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MEASURE OF DEPARTURE FROM QUASI-SYMMETRY AND BRADLEY-TERRY MODELS FOR SQUARE CONTINGENCY TABLES WITH NOMINAL CATEGORIES

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ABSTRACT

For square contingency tables with nominal categories, this paper proposes a measure to represent the degree of departure from the quasi-symmetry (QS) model and the Bradley-Terry (BT) model. The measure proposed is expressed by using the Cressie and Read (1984)'s power-divergence or Patil and Taillie (1982)'s diversity index. The measure lies between 0 and 1, and it is useful for comparing the degree of departure from QS or BT in several tables.

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1. INTRODUCTION

For an $R \times R$ square contingency table with the same nominal row and column classifications, let p_{ij} denote the probability that an observation will fall in the i^{th} row and j^{th} column of the table ($i = 1, 2, \ldots, R$; $j = 1, 2, \ldots, R$). The symmetry (S) model is defined by

$$p_{ij} = \phi_{ij} \quad \text{for} \quad i = 1, 2, \ldots, R;\ j = 1, 2, \ldots, R,$$

where $\phi_{ij} = \phi_{ji}$ (Bishop *et al.*, 1975, p. 282). The quasi-symmetry (QS) model, considered by Caussinus (1965), is defined by

$$p_{ij} = \alpha_i \beta_j \psi_{ij} \quad \text{for} \quad i = 1, 2, \ldots, R;\ j = 1, 2, \ldots, R.$$

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where $\psi_{ij} = \psi_{ji}$. Putting $\gamma_i = \alpha_i/\beta_i$ and $\psi^*_{ij} = \beta_i\beta_j\psi_{ij}$, this model may be expressed as

$$p_{ij} = \gamma_i\psi^*_{ij} \quad \text{for } i = 1,2,\ldots,R;\ j = 1,2,\ldots,R,$$

where $\psi^*_{ij} = \psi^*_{ji}$. A special case of this model obtained by putting $\{\alpha_i = \beta_i\}$ or $\{\gamma_i = 1\}$ is the S model. The QS model may also be expressed as

$$D_{ijk} = D_{kji} \quad \text{for } i < j < k, \tag{1.1}$$

where

$$D_{ijk} = p_{ij}p_{jk}p_{ki}, \quad D_{kji} = p_{kj}p_{ji}p_{ik}.$$

From (1.1), the QS model is further expressed as, using the odds-ratios, *e.g.*,

$$\theta_{(i<j;j<k)} = \theta_{(j<k;i<j)} \quad \text{for } i < j < k,$$

where

$$\theta_{(i<j;j<k)} = \frac{p_{ij}p_{jk}}{p_{jj}p_{ik}}, \quad \theta_{(j<k;i<j)} = \frac{p_{ji}p_{kj}}{p_{ki}p_{jj}}.$$

This indicates the symmetry of odds-ratios with respect to the main diagonal of the square table (though the S model indicates the symmetry of cell probabilities $\{p_{ij}\}$).

Let

$$p^c_{ij} = \frac{p_{ij}}{p_{ij} + p_{ji}} \quad \text{for } i \neq j.$$

This indicates the conditional probability that an observation will fall in cell (i,j) on condition that it falls in cell (i,j) or (j,i), $i \neq j$. Then the QS model may be furthermore expressed as

$$p^c_{ij} = \frac{\gamma_i}{\gamma_i + \gamma_j} \quad \text{for } i \neq j. \tag{1.2}$$

From (1.1) and (1.2), the QS model may be expressed as

$$Q_{ijk} = Q_{kji} \quad \text{for } i < j < k, \tag{1.3}$$

where

$$Q_{ijk} = p^c_{ij}p^c_{jk}p^c_{ki}, \quad Q_{kji} = p^c_{kj}p^c_{ji}p^c_{ik}.$$

The QS model is essentially equivalent to the Bradley-Terry (BT) model applied to a set of data from $R(R-1)/2$ paired comparison (Bradley and Terry, 1952). For example, consider the athletic competitions with the outcome for the play of any two teams of R teams. Then, let π_{ij} for $i < j$ denote the probability that team i defeats team j when team i plays team j, and let π_{ji} $(= 1 - \pi_{ij})$ for $i < j$ denote the probability that team j defeats team i when team i plays team j. Then the BT model is defined by (1.2) or (1.3) with $\{p^c_{ij}\}$ replaced by $\{\pi_{ij}\}$, $i \neq j$. The BT model indicates that for the plays of any two teams of teams i, j and k, the probability that team i defeats team j, team j defeats team k, and team k defeats team i, is equal to the probability that j defeats i, i defeats k, and k defeats j.

By the way, Tomizawa (1994) and Tomizawa *et al.* (1998) considered the measures which represent the degree of departure from the S model for square tables with *nominal* categories. Also, Tomizawa *et al.* (2001) considered the measure which represents the degree of departure from the S model for square tables with *ordered* categories. We are now interested in a measure which represents the degree of departure from the QS model and the BT model. Note that Tomizawa (1994) also considered the measures for the other symmetry models; see, *e.g.*, Tomizawa and Saitoh (1998), and Tomizawa and Saitoh (1999).

The purpose of this paper is to propose a power-divergence type measure which represents the degree of departure from the QS model and the BT model for square contingency tables with *nominal* categories. When the QS model (the BT model) does not fit the data well in terms of the goodness-of-fit test, as the next step, it is meaningful to measure what degree the departure from the QS model (the BT model) is, in terms of the measure. The measure proposed would be useful for *comparing* the degree of departure from the QS model (the BT model) in several tables.

2. MEASURE OF DEPARTURE FROM QUASI-SYMMETRY AND BRADLEY-TERRY MODELS

2.1. Measure for quasi-symmetry

For an $R \times R$ square contingency table with *nominal* categories, let n_{ij} denote the observed frequency in the i^{th} row and j^{th} column of the table ($i = 1, 2, \ldots, R$; $j = 1, 2, \ldots, R$). We assume that $\{n_{ij}\}$ have a multinomial distribution,

$$\frac{n!}{\prod_{i=1}^{R}\prod_{j=1}^{R} n_{ij}!}\prod_{i=1}^{R}\prod_{j=1}^{R} p_{ij}^{n_{ij}}, \tag{2.1}$$

where $n = \sum_{i=1}^{R}\sum_{j=1}^{R} n_{ij}$.

Let

$$\Delta = \sum_{i<j<k} (Q_{ijk} + Q_{kji}),$$

and

$$Q^*_{ijk} = \frac{Q_{ijk}}{\Delta}, \quad Q^*_{kji} = \frac{Q_{kji}}{\Delta}, \quad C^*_{ijk} = C^*_{kji} = \frac{1}{2}(Q^*_{ijk} + Q^*_{kji}) \quad \text{for } i < j < k.$$

Assuming that $Q_{ijk} + Q_{kji} \neq 0$ for $i < j < k$, consider a measure defined by

$$\Phi^{(\lambda)}_{QS} = \frac{\lambda(\lambda+1)}{2^\lambda - 1} I^{(\lambda)} \quad \text{for } \lambda > -1,$$

where

$$I^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{i<j<k} \left[Q^*_{ijk} \left\{ \left(\frac{Q^*_{ijk}}{C^*_{ijk}} \right)^\lambda - 1 \right\} + Q^*_{kji} \left\{ \left(\frac{Q^*_{kji}}{C^*_{kji}} \right)^\lambda - 1 \right\} \right],$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. Thus,

$$\Phi^{(0)}_{QS} = \frac{1}{\log 2} I^{(0)},$$

where

$$I^{(0)} = \sum_{i<j<k} \left\{ Q^*_{ijk} \log \left(\frac{Q^*_{ijk}}{C^*_{ijk}} \right) + Q^*_{kji} \log \left(\frac{Q^*_{kji}}{C^*_{kji}} \right) \right\}.$$

Note that $I^{(\lambda)}$ is the power-divergence between $\{Q^*_{ijk}\}$ and $\{C^*_{ijk}\}$, (for $i < j < k$ or $i > j > k$) and especially, $I^{(0)}$ is the Kullback-Leibler information between them. For more details of the power-divergence $I^{(\lambda)}$, see Cressie and Read (1984), and Read and Cressie (1988, p. 15). Also, note that $I^{(\lambda)} = 0$ when the QS model holds, and a real value λ is chosen by the user.

Let

$$Q^c_{ijk} = \frac{Q_{ijk}}{Q_{ijk} + Q_{kji}}, \quad Q^c_{kji} = \frac{Q_{kji}}{Q_{ijk} + Q_{kji}} \quad \text{for } i < j < k.$$

Then the measure may also be expressed as

$$\Phi^{(\lambda)}_{QS} = 1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} \sum_{i<j<k} \left(Q^*_{ijk} + Q^*_{kji} \right) H^{(\lambda)}_{ijk} \quad \text{for } \lambda > -1,$$

where

$$H^{(\lambda)}_{ijk} = \frac{1}{\lambda} \left\{ 1 - \left(Q^c_{ijk}\right)^{\lambda+1} - \left(Q^c_{kji}\right)^{\lambda+1} \right\}.$$

When $\lambda = 0$,

$$\Phi_{QS}^{(0)} = 1 - \frac{1}{\log 2} \sum_{i<j<k} \left(Q_{ijk}^{*} + Q_{kji}^{*}\right) H_{ijk}^{(0)},$$

where

$$H_{ijk}^{(0)} = -Q_{ijk}^{c} \log Q_{ijk}^{c} - Q_{kji}^{c} \log Q_{kji}^{c}.$$

Note that $H_{ijk}^{(\lambda)}$ is the Patil and Taillie (1982)'s diversity index of degree-λ for $\{Q_{ijk}^{c}, Q_{kji}^{c}\}$, which includes the Shannon entropy when $\lambda = 0$, and the Gini concentration when $\lambda = 1$. The measure $\Phi_{QS}^{(\lambda)}$ would represent essentially the weighted sum of the diversity index $H_{ijk}^{(\lambda)}$.

We see that the measure $\Phi_{QS}^{(\lambda)}$ must lie between 0 and 1, and for each λ, (i) $\Phi_{QS}^{(\lambda)} = 0$ if and only if there is a structure of QS in the $R \times R$ table, and (ii) $\Phi_{QS}^{(\lambda)} = 1$ if and only if the degree of departure from QS is the largest, in the sense that $Q_{ijk}^{c} = 0$ (then $Q_{kji}^{c} = 1$) or $Q_{kji}^{c} = 0$ (then $Q_{ijk}^{c} = 1$) for any $i < j < k$.

We point out that $\Phi_{QS}^{(\lambda)} = 1$ indicates that for any $i < j < k$, $p_{ij}p_{jk}p_{ki} = 0$ or $p_{ji}p_{kj}p_{ik} = 0$ holds. Thus, this indicates that for any $i < j < k$, at least one of p_{ij}^{c}, p_{jk}^{c} and p_{ki}^{c} is equal to zero, or at least one of p_{ji}^{c}, p_{kj}^{c} and p_{ik}^{c} is equal to zero; namely, this indicates that for any $i < j < k$, the complete asymmetry arises for at least one pair of symmetric cells, *e.g.*, as $p_{ij}^{c} = 0$, *i.e.*, $p_{ji}^{c} = 1$ (which may indicate the partial complete asymmetry of cell probabilities).

In addition, we note that the QS model is expressed as

$$\frac{\theta_{(i<j;j<k)}}{\theta_{(j<k;i<j)}} = \frac{D_{ijk}}{D_{kji}} = 1 \quad \text{for} \quad i < j < k.$$

Namely, this describes the symmetry of odds-ratios. So, $\Phi_{QS}^{(\lambda)} = 1$ means that $Q_{ijk} = 0$ (then $Q_{kji} > 0$) or $Q_{kji} = 0$ (then $Q_{ijk} > 0$), *i.e.*, $D_{ijk} = 0$ (then $D_{kji} > 0$) or $D_{kji} = 0$ (then $D_{ijk} > 0$); namely this means that $D_{ijk}/D_{kji} = 0$ or ∞ for $i < j < k$ which would indicate the complete *asymmetry* of odds-ratios. Thus, it would be natural to consider that then the departure from the symmetry of odds-ratios (*i.e.*, from QS) is the largest.

According to the power-divergence or the weighted sum of the Patil-Taillie diversity index, $\Phi_{QS}^{(\lambda)}$ represents the degree of departure from QS, and the degree increases as the value of $\Phi_{QS}^{(\lambda)}$ increases.

2.2. Measure for Bradley-Terry model

Consider a set of data from $R(R-1)/2$ paired comparison experiments for R treatments. Let r_{ij} be the number of comparisons for the treatment pair (i,j), and n_{ij} the number that the treatment i exceeds the treatment j in the r_{ij} comparisons. Assuming that there is no tie we have $r_{ij} = r_{ji} = n_{ij} + n_{ji}$. Let π_{ij} be the probability that the treatment i exceeds the treatment j in a single comparison of the pair. We have $\pi_{ij} + \pi_{ji} = 1$ excluding the possibility of ties. The probability for $\{n_{ij}\}$, $i \neq j$, is then the product of $R(R-1)/2$ binomials,

$$\prod_{1 \le i < j \le R} \frac{r_{ij}!}{n_{ij}! n_{ji}!} \pi_{ij}^{n_{ij}} \pi_{ji}^{n_{ji}}. \tag{2.2}$$

The BT model is defined by

$$\pi_{ij} = \frac{\lambda_i}{\lambda_i + \lambda_j} \quad \text{for} \quad i \neq j.$$

Also, this model may be expressed as

$$G_{ijk} = G_{kji} \quad \text{for} \quad i < j < k,$$

where

$$G_{ijk} = \pi_{ij}\pi_{jk}\pi_{ki}, \; G_{kji} = \pi_{kj}\pi_{ji}\pi_{ik}.$$

We assume that $G_{ijk} + G_{kji} \neq 0$ for $i < j < k$. Then, the BT model may be expressed as

$$G^c_{ijk} = G^c_{kji} \quad \text{for} \quad i < j < k,$$

where

$$G^c_{ijk} = \frac{G_{ijk}}{G_{ijk} + G_{kji}}, \; G^c_{kji} = \frac{G_{kji}}{G_{ijk} + G_{kji}}.$$

For example, consider the athletic competitions described in Section 1. The BT model indicates that the probability that team i defeats team j, team j defeats team k, and team k defeats team i, is equal to the probability that j defeats i, i defeats k, and k defeats j. We shall now say that the *stochastic* three-way deadlock arises when the probability that i defeats j, j defeats k, and k defeats i is larger or smaller than (*i.e.*, not equal to) the probability for the reverse order. Thus, the BT model may indicate that for any three teams of R teams, the stochastic three-way deadlock does not arise.

It is seen that the BT model is essentially equivalent to the QS model. Thus, we shall define the measure $\Phi_{BT}^{(\lambda)}$ (for $\lambda > -1$), which represents the degree of departure from the BT model, by $\Phi_{QS}^{(\lambda)}$ with $\{p_{ij}^c\}$ replaced by $\{\pi_{ij}\}$.

We note that the measure $\Phi_{BT}^{(\lambda)}$ lies between 0 and 1, and also (i) $\Phi_{BT}^{(\lambda)} = 0$ if and only if the BT model holds, and (ii) $\Phi_{BT}^{(\lambda)} = 1$ if and only if the degree of departure from the BT model is a maximum. Thus, *e.g.*, for the case of athletic competitions, (i) $\Phi_{BT}^{(\lambda)} = 0$ indicates that for any three teams of R teams, the stochastic three-way deadlock does not arise, however, (ii) $\Phi_{BT}^{(\lambda)} = 1$ indicates that for any three teams of R teams, the *strongest* stochastic three-way deadlock arises; namely, the conditional probability that i defeats j, j defeats k, and k defeats i on condition that i defeats j, j defeats k, and k defeats i, or j defeats i, i defeats k, and k defeats j, is 1 or 0; because then $G_{ijk}^c = 1$ (then $G_{kji}^c = 0$) or $G_{kji}^c = 1$ (then $G_{ijk}^c = 0$).

3. Approximate Confidence Intervals for Measures

We shall consider an approximate standard error and large-sample confidence interval for the measures $\Phi_{QS}^{(\lambda)}$ and $\Phi_{BT}^{(\lambda)}$ (say, $\Phi^{(\lambda)}$), using the delta method, as described by Bishop *et al.* (1975, Section 14.6) and Agresti (1990, Section 12.1). The sample version of $\Phi_{QS}^{(\lambda)}$, *i.e.*, $\widehat{\Phi}_{QS}^{(\lambda)}$, is given by $\Phi_{QS}^{(\lambda)}$ with $\{p_{ij}\}$ replaced by $\{\widehat{p}_{ij}\}$, where $\widehat{p}_{ij} = n_{ij}/n$. Similarly, the $\widehat{\Phi}_{BT}^{(\lambda)}$ is given by $\Phi_{BT}^{(\lambda)}$ with $\{\pi_{ij}\}$ replaced by $\{\widehat{\pi}_{ij}\}$, where $\widehat{\pi}_{ij} = n_{ij}/r_{ij}$. Using the delta method, $\widehat{\Phi}^{(\lambda)}$ has asymptotically a normal distribution with mean $\Phi^{(\lambda)}$ and variance $\sigma^2[\widehat{\Phi}^{(\lambda)}]$. The $\sigma^2[\widehat{\Phi}^{(\lambda)}]$ are given in Appendix.

The measure $\widehat{\Phi}_{QS}^{(\lambda)}$ is applied to a multinomial sampling, and $\widehat{\Phi}_{BT}^{(\lambda)}$ is applied to the independent binomial sampling. So, $\sigma^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ with $\{p_{ij}^c\}$ replaced by $\{\pi_{ij}\}$, $i \neq j$, is not always identical to $\sigma^2[\widehat{\Phi}_{BT}^{(\lambda)}]$ except when $\{p_{ij} + p_{ji}\}$ are equal to $\{(n_{ij}+n_{ji})/n\}$ in $\sigma^2[\widehat{\Phi}_{QS}^{(\lambda)}]$; see Appendix. Let $\widehat{\sigma}^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ denote $\sigma^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ with $\{p_{ij}\}$ replaced by $\{\widehat{p}_{ij}\}$. Similarly, let $\widehat{\sigma}^2[\widehat{\Phi}_{BT}^{(\lambda)}]$ denote $\sigma^2[\widehat{\Phi}_{BT}^{(\lambda)}]$ with $\{\pi_{ij}\}$ replaced by $\{\widehat{\pi}_{ij}\}$. Noting that $\{\widehat{p}_{ij} + \widehat{p}_{ji} = (n_{ij} + n_{ji})/n\}$ in $\widehat{\sigma}^2[\widehat{\Phi}_{QS}^{(\lambda)}]$, we point out that the estimated variance $\widehat{\sigma}^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ is theoretically identical to the estimated variance $\widehat{\sigma}^2[\widehat{\Phi}_{BT}^{(\lambda)}]$; see Appendix.

The approximate confidence intervals for the measure $\Phi^{(\lambda)}$ can be obtained using the $\widehat{\sigma}[\widehat{\Phi}^{(\lambda)}]$.

TABLE 1 *Cross-classification of father's and his son's social class; taken from Hashimoto (1999, p. 151)*

(a) Examined in 1955.

	Son's class					
Father's class	*Capitalist*	*New-Middle*	*Labor*	*Self-Support*	*Peasantry*	*Total*
Capitalist	39	39	39	57	23	197
New-Middle	12	78	23	23	37	173
Labor	6	16	78	23	20	143
Self-Support	18	80	79	126	31	334
Peasantry	28	106	136	122	628	1020
Total	103	319	355	351	739	1867

(b) Examined in 1995

	Son's class					
Father's class	*Capitalist*	*New-Middle*	Labor	*Self-Support*	*Peasantry*	*Total*
Capitalist	68	48	36	23	1	176
New-Middle	33	191	102	33	3	362
Labor	25	147	229	34	2	437
Self-Support	48	119	146	129	5	447
Peasantry	40	126	192	82	88	528
Total	214	631	705	301	99	1950

4. EXAMPLES

4.1. Example 1

Consider the data in Table 1, taken from Hashimoto (1999, p. 151). These data describe the cross-classification of father's and his son's social class in Japan which were examined in 1955 and 1995.

For these data, we shall apply the measure $\Phi^{(\lambda)}_{QS}$. Since the confidence intervals for the measure $\Phi^{(\lambda)}_{QS}$ applied to the data in Table 1(a) do not contain zero for each λ (see Table 2(a)), these would indicate that there is not a structure of the QS model between the social ranks of father-son pairs. On the other hand, since the confidence intervals for the measure $\Phi^{(\lambda)}_{QS}$ applied to the data in Table 1(b) contain zero for each λ (see Table 2(b)), these may indicate that there is a structure of the QS model between them.

When the degrees of departure from the QS model in Tables 1(a) and 1(b) are compared using the estimated measure $\widehat{\Phi}^{(\lambda)}_{QS}$ (see Table 2), for each λ, the value of $\widehat{\Phi}^{(\lambda)}_{QS}$ is greater for Table 1(a) than for Table 1(b). So, the degree of departure

TABLE 2 *Estimate of measure* $\Phi_{QS}^{(\lambda)}$, *estimated approximate standard error for* $\widehat{\Phi}_{QS}^{(\lambda)}$, *and approximate 95% confidence interval for* $\Phi_{QS}^{(\lambda)}$, *applied to Tables 1(a) and 1(b)*

(a) For Table 1(a)

λ	*Estimated measure*	*Standard error*	*Confidence interval*
−0.2	0.078	0.032	(0.015, 0.141)
0.0	0.089	0.036	(0.018, 0.160)
0.2	0.098	0.039	(0.021, 0.175)
0.6	0.110	0.043	(0.026, 0.195)
1.0	0.117	0.045	(0.028, 0.205)
1.8	0.118	0.045	(0.029, 0.207)
2.4	0.113	0.044	(0.026, 0.199)

(a) For Table 1(b)

λ	*Estimated measure*	*Standard error*	*Confidence interval*
−0.2	0.023	0.020	(−0.015, 0.062)
0.0	0.027	0.023	(−0.018, 0.072)
0.2	0.030	0.025	(−0.020, 0.080)
0.6	0.035	0.029	(−0.022, 0.092)
1.0	0.037	0.031	(−0.024, 0.098)
1.8	0.038	0.031	(−0.024, 0.099)
2.4	0.036	0.030	(−0.023, 0.094)

from the QS model is greater for Table 1(a) than for Table 1(b). Namely, the data in Table 1(a) rather than in Table 1(b) are estimated to be close to the *maximum* departure from the QS model, *i.e.*, the complete asymmetry of odds-ratios (or the partial complete asymmetry of cell probabilities).

We note that the *maximum* departure from the QS model indicates that for any three father-son pairs with the social ranks (i,j), (j,k) and (k,i), $i \neq j$, $j \neq k$, $k \neq i$, the probability that the ranks for first father-son pair moved to the son's rank j from his father's rank i, those for second pair moved to k from j, and those for third pair moved to i from k, is zero (not zero), however, the probability that those for first pair moved to the son's rank i from his father's rank j, those for second pair moved to k from i, and those for third pair moved to j from k, is not zero (zero); namely, the *stochastic circular* social mobility arises among any three father-son pairs (when the degree of departure from the QS model is a maximum).

TABLE 3 *Values of power-divergence statistic $W_{QS}^{(\lambda)}$ (with 6 degrees of freedom), applied to Tables 1(a) and 1(b)*

λ	*For Table 1(a)*	*For Table 1(b)*
−0.2	22.23	5.83
0.0	22.13	5.83
0.2	22.06	5.83
0.6	21.95	5.83
1.0	21.90	5.85
1.8	21.99	5.91

NOTE. We denote the power-divergence statistic for testing goodness-of-fit of the QS model with $(R-1)(R-2)/2 = 6$ degrees of freedom by $W_{QS}^{(\lambda)}$. See Cressie and Read (1984) and Read and Cressie (1988, p. 15) for details of the power-divergence test statistic. In particular, $W_{QS}^{(0)}$ and $W_{QS}^{(1)}$ are the likelihood ratio and the Pearson's chi-squared statistics, respectively. Table 3 gives the values of $W_{QS}^{(\lambda)}$ applied to the data in Tables 1(a) and 1(b). The data in Table 1(a) fit the QS model poorly, however, the data in Table 1(b) fit the QS model well.

TABLE 4 *Score sheet of the Pacific League in Japan in 1995 and 2002*

(a) 1995

	Lions	*Buffaloes*	*Hawks*	*Marines*	*Fighters*	*Bluewave*	*Total*
Lions	-	18	14	15	15	5	67
Buffaloes	7	-	10	13	11	8	49
Hawks	9	16	-	5	11	13	54
Marines	10	12	21	-	13	13	69
Fighters	10	14	14	13	-	8	59
Bluewave	21	18	13	12	18	-	82
Total	57	78	72	58	68	47	380

(b) 2002

	Lions	*Buffaloes*	*Hawks*	*Marines*	*Fighters*	*Bluewave*	*Total*
Lions	-	15	16	19	18	22	90
Buffaloes	13	-	14	12	15	19	73
Hawks	12	13	-	18	15	15	73
Marines	8	16	10	-	18	15	67
Fighters	10	13	12	10	-	16	61
Bluewave	6	8	13	13	10	-	50
Total	49	65	65	72	76	87	414

4.2. Example 2

The data in Tables 4 gives the results of the professional baseball league in Japan in 1995 and 2002. For instance, from Lions's perspective, the (Lions, Buffaloes) results in 1995 correspond to 18 successes and 7 failures in 25 trials.

TABLE 5 *Estimate of measure* $\Phi_{BT}^{(\lambda)}$, *estimated approximate standard error for* $\widehat{\Phi}_{BT}^{(\lambda)}$, *and approximate 95% confidence interval for* $\Phi_{BT}^{(\lambda)}$, *applied to Tables 4(a) and 4(b)*

(a) For Table 4(a)

λ	*Estimated measure*	*Standard error*	*Confidence interval*
−0.2	0.152	0.058	(0.039, 0.266)
0.0	0.173	0.064	(0.047, 0.299)
0.2	0.189	0.068	(0.055, 0.323)
0.6	0.210	0.074	(0.066, 0.354)
1.0	0.221	0.076	(0.072, 0.370)
1.8	0.223	0.076	(0.074, 0.372)
2.4	0.215	0.075	(0.068, 0.362)

(b) For Table 4(b)

λ	*Estimated measure*	*Standard error*	*Confidence interval*
−0.2	0.057	0.038	(−0.017, 0.132)
0.0	0.066	0.044	(−0.019, 0.152)
0.2	0.074	0.048	(−0.021, 0.168)
0.6	0.084	0.054	(−0.022, 0.190)
1.0	0.089	0.057	(−0.023, 0.201)
1.8	0.090	0.058	(−0.023, 0.204)
2.4	0.086	0.055	(−0.023, 0.194)

For these data, we shall apply the measure $\Phi_{BT}^{(\lambda)}$. Since the confidence intervals for the measure $\Phi_{BT}^{(\lambda)}$ applied to the data in Table 4(a) do not contain zero for each λ (see Table 5(a)), these would indicate that there is not a structure of the BT model between the teams in Pacific League in 1995. On the other hand, since the confidence intervals for the measure $\Phi_{BT}^{(\lambda)}$ applied to the data in Table 4(b) contain zero for each λ (see Table 5(b)), these may indicate that there is a structure of the BT model between the teams in Pacific League in 2002.

When the degrees of departure from the BT model in Tables 4(a) and 4(b) are compared using the estimated measure $\widehat{\Phi}_{BT}^{(\lambda)}$, for each λ, the value of $\widehat{\Phi}_{BT}^{(\lambda)}$ is greater for Table 4(a) than for Table 4(b). So, the degree of departure from the

BT model is greater for Table 4(a) than for Table 4(b). Namely, the data in 1995 rather than in 2002 are estimated to be close to a situation with the *strongest* stochastic three-way deadlock, which indicates that for any three teams, i, j and k, the conditional probability that team i defeats team j, team j defeats team k, and team k defeats team i on condition that i defeats j, j defeats k, and k defeats i, or j defeats i, i defeats k, and k defeats j, is 1 or 0.

TABLE 6 *Values of power-divergence statistic* $W_{BT}^{(\lambda)}$ *(with 10 degrees of freedom), applied to Tables 4(a) and 4(b)*

λ	*For Table 4(a)*	*For Table 4(b)*
−0.2	21.81	8.40
0.0	21.55	8.38
0.2	21.33	8.37
0.6	20.98	8.35
1.0	20.75	8.34
1.8	20.61	8.34

NOTE. We denote the power-divergence statistic for testing goodness-of-fit of the BT model with $(R-1)(R-2)/2 = 10$ degrees of freedom by $W_{BT}^{(\lambda)}$. Table 6 gives the values of $W_{BT}^{(\lambda)}$ applied to the data in Tables 4(a) and 4(b). The data in Table 4(a) fit the BT model poorly, however, the data in Table 4(b) fit the BT model well.

5. CONCLUDING REMARKS

Since the measures $\Phi_{QS}^{(\lambda)}$ ($\Phi_{BT}^{(\lambda)}$) always range between 0 and 1 independent of the dimension R and sample size n ($\{r_{ij}\}$), those may be useful for *comparing* the degree of departure from the QS (BT) model in several tables.

The measure $\Phi_{QS}^{(\lambda)}$ would be useful when we want to see with a single summary measure what degree the departure from the QS model (*i.e.*, the symmetry of odds-ratios) is toward the complete asymmetry of odds-ratios or the partial complete asymmetry of cell probabilities.

The measure $\Phi_{BT}^{(\lambda)}$ would be useful when we want to see with a single summary measure, for example, for the athletic competitions, how strong the stochastic three-way deadlock for any three teams of R teams arises toward a situation with the *strongest* stochastic three-way deadlock, which indicates that the conditional probability that i defeats j, j defeats k, and k defeats i on condition that i defeats

j, j defeats k, and k defeats i, or j defeats i, i defeats k, and k defeats j, is 1 or 0; thus, this indicates that at least one team, *e.g.*, team i, among any three teams i, j and k of R teams, is always defeated by one of the other teams.

The measures $\Phi_{QS}^{(\lambda)}$ ($\Phi_{BT}^{(\lambda)}$) are invariant under the arbitrary simultaneous permutations of row and column categories, and therefore it is possible to apply these measures for analyzing the data on a nominal scale, and also possible for analyzing the data on an ordinal scale if one may not use the information about the order of listing the categories.

We note that the properties of measures $\Phi_{QS}^{(\lambda)}$ ($\Phi_{BT}^{(\lambda)}$) are similar to the previous works of Tomizawa in References; and so we omit the various properties of the measures.

TABLE 7 *Artificial data (Tables 7(a) and 7(b)), the corresponding estimated measure $\widehat{\Phi}_{QS}^{(\lambda)}$, estimated approximate standard error for $\widehat{\Phi}_{QS}^{(\lambda)}$, approximate 95% confidence interval for $\Phi_{QS}^{(\lambda)}$, and the values of power-divergence statistic $W_{QS}^{(\lambda)}$ (with 3 degrees of freedom), applied to Tables 7(a) and 7(b)*

(a) n=11520 (sample size)

	(1)	*(2)*	*(3)*	*(4)*
(1)	2004	600	540	360
(2)	120	1572	600	828
(3)	540	600	240	1200
(4)	144	600	576	996

(b) n=51 (sample size)

	(1)	*(2)*	*(3)*	*(4)*
(1)	1	2	1	4
(2)	1	3	3	1
(3)	6	2	1	7
(4)	8	5	4	2

(c) Values of $\Phi_{QS}^{(\lambda)}$

	For Table 7(a)			*For Table 7(b)*		
λ	*Estimated measure*	*Standard error*	*Confidence interval*	*Estimated measure*	*Standard error*	*Confidence interval*
−0.2	0.125	0.017	(0.091, 0.160)	0.424	0.288	(−0.140, 0.988)
0.0	0.143	0.019	(0.105, 0.181)	0.464	0.294	(−0.112, 1.039)
0.6	0.175	0.023	(0.131, 0.220)	0.523	0.289	(−0.044, 1.090)
1.0	0.185	0.023	(0.139, 0.231)	0.536	0.285	(−0.022, 1.095)
1.6	0.188	0.024	(0.141, 0.234)	0.540	0.283	(−0.015, 1.095)

(d) Values of $W_{QS}^{(\lambda)}$

λ	*For Table 7(a)*	*For Table 7(b)*
−0.2	173.18	4.36
0.0	170.91	4.36
0.6	165.25	4.40
1.0	162.33	4.50
1.6	159.04	4.74

As a referee comments, we note that it may be dangerous to compare two data sets (say, A and B) with the great different sample sizes by using only the estimated measure $\widehat{\Phi}^{(\lambda)}_{QS}$ (or $\widehat{\Phi}^{(\lambda)}_{BT}$). For example, consider the data in Tables 7(a) and 7(b) (the sample sizes are $n = 11520$ for Table 7(a) and $n = 51$ for Table 7(b)). We see now that *e.g.*, for $\lambda = 1$, the values of $\widehat{\Phi}^{(1)}_{QS}$ are 0.185 for Table 7(a) and 0.536 for Table 7(b) (see Table 7(c)); however, the values of power-divergence test statistic $W^{(1)}_{QS}$ are 162.33 for Table 7(a) and 4.50 for Table 7(b) with 3 degrees of freedom (see Table 7(d)). So, there is a situation that the data set A is statistical significant and the data set B is not, but the values of $\widehat{\Phi}^{(\lambda)}_{QS}$ indicate the reverse relation, when the sample size of data set A is far bigger than the sample size of data set B. It seems difficult that we describe the relation between the degree of departure from the model and the results of testing hypothesis of the model. Since the measure $\widehat{\Phi}^{(\lambda)}_{QS}$ does not take account of the sample sizes while the p values do, it may be dangerous to compare the results of testing hypothesis of the model using only the measure $\widehat{\Phi}^{(\lambda)}_{QS}$.

When we consider the values of standard error for the measure $\widehat{\Phi}^{(\lambda)}_{QS}$ and compare the values of confidence interval of $\Phi^{(\lambda)}_{QS}$ for Tables 7(a) and 7(b) (see Table 7(c)), those for Table 7(a) do not contain the zero, but those for Table 7(b) do. These would indicate that Table 7(a) does not have a structure of QS but Table 7(b) may have it. Therefore, we should pay attention to compare the degree of departure from QS by using only the values of estimated measure $\widehat{\Phi}^{(\lambda)}_{QS}$ (not the confidence interval of $\Phi^{(\lambda)}_{QS}$) when we compare the two data sets with the great different sample sizes (with the great different standard errors).

For analyzing the degree of departure from the QS (BT) model, the analyst would check whether or not the QS (BT) model holds by using a test statistic, such as $W^{(\lambda)}_{QS}$ ($W^{(\lambda)}_{BT}$). We note then that even if it is judged that there is a structure of the QS (BT) model in the table by the test statistic, it would be meaningful to measure the degree of departure from the QS (BT) model toward the maximum departure by using the estimated measure $\widehat{\Phi}^{(\lambda)}_{QS}$ ($\widehat{\Phi}^{(\lambda)}_{BT}$).

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APPENDIX

A. Appendix 1

First, we shall consider $\sigma^2[\widehat{\Phi}^{(\lambda)}_{BT}]$. Suppose that the probability for the data $\{n_{ij}\}$, $i \neq j$, is the product of $R(R-1)/2$ binomials given by (2.2). Then let $\widehat{\boldsymbol{\pi}}'$ be the $1 \times R(R-1)$ vector

$$\widehat{\boldsymbol{\pi}}' = \left(\widehat{\boldsymbol{\pi}}'_{(12)}, \widehat{\boldsymbol{\pi}}'_{(13)}, \ldots, \widehat{\boldsymbol{\pi}}'_{(R-1,R)}\right),$$

where "$'$" denotes the transpose and

$$\begin{aligned} \widehat{\boldsymbol{\pi}}'_{(ij)} &= (\widehat{\pi}_{ij}, \widehat{\pi}_{ji}),\ i < j, \\ \widehat{\pi}_{ij} &= \frac{n_{ij}}{r_{ij}} = \frac{n_{ij}}{n_{ij} + n_{ji}}. \end{aligned}$$

Also, let us define the vector $\boldsymbol{\pi}$ in terms of π_{ij}'s in the same way as $\widehat{\boldsymbol{\pi}}$. Then $\widehat{\boldsymbol{\pi}}$ is asymptotically distributed as normal $N(\boldsymbol{\pi}, \boldsymbol{V}(\boldsymbol{\pi}))$, where $\boldsymbol{V}(\boldsymbol{\pi})$ is the $R(R-1) \times R(R-1)$ matrix,

$$\boldsymbol{V}(\boldsymbol{\pi}) = \begin{pmatrix} \boldsymbol{V}_{12}(\boldsymbol{\pi}) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{V}_{13}(\boldsymbol{\pi}) & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{V}_{R-1,R}(\boldsymbol{\pi}) \end{pmatrix},$$

where

$$\boldsymbol{V}_{ij}(\boldsymbol{\pi}) = \frac{1}{r_{ij}} \begin{pmatrix} \pi_{ij}(1-\pi_{ij}) & -\pi_{ij}\pi_{ji} \\ -\pi_{ij}\pi_{ji} & \pi_{ji}(1-\pi_{ji}) \end{pmatrix}, \quad i < j.$$

We also obtain

$$\widehat{\Phi}^{(\lambda)}_{BT} = \Phi^{(\lambda)}_{BT} + d_1(\boldsymbol{\pi})(\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}) + o(\|\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}\|),$$

where $d_1(\boldsymbol{\pi}) = \partial\Phi^{(\lambda)}_{BT}/\partial\boldsymbol{\pi}'$ being the $1 \times R(R-1)$ vector. Using the delta method, $\widehat{\Phi}^{(\lambda)}_{BT}$ is asymptotically distributed as normal $N(\Phi^{(\lambda)}_{BT}, \sigma^2[\widehat{\Phi}^{(\lambda)}_{BT}])$, where

$$\begin{aligned} \sigma^2[\widehat{\Phi}^{(\lambda)}_{BT}] &= d_1(\boldsymbol{\pi})\boldsymbol{V}(\boldsymbol{\pi})d_1(\boldsymbol{\pi})' \\ &= \frac{1}{(\Delta^{BT})^2} \sum_{s=1}^{R-1} \sum_{t=s+1}^{R} \frac{1}{r_{st}} \left\{ \frac{1}{\pi_{st}} \left(W^{(\lambda)}_{st}\right)^2 + \frac{1}{\pi_{ts}} \left(V^{(\lambda)}_{ts}\right)^2 - \left(W^{(\lambda)}_{st} + V^{(\lambda)}_{ts}\right)^2 \right\}. \end{aligned}$$

where $\Delta^{BT} = \sum_{i<j<k} (G_{ijk} + G_{kji})$, and for $\lambda > -1$ and $\lambda \neq 0$,

$$
\begin{aligned}
W_{st}^{(\lambda)} = & \Bigg[\sum_{k=t+1}^{R} \left\{ G_{stk} \left(G_{stk}^{c}\right)^{\lambda} + \lambda G_{stk} G_{kts}^{c} \left(\left(G_{stk}^{c}\right)^{\lambda} - \left(G_{kts}^{c}\right)^{\lambda} \right) \right\} \\
& + \sum_{i=1}^{s-1} \left\{ G_{ist} \left(G_{ist}^{c}\right)^{\lambda} + \lambda G_{ist} G_{tsi}^{c} \left(\left(G_{ist}^{c}\right)^{\lambda} - \left(G_{tsi}^{c}\right)^{\lambda} \right) \right\} \\
& + \sum_{j=s+1}^{t-1} \left\{ G_{tjs} \left(G_{tjs}^{c}\right)^{\lambda} + \lambda G_{tjs} G_{sjt}^{c} \left(\left(G_{tjs}^{c}\right)^{\lambda} - \left(G_{sjt}^{c}\right)^{\lambda} \right) \right\} \\
& - \left(\sum_{k=t+1}^{R} G_{stk} + \sum_{i=1}^{s-1} G_{ist} + \sum_{j=s+1}^{t-1} G_{tjs} \right) \cdot \frac{\left(2^{\lambda} - 1\right) \Phi_{BT}^{(\lambda)} + 1}{2^{\lambda}} \Bigg] \frac{2^{\lambda}}{2^{\lambda} - 1},
\end{aligned}
$$

$$
\begin{aligned}
V_{ts}^{(\lambda)} = & \Bigg[\sum_{k=t+1}^{R} \left\{ G_{kts} \left(G_{kts}^{c}\right)^{\lambda} + \lambda G_{kts} G_{stk}^{c} \left(\left(G_{kts}^{c}\right)^{\lambda} - \left(G_{stk}^{c}\right)^{\lambda} \right) \right\} \\
& + \sum_{i=1}^{s-1} \left\{ G_{tsi} \left(G_{tsi}^{c}\right)^{\lambda} + \lambda G_{tsi} G_{ist}^{c} \left(\left(G_{tsi}^{c}\right)^{\lambda} - \left(G_{ist}^{c}\right)^{\lambda} \right) \right\} \\
& + \sum_{j=s+1}^{t-1} \left\{ G_{sjt} \left(G_{sjt}^{c}\right)^{\lambda} + \lambda G_{sjt} G_{tjs}^{c} \left(\left(G_{sjt}^{c}\right)^{\lambda} - \left(G_{tjs}^{c}\right)^{\lambda} \right) \right\} \\
& - \left(\sum_{k=t+1}^{R} G_{kts} + \sum_{i=1}^{s-1} G_{tsi} + \sum_{j=s+1}^{t-1} G_{sjt} \right) \cdot \frac{\left(2^{\lambda} - 1\right) \Phi_{BT}^{(\lambda)} + 1}{2^{\lambda}} \Bigg] \frac{2^{\lambda}}{2^{\lambda} - 1},
\end{aligned}
$$

and where for $\lambda = 0$,

$$
\begin{aligned}
W_{st}^{(0)} = & \Bigg[\sum_{k=t+1}^{R} G_{stk} \log G_{stk}^{c} + \sum_{i=1}^{s-1} G_{ist} \log G_{ist}^{c} + \sum_{j=s+1}^{t-1} G_{tjs} \log G_{tjs}^{c} \\
& - \left\{ \sum_{k=t+1}^{R} G_{stk} + \sum_{i=1}^{s-1} G_{ist} + \sum_{j=s+1}^{t-1} G_{tjs} \right\} \left(\Phi_{BT}^{(0)} - 1 \right) \log 2 \Bigg] \frac{1}{\log 2},
\end{aligned}
$$

$$
\begin{aligned}
V_{ts}^{(0)} = & \Bigg[\sum_{k=t+1}^{R} G_{kts} \log G_{kts}^{c} + \sum_{i=1}^{s-1} G_{tsi} \log G_{tsi}^{c} + \sum_{j=s+1}^{t-1} G_{sjt} \log G_{sjt}^{c} \\
& - \left\{ \sum_{k=t+1}^{R} G_{kts} + \sum_{i=1}^{s-1} G_{tsi} + \sum_{j=s+1}^{t-1} G_{sjt} \right\} \left(\Phi_{BT}^{(0)} - 1 \right) \log 2 \Bigg] \frac{1}{\log 2}.
\end{aligned}
$$

The estimate of $\sigma^2[\widehat{\Phi}^{(\lambda)}_{BT}]$ is given by $\widehat{\sigma}^2[\widehat{\Phi}^{(\lambda)}_{BT}]$ which is $\sigma^2[\widehat{\Phi}^{(\lambda)}_{BT}]$ with $\{\pi_{ij}\}$ replaced by $\{\widehat{\pi}_{ij}\}$.

B. Appendix 2

Secondly, we shall consider $\sigma^2[\widehat{\Phi}^{(\lambda)}_{QS}]$. Suppose that the data $\{n_{ij}\}$ have a multinomial distribution given by (2.1). Let $\widehat{\boldsymbol{p}}'$ be the $1 \times R^2$ vector

$$\widehat{\boldsymbol{p}}' = \left(\widehat{\boldsymbol{p}}'_{(12)}, \widehat{\boldsymbol{p}}'_{(13)}, \ldots, \widehat{\boldsymbol{p}}'_{(R-1,R)}, \widehat{p}_{11}, \ldots, \widehat{p}_{RR}\right),$$

where

$$\begin{aligned} \widehat{\boldsymbol{p}}'_{(ij)} &= \left(\widehat{p}_{ij}, \widehat{p}_{ji}\right), \quad i<j, \\ \widehat{p}_{ij} &= \frac{n_{ij}}{n}, \ \widehat{p}_{ii} = \frac{n_{ii}}{n}. \end{aligned}$$

Also, let us define the vector $\boldsymbol{p}$ in terms of p_{ij}'s in the same way as $\widehat{\boldsymbol{p}}$. Then $\widehat{\boldsymbol{p}}$ is asymptotically distributed as normal $N(\boldsymbol{p}, \boldsymbol{\Sigma}_1(\boldsymbol{p}))$, where $\boldsymbol{\Sigma}_1(\boldsymbol{p})$ is the $R^2 \times R^2$ matrix,

$$\boldsymbol{\Sigma}_1(\boldsymbol{p}) = \frac{1}{n}\left(D\left(\boldsymbol{p}\right) - \boldsymbol{p}\boldsymbol{p}'\right),$$

where $D(\boldsymbol{p})$ denotes a diagonal matrix with the i^{th} element of $\boldsymbol{p}$ as the i^{th} diagonal element. We also obtain

$$\widehat{\Phi}^{(\lambda)}_{QS} = \Phi^{(\lambda)}_{QS} + d_2(\boldsymbol{p})(\widehat{\boldsymbol{p}} - \boldsymbol{p}) + o(\|\widehat{\boldsymbol{p}} - \boldsymbol{p}\|),$$

where $d_2(\boldsymbol{p}) = \partial\Phi^{(\lambda)}_{QS}/\partial\boldsymbol{p}'$ being the $1 \times R^2$ vector. Thus, $\widehat{\Phi}^{(\lambda)}_{QS}$ is asymptotically distributed as normal $N(\Phi^{(\lambda)}_{QS}, \sigma^2[\widehat{\Phi}^{(\lambda)}_{QS}])$, where

$$\sigma^2[\widehat{\Phi}^{(\lambda)}_{QS}] = d_2(\boldsymbol{p})\boldsymbol{\Sigma}_1(\boldsymbol{p})d_2(\boldsymbol{p})'.$$

Let $\boldsymbol{p}^{c'}$ be the $1 \times R(R-1)$ vector

$$\boldsymbol{p}^{c'} = \left(\boldsymbol{p}^{c'}_{(12)}, \boldsymbol{p}^{c'}_{(13)}, \ldots, \boldsymbol{p}^{c'}_{(R-1,R)}\right),$$

where

$$\begin{aligned} \boldsymbol{p}^{c'}_{(ij)} &= \left(p^c_{ij}, p^c_{ji}\right), \quad i<j, \\ p^c_{ij} &= \frac{p_{ij}}{p_{ij} + p_{ji}}. \end{aligned}$$

Then, noting that $\Phi_{QS}^{(\lambda)}$ is a function of only $\{p_{ij}^c\}$, we obtain

$$d_2(\boldsymbol{p}) = \frac{\partial \Phi_{QS}^{(\lambda)}}{\partial \boldsymbol{p}^{c'}} \cdot \frac{\partial \boldsymbol{p}^c}{\partial \boldsymbol{p}'}.$$

Note that $\partial \Phi_{QS}^{(\lambda)}/\partial \boldsymbol{p}^{c'}$ is the $1 \times R(R-1)$ vector and $\partial \boldsymbol{p}^c/\partial \boldsymbol{p}'$ is the $R(R-1) \times R^2$ matrix. By obtaining $\partial \boldsymbol{p}^c/\partial \boldsymbol{p}'$, we can see that $\sigma^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ is expressed as

$$\sigma^2[\widehat{\Phi}_{QS}^{(\lambda)}] = \frac{\partial \Phi_{QS}^{(\lambda)}}{\partial \boldsymbol{p}^{c'}} \cdot \boldsymbol{\Sigma}_2(\boldsymbol{p}) \cdot \frac{\partial \Phi_{QS}^{(\lambda)}}{\partial \boldsymbol{p}^{c}},$$

where $\boldsymbol{\Sigma}_2(\boldsymbol{p})$ is the $R(R-1) \times R(R-1)$ matrix,

$$\boldsymbol{\Sigma}_2(\boldsymbol{p}) = \begin{pmatrix} \boldsymbol{\Sigma}_{12}(\boldsymbol{p}) & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{13}(\boldsymbol{p}) & & \vdots \\ \vdots & & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{\Sigma}_{R-1,R}(\boldsymbol{p}) \end{pmatrix},$$

where

$$\boldsymbol{\Sigma}_{ij}(\boldsymbol{p}) = \frac{1}{n\,(p_{ij}+p_{ji})} \begin{pmatrix} p_{ij}^c(1-p_{ij}^c) & -p_{ij}^c p_{ji}^c \\ -p_{ij}^c p_{ji}^c & p_{ji}^c(1-p_{ji}^c) \end{pmatrix}, \quad i<j.$$

Therefore $\sigma^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ is identical to $\sigma^2[\widehat{\Phi}_{BT}^{(\lambda)}]$ with $\{\pi_{ij}\}$ and $\{r_{ij}(= n_{ij} + n_{ji})\}$ replaced by $\{p_{ij}^c\}$ and $\{n(p_{ij} + p_{ji})\}$, respectively. The estimate of $\sigma^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ is given by $\widehat{\sigma}^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ which is $\sigma^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ with p_{ij} replaced by $\widehat{p}_{ij} = n_{ij}/n$ (thus $\widehat{p}_{ij}^c = n_{ij}/(n_{ij} + n_{ji})$). Therefore, we point out that $\widehat{\sigma}^2[\widehat{\Phi}_{QS}^{(\lambda)}]$ is identical to $\widehat{\sigma}^2[\widehat{\Phi}_{BT}^{(\lambda)}]$.

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