|^m\big)^{1/m}+m-1\Big\}, \end{aligned}$$

where $-\infty < x, y < \infty$, $1 \le m \le 2$, *and* $P(x,y) = \exp\big\{-(x^m+y^m)^{1/m}\big\}$.

The third type of the bivariate Laplace *pdf* are shown at Figure 3.11 and Figure 3.12. In order to get the *cdf* of the third types, complicated integration methods are demanded, so that we left this to readers.

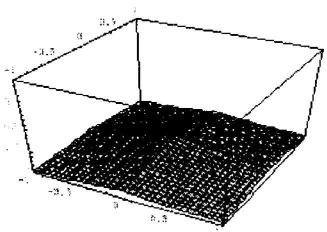

FIGURE 3.11 *pdf with* $m = 1$

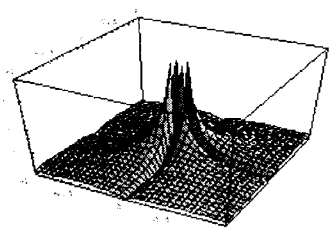

FIGURE 3.12 *pdf with* $m = 1.6$

4. APPLICATIONS

4.1. Random vector generation

Since we knew *cdf*s of the first and second types of the bivariate Laplace distribution, we could generate random vectors of $\mathbf{X}$ and $\mathbf{Y}$ which follow the bivariate Laplace distribution by using the well-known inverse probability integral transformation method.

SKETCH OF ALGORITHM.

Step 1. Generate $u_1 \sim U(0,1)$.

Step 2. If $0 < u_1 < 1/2$, then generate $u_2 \sim U(0,1/2)$. Get a negative x such that

$$x = F_X^{-1}(u_2)$$

and if $1/2 \leq u_1 < 1$, then generate $u_2 \sim U(1/2,1)$. Get a non-negative x such that

$$x = F_X^{-1}(u_2).$$

Step 3. With given values u_1 and x obtained at Step 1 and 2, respectively, generate

$$y = F_{Y|x}^{-1}(u_1).$$

Step 4. Once appropriate x and y are obtained, then go to Step 1 until random vectors of given sample size are collected. If one fails to obtain an appropriate value of Y, then go back to the Step 2, so that get another value of $x = F_X^{-1}(u_2)$. And try to get an appropriate value y.

4.2. Simulation of comparison variance-covariance

From the random samples collected for standard normal and Laplace distributions, the ratio of the asymptotic variances of the sample mean $\overline{X}$ and sample median M_X are well-known such that

1. $X_1, X_2, \ldots, X_n \sim N(0,1)$,

$$\frac{V(\overline{X})}{V(M_X)} = \frac{2}{\pi} < 1;$$

2. $X_1, X_2, \ldots, X_n \sim \text{Laplace}(0,1)$.

$$\frac{V(\overline{X})}{V(M_X)} = 2 > 1.$$

Now we generate random vector of $\mathbf{X}$ and $\mathbf{Y}$ whose distribution is the bivariate Laplace distribution with $\mu_X = \mu_Y = 0$. Then the variance and covariance matrices of the sample mean $(\overline{X}, \overline{Y})$, the MLE of (μ_X, μ_Y), and sample median (M_X, M_Y) from the random vectors of $\mathbf{X}$ and $\mathbf{Y}$ could be obtained and will be compared with those of the random vectors whose distribution is the bivariate normal distribution.

First of all, let us get the variance and covariance matrices of the sample means and medians from the random vectors which are generated from bivariate normal distribution. Consider the standard bivariate normal distribution with a given correlation coefficient ρ. The values of ρ are given from -0.9 to 0.9 by the increment 0.1. For each ρ, the size of random vector is 50 and collect $1,000$ random samples. Then the variances and covariances matrices of the sample means and the sample medians are calculated and analyzed at Table 4.1. We could find that the ratio of the determinant of the variance and covariance matrices is also less than 1. The ratio is increasing up to 0.44 as absolute values of ρ get smaller.

Consider the first and second types of the bivariate Laplace distribution with a given value of θ and α are given from 0 to 1.0 and -1.0 to 1.0 by increment 0.1, respectively. For each θ and α, random vectors are of size 50, and take $1,000$ repetitions. Table 4.2 shows that since the correlation the first type of the bivariate Laplace distribution is zero, the variance-covariance matrices are also larger than 1. With comparing of Table 4.3, these ratios of the first type of the bivariate Laplace distribution are larger than those of the second type.

5. CONCLUSION

The multivariate law suggested by Kotz *et al.* (2000) is somewhat ambiguous as mentioned in Section 2. When two dimensional distributions among multivariate ones are considered, one can find that several types of distributions are overlapped in each cases. In particular, 1, 3, 4 and 5 cases have elliptical contour curves.

In this work, three types of bivariate Laplace distributions with the same marginals are derived, whose contours are non-elliptical and non-hyperbolic like in Figure 3.6 and Figure 3.10. The first type of the bivariate Laplace distribution has zero correlation coefficient. The oddth conditional moments are always '0', so that the conditional *pdf* is symmetric with respect to its mean. The second type does not have zero correlation coefficient. The correlation coefficient belongs to $[-9/32, 9/32]$, which is much narrow than that of the bivariate normal but a little

TABLE 4.1 *Results of the bivariate normal distribution*

ρ	$V(\overline{X})$	$V(M_X)$	$\frac{V(\overline{X})}{V(M_X)}$	$Cov(\overline{X},\overline{Y})$	$Cov(M_X, M_Y)$	$\frac{Det(\sum_{\overline{X}\overline{Y}})}{Det(\sum_{M_X M_Y})}$
−1.0	0.0209	0.0332	0.6308			
−0.9	0.0199	0.0323	0.6162	−0.0183	−0.0232	0.1476
−0.8	0.0205	0.0315	0.6501	−0.0159	−0.0196	0.2400
−0.7	0.0207	0.0299	0.6921	−0.0149	−0.0147	0.3088
−0.6	0.0199	0.0296	0.6716	−0.0121	−0.0128	0.3511
−0.5	0.0227	0.0335	0.6782	−0.0110	−0.0106	0.3683
−0.4	0.0188	0.0290	0.6500	−0.0081	−0.0086	0.3831
−0.3	0.0208	0.0301	0.6918	−0.0072	−0.0060	0.4041
−0.2	0.0203	0.0303	0.6693	−0.0028	−0.0026	0.4178
−0.1	0.0209	0.0323	0.6473	−0.0021	−0.0026	0.4387
0.0	0.0189	0.0283	0.6685	−0.0001	−0.0001	0.4371
0.1	0.0192	0.0304	0.6327	0.0022	0.0017	0.4203
0.2	0.0202	0.0317	0.6376	0.0047	0.0044	0.3857
0.3	0.0214	0.0327	0.6542	0.0067	0.0075	0.4125
0.4	0.0208	0.0324	0.6419	0.0082	0.0085	0.3928
0.5	0.0193	0.0302	0.6396	0.0095	0.0107	0.3672
0.6	0.0206	0.0312	0.6606	0.0113	0.0134	0.3649
0.7	0.0194	0.0298	0.6522	0.0135	0.0145	0.3228
0.8	0.0211	0.0313	0.6722	0.0173	0.0196	0.2559
0.9	0.0202	0.0314	0.6409	0.0182	0.0223	0.1627
1.0	0.0204	0.0332	0.6155			

TABLE 4.2 *Results of the first type of the bivariate Laplace distribution*

ρ	$V(\overline{X})$	$V(M_X)$	$\frac{V(\overline{X})}{V(M_X)}$	$Cov(\overline{X},\overline{Y})$	$Cov(M_X, M_Y)$	$\frac{Det(\sum_{\overline{X}\overline{Y}})}{Det(\sum_{M_X M_Y})}$
0.0	0.0191	0.0087	2.1881	0.0001	0.0002	4.2139
0.1	0.0160	0.0081	1.9620	−0.0009	0.0003	3.3897
0.2	0.0154	0.0081	1.8929	−0.0006	−0.0003	3.6100
0.3	0.0146	0.0080	1.9402	0.0000	0.0002	3.7472
0.4	0.0151	0.0077	1.9655	−0.0005	−0.0004	3.3323
0.5	0.0148	0.0077	1.9270	−0.0007	−0.0001	3.3528
0.6	0.0133	0.0074	1.7948	−0.0003	0.0003	3.6479
0.7	0.0139	0.0075	1.8550	−0.0013	−0.0001	3.5876
0.8	0.0137	0.0074	1.8491	−0.0006	−0.0005	3.2745
0.9	0.0132	0.0075	1.7603	−0.0005	0.0003	3.2835
1.0	0.0132	0.0073	1.8132	0.0004	0.0000	3.4505

TABLE 4.3 *Results of the second type of the bivariate Laplace distribution*

ρ	$V(\overline{X})$	$V(M_X)$	$\frac{V(\overline{X})}{V(M_X)}$	$Cov(\overline{X},\overline{Y})$	$Cov(M_X,M_Y)$	$\frac{Det(\sum_{\overline{X}\overline{Y}})}{Det(\sum_{M_X M_Y})}$
−1.0	0.0417	0.0362	1.3958	−0.0139	−0.0079	1.8362
−0.9	0.0394	0.0279	1.4128	−0.0122	−0.0066	1.9663
−0.8	0.0406	0.0320	1.5356	−0.0117	−0.0059	1.9882
−0.7	0.0397	0.0282	1.4179	−0.0096	−0.0040	2.1993
−0.6	0.0415	0.0310	1.4027	−0.0083	−0.0049	2.0152
−0.5	0.0422	0.0297	1.4033	−0.0075	−0.0052	2.4178
−0.4	0.0417	0.0303	1.3166	−0.0059	−0.0032	2.2028
−0.3	0.0395	0.0255	1.2567	−0.0043	−0.0025	2.5356
−0.2	0.0358	0.0259	1.4754	−0.0045	−0.0008	2.0198
−0.1	0.0378	0.0267	1.4131	−0.0028	−0.0013	2.0848
0.0	0.0368	0.0227	1.5037	−0.0003	0.0001	2.7077
0.1	0.0359	0.0265	1.3793	−0.0018	0.0008	1.9703
0.2	0.0343	0.0244	1.4047	−0.0011	0.0013	2.0391
0.3	0.0389	0.0269	1.3742	0.0015	0.0008	2.2219
0.4	0.0375	0.0271	1.3353	0.0037	0.0018	2.3266
0.5	0.0433	0.0262	1.3973	0.0060	0.0006	2.6630
0.6	0.0388	0.0286	1.3005	0.0064	0.0038	2.2621
0.7	0.0388	0.0278	1.2744	0.0057	0.0045	2.2858
0.8	0.0371	0.0276	1.2982	0.0080	0.0038	2.3759
0.9	0.0389	0.0316	1.2765	0.0103	0.0048	1.9165
1.0	0.0423	0.0314	1.3265	0.0114	0.0060	2.1414

larger than that of bivariate exponential. The eventh conditional moments has a constant value. The value of the conditional kurtosis is 3 which is larger than that of the conditional normal distribution.

With known *cdf*s of the bivariate Laplace distribution, random vectors could be generated by the inverse probability transformation integral method. The variance-covariance matrices of the sample means and medians obtained from generated random vectors are calculated and compared with each other. We might conclude that the determinant of the variance-covariance matrix of the sample means collected from bivariate Laplace distribution is larger than that of the sample medians for both of the first and second type of the bivariate Laplace distributions.

These multivariate Laplace distributional properties might be used for some statistical inference in a near future such as real problem applications including rare event analysis, econometrics and finance analysis.

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