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SIMPLE MODELS TO INVESTIGATE THE EFFECT OF VELOCITY DEPENDENT FRICTION ON THE DISC BRAKE SQUEAL NOISE

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ABSTRACT–This paper suggests two simple two-degree-of-freedom models to describe the dynamical interaction between the pad and the disc of a disc brake system. Separate models for in-plane and out-of-plane vibration are described. Although a brake pad and disc have many modes of vibration, the interaction between a single mode of each component is considered as this is thought to be crucial for brake noise. For both models, the pad and the disc are connected by a sliding friction interface having a velocity dependent friction coefficient. In this paper, it is shown that this friction model acts as negative damping in the system that describes the in-plane vibration, and as negative stiffness in system that describes the out-of-plane vibration. Stability analysis is performed to investigate the conditions under which the systems become unstable. The results of the stability analysis show that the damping is the most important parameter for in-plane vibration, whereas the stiffness is the most important parameter for the out-of-plane vibration.

KEY WORDS : Disc brake, Squeal, Stability analysis, Velocity dependent friction coefficient, Stick-slip, Negative damping, Negative stiffness

1. INTRODUCTION

Dynamic models that have a friction interface to describe the mechanisms of disc brake squeal noise have been studied extensively over the last few decades. Much research uses either a lumped parameter approach (Brooks *et al.*, 1993; Crolla and Lang, 1990; Earles and Chambers, 1987, 1988; Jarvis and Mills, 1963-64; Lang and Smales, 1983; North, 1976) or the finite element method (Ghesqueiere, 1992; Lang *et al.*, 1993; Lee *et al.*, 1999; Liies, 1989; Murakami *et al.*, 1984). Matsui *et al.* (1992) has combined these two methods, and other methods have also been introduced such as the holographic image method (Fieldhouse and Newcomb, 1993) and an experimental method (Nosseir *et al.*, 1998). The method described in this paper uses the lumped parameter approach.

Simple models to separately describe in-plane and out-of-plane vibration are suggested, and then stability analysis is performed by finding the real parts of the eigenvalues of the characteristic equation. If these are negative then the system is stable, but if they are positive then the system is unstable.

Considering the nature of the friction between the pad and the disc, the friction force is distributed over the contact area in a complicated manner according to the dynamic modes of pad and disc. Ideally the distributed friction force may be found by measuring the pressure applied to the pad and the mode shapes of pad and disc in operation. However it is very difficult to measure the individual modes when the disc is rotating.

Alternatively, each mode of the pad and disc can be found by modal testing (or FEM) in a free condition. Recently a finite element model incorporating the distributed friction force has also been introduced (Lee *et al.*, 1999). Research works that use many modes and more realistic friction mechanisms have greatly been increased in number.

However, the aim of the mathematical model, in this paper, is a description of the basic dynamics of the friction interface and the corresponding stability analysis. Although a many degrees of freedom model may be considered, the results may be very complicated to interpret. Thus it is very important to study a simple model but not over-simplified, such as proposed two-degree-of-freedom models in this paper. This may facilitate greater understanding of the effect of the friction interface on brake squeal noise.

A simple one-degree-of-freedom system has been described by Matsui *et al.* (1992) and Crolla and Lang (1990). However, this system is over-simplified and does not adequately represent the practical physical phenomena.

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As with many other research works, it is mainly concerned with the dynamical behavior of the pad only. Shin *et al.* (2002) extended the simple model to a two-degree-of-freedom model, which emulates a single mode interaction between the pad and the disc. However, the model describes the in-plane dynamic behavior only. The out-of-plane motion is very important since it is directly related to the radiated squeal noise. Thus, in this paper, a two-degree-of-freedom out-of-plane model is also considered, which describes the relationship between the in-plane and the out-of plane vibrations of a disc brake system.

By comparing the in-plane model and the out-of-plane model, it is shown that the friction mechanism acts as a negative damping in the model for in-plane vibration, and acts as a negative stiffness in the model for out-of-plane vibration. Stability analysis is then carried out for both models to find the conditions for instability. The results show whch parameters are important to control both in-plnae and out-of-plane vibrations.

2. TWO-DEGREE-OF-FREEDOM IN-PLANE VIBRATION MODEL

In this section the model for in-plane vibration is described. It is generally known that one of many causes of the brake noise results from 'stick-slip' nonlinear vibration (Popp and Stelter, 1990). The stick-slip motion is usually described as a limit cycle in phase space, and requires a complex analysis to understand its non-linear dynamics. However, linear stability analysis can be used by assuming that the existence of limit cycle is related to the squeal noise, because the existence and non-existence of a limit cycle depends on the stability of an equilibrium point inside the limit cycle. Based on this assumption, the stability analysis of a two-degree-of-freedom in-plane model is described.

Suppose that the pad and the disc have their own vibrating modes connected through a friction interface which is the source of an external forcing term. If only one mode interaction of a pad-disc pair is considered, the two-degree-of-freedom in-plane model may be described as in Figure 1.

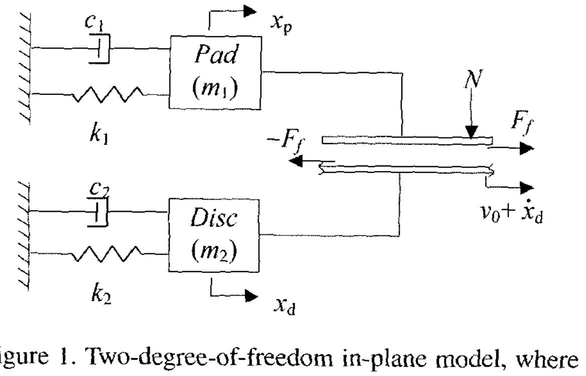

Figure 1. Two-degree-of-freedom in-plane model, where v_0: disc speed, N: braking force, F_f: friction force, v_r: relative velocity ($v_r = v_0 + \dot{x}_d - \dot{x}_p$).

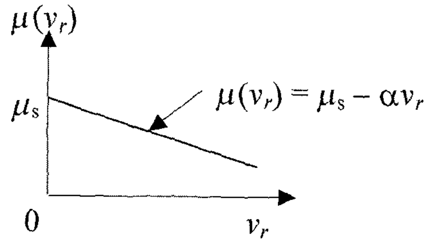

Figure 2. Velocity dependent friction coefficient.

As shown in Figure 1, the contact surface is the source of the external friction force that is dependent on the dynamic friction coefficient. The normal force (braking force) acting on the pad is constant. The coefficient of friction is a function of the relative velocity as shown in Figure 2. This shows that $\mu(v_r)$ has a negative gradient for the relative velocities, and is linear. The dynamics of this system may vary according to the friction model chosen, and there are many models for the friction coefficient to represent a realistic dynamic friction force. However, in this paper, a simple linear friction model is selected, as it has been found to be sufficient for understanding the effect of the friction mechanism on the system.

Provided that the relative velocity is always positive, the friction force can be written as

$$F_f = N(\mu_s - \alpha v_r) = N(\mu_s - \alpha v_0) + N\alpha(\dot{x}_p - \dot{x}_d) \quad (1)$$

Thus, the equations of motion are given by

$$m_1\ddot{x}_p + c_1\dot{x}_p + k_1x_p = N(\mu_s - \alpha v_0) + N\alpha(\dot{x}_p - \dot{x}_d)$$
$$m_2\ddot{x}_d + c_2\dot{x}_d + k_2x_d = -N(\mu_s - \alpha v_0) + N\alpha(\dot{x}_d - \dot{x}_p) \quad (2)$$

Rearranging these equations gives

$$m_1\ddot{x}_p + c_1\dot{x}_p - N\alpha(\dot{x}_p - \dot{x}_d) + k_1x_p = N(\mu_s - \alpha v_0)$$
$$m_2\ddot{x}_d + c_2\dot{x}_d - N\alpha(\dot{x}_d - \dot{x}_p) + k_2x_d = -N(\mu_s - \alpha v_0) \quad (3)$$

Based on Equation (3), the system may be redrawn as shown in Figure 3, where the frictional forcing term is split into two parts: one is associated with a state variable

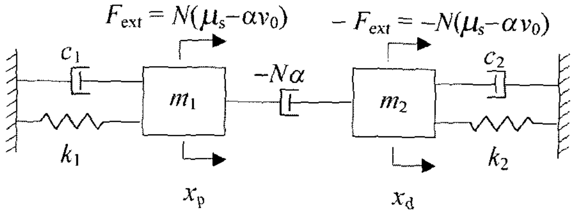

Figure 3. Modified two-degree-of-freedom in-plane model.

(damping in this case), and the other can be considered as an independent external force directly related to the dynamic friction coefficient. This independent explicit forcing term is denoted as 'F_{ext}' in Figure 3.

The '$N\alpha$' term is the most important parameter to be considered. It has the effect of 'negative damping' which supplies energy to the system. For stability analysis, the characteristic equation becomes.

$$\det\begin{bmatrix} \lambda^2 + c_{11}\lambda + k_{11} & c_{12}\lambda \\ c_{21}\lambda & \lambda^2 + c_{22}\lambda + k_{22} \end{bmatrix} = 0 \quad (4)$$

where

$$c_{11} = \frac{c_1 - N\alpha}{m_1},\ c_{22} = \frac{c_1 - N\alpha}{m_2},\ c_{12} = \frac{N\alpha}{m_1},\ c_{21} = \frac{N\alpha}{m_2},$$
$$k_{11} = \frac{k_1}{m_1},\ k_{22} = \frac{k_2}{m_2}.$$

Equation (4) has the following form.

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \quad (5)$$

Thus, an array can be formed from which the stability requirement can be obtained as below.

$$\begin{bmatrix} 1 & a_2 & a_4 \\ a_1 & a_3 & 0 \\ \dfrac{a_1a_2 - a_3}{a_1} & a_4 & 0 \\ \dfrac{a_1a_2a_3 - a_1^2a_4 - a_3^2}{a_1a_2 - a_3} & 0 & 0 \end{bmatrix} \quad (6)$$

where

$a_1 = c_{11} + c_{22}$, $a_2 = c_{11}c_{22} - c_{12}c_{21} + k_{11} + k_{22}$,
$a_3 = k_{11}c_{22} + k_{22}c_{11}$, $a_4 = k_{11}k_{22}$

Thus, the system becomes unstable when the following conditions are met.

$$a_1 < 0, \text{ or } a_2 < 0 \text{ or } a_3 < 0, \text{ or } a_4 < 0 \text{ or}$$
$$a_1a_2 - a_3 < 0 \text{ or} \quad (7)$$
$$a_1a_2a_3 - a_1^2a_4 - a_3^2 < 0$$

This means that the brake assembly would become unstable and hence noise occurs.

Squeal noise occurs in a high frequency range usually between 2 kHz to 10 kHz, and generally results from the coupling between the pad and the disc vibrations. A 'damping shim' is often attached to the brake pad to provide some damping to the system. However, it is often reported that squeal noise occurs even with a high degree of damping treatment on the pad. It is now shown that the damping of disc is as much important as that of pad, and this may explain that the damping shim is not always a solution to squeal noise.

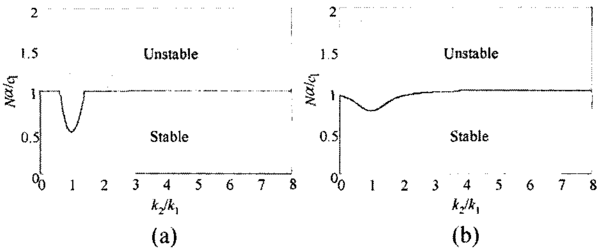

Figure 4. Stability area of the '$N\alpha/c_1$' versus normalized stiffness 'k_2/k_1': (a) $m_2 = m_1$, $c_2 = c_1$; (b) $m_2 = m_1$, $c_2 = 3c_1$.

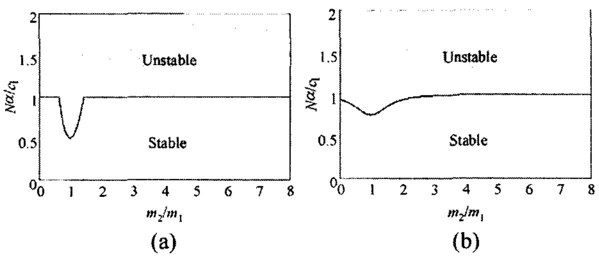

Figure 5. Stability area of the '$N\alpha/c_1$' versus normalized mass m_2/m_1: (a) $k_2 = k_1$, $c_2 = c_1$; (b) $k_2 = k_1$, $c_2 = 3c_1$.

Stability analysis is conducted for various conditions with special emphasis on the '$N\alpha$' term. It is normalized with respect to 'c_1' (i.e., $N\alpha/c_1$) for convenience. The way in which the stability is affected by the normalized '$N\alpha$' term and other normalized system parameters (m_2/m_1, k_2/k_1 and c_2/c_1). Figure 4 shows the relationship between the '$N\alpha$' term and stiffness terms (k_1 and k_2). It can be seen that the system is more stable as the difference between two stiffness terms becomes larger and as the damping increases.

The '$N\alpha$' term is also compared with the mass terms (m_1 and m_2), in Figure 5. Similar to the results in Figure 4, it is shown that the system is less stable when the natural frequencies of the pad and the disc are the same. Thus, it is desirable to avoid any coincidences between the pad modes and the disc modes. From the results shown in Figures 4 and 5, it can be seen that for a stable condition the normalized '$N\alpha$' term should never be greater than c_1 or c_2. Thus, a stability criterion can be obtained for the case when the difference between two natural frequencies is large. That is, the system is stable if the value of '$N\alpha$' term is smaller than both damping parameters (c_1 and c_2) of the pad and the disc.

Finally, the normalized '$N\alpha$' term is compared with the normalized damping parameter of the disc (i.e., c_2/c_1) for various combinations of the system parameters (mass and stiffness terms). The results are shown in Figure 6. This further verifies the previous results shown in Figures 4 and 5, which shows that the system is more likely to

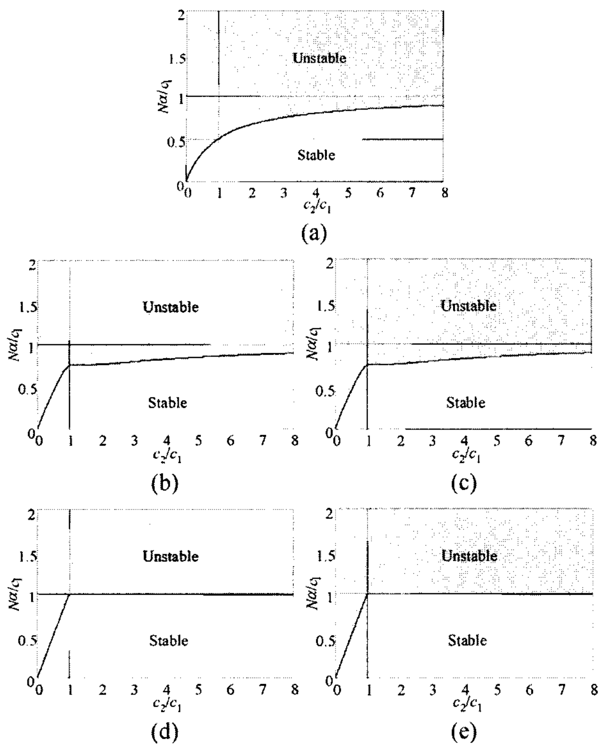

Figure 6. Stability area of the '$N\alpha/c_1$' versus normalized damping 'c_2/c_1': (a) $m_2 = m_1$, $k_2 = k_1$; (b) $m_2 = 1.3m_1$, $k_2 = k_1$; (c) $m_2 = m_1$, $k_2 = 1.3k_1$; (d) $m_2 = 5m_1$, $k_2 = k_1$; (e) $m_2 = m_1$, $k_2 = 5k_1$.

become unstable when natural frequencies of the disc and the pad are in close proximity. It also shows that the system becomes more stable as the damping is increased.

From careful examination of the figures, it can be seen that the stability area never exceeds the value of '1' in the vertical axis (corresponding to the normalized '$N\alpha$' term), and the normalized damping 'c_2/c_1' is never smaller than twice the normalized '$N\alpha$' term while maintaining its stability (see Figure 6(a) in the horizontal axis of '1' and the vertical axis of '0.5'). Also comparing the same points in Figures 4(a) and 5(a), a criterion can be derived to guarantee the system stability. The system is guaranteed to be always stable if '$2N\alpha$' is smaller than both damping parameters (c_1 and c_2). This also implies that the system may become unstable if any one of the damping parameters is smaller than '$2N\alpha$'.

The significance of the '$N\alpha$' term may be explained by the fact that it acts as negative damping, which constantly supplies energy to the system. That is, if the energy dissipated by the system damping (c_1 and c_2) is smaller than the energy supplied by the friction interface (the '$N\alpha$' term), the system becomes unstable.

The results of the stability analysis are summarized below:

(1) The system becomes more prone to instability when the natural frequencies of two modes (pad and disc) are close, i.e., it has the smallest stability area.
(2) Increasing damping results in a larger stability area.
(3) For the system to be stable, two criteria can be stated when the two natural frequencies are very different,

$$\min(c_1, c_2) > N\alpha \tag{8}$$

when the two natural frequencies are close,

$$\min(c_1, c_2) > 2N\alpha \tag{9}$$

where 'min(,)' denotes the minimum value. Note that twice the damping is required when the two natural frequencies are same to meet the criterion. These criteria imply that no matter how much damping is added to the pad, the system can become unstable unless an appropriate amount of damping is added to the disc.

3. TWO DEGREE OF FREEDOM OUT-OF-PLANE VIBRATION MODEL

In the previous section, the instability resulting from the in-plane motion of the pad-disc pair has been considered, where the dynamic friction coefficient was a function of relative velocity and the normal force was a constant. If the normal force varies with the vertical relative displacement, i.e., $N(y)$, and the friction coefficient varies with the normal force, i.e., $\mu(v_r, N)$, then a two-degree-of-freedom out-of-plane model as shown in Figure 7 can be considered, where the contact stiffness k_c is introduced.

Since the friction force acting along the horizontal axis does not influence on the vertical vibration, the equations of motion can be written as

$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K + K_c(y)]\{y\} = 0 \tag{10}$$

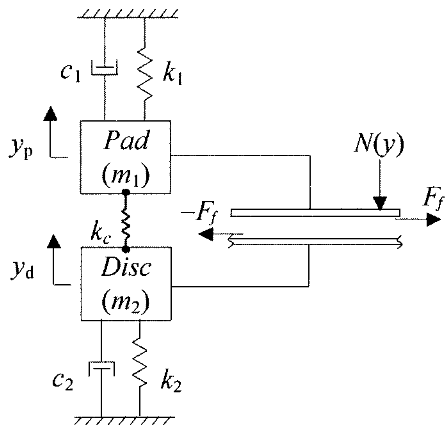

Figure 7. Two-degree-of-freedom out-of-plane model.

where, $[M]$, $[C]$, $[K]$ and $[K_c]$ are the mass, damping, stiffness and contact stiffness matrices respectively, and $\{y\}$ is the vertical displacement vector. In this case, the normal force can be written as

$$N(y) = k_c(y_p - y_d) \tag{11}$$

Thus, the friction force becomes

$$F_f = -\mu(v_r, N) \cdot k_c \cdot (y_p - y_d) \tag{12}$$

Although the friction force acts along the horizontal axis, it depends on the vertical relative displacement. As a result, the effect of friction force is to influence the contact stiffness. The linearized contact stiffness matrix can be found by (Nack, 1999).

$$[K_c] = \begin{bmatrix} \frac{\partial F_f}{\partial y_p} & -\frac{\partial F_f}{\partial y_p} \\ -\frac{\partial F_f}{\partial y_d} & \frac{\partial F_f}{\partial y_d} \end{bmatrix} \tag{13}$$

where,

$$\frac{\partial F_f}{\partial y_p} = -\frac{\partial \mu}{\partial y_p} k_c \cdot (y_p - y_d) - \mu k_c,$$

$$\frac{\partial F_f}{\partial y_d} = -\frac{\partial \mu}{\partial y_d} k_c \cdot (y_p - y_d) - \mu k_c$$

and

$$\frac{\partial \mu}{\partial y_p} = \frac{\partial \mu}{\partial N}\frac{\partial N}{\partial y_p} = \frac{\partial \mu}{\partial N} k_c, \quad \frac{\partial \mu}{\partial y_d} = \frac{\partial \mu}{\partial N}\frac{\partial N}{\partial y_d} = \frac{\partial \mu}{\partial N} k_c$$

Thus, the contact stiffness matrix can be written as

$$[K_c] = -\mu(v_r, N) \begin{bmatrix} k_c & -k_c \\ -k_c & k_c \end{bmatrix} - k_c \begin{bmatrix} \frac{\partial \mu}{\partial N} k_c (y_p - y_d) & -\frac{\partial \mu}{\partial N} k_c (y_p - y_d) \\ \frac{\partial \mu}{\partial N} k_c (y_p - y_d) & -\frac{\partial \mu}{\partial N} k_c (y_p - y_d) \end{bmatrix} \tag{14}$$

If the friction coefficient is a function of relative velocity only, i.e., $\mu(v_r)$ as in the previous section, then the contact stiffness matrix becomes

$$[K_c] = -\mu(v_r) \begin{bmatrix} k_c & -k_c \\ -k_c & k_c \end{bmatrix} \tag{15}$$

Since the friction coefficient is always positive, $\mu(v_r) > 0$, the stiffness elements in the contact stiffness matrix $[K_c]$ all act as a 'negative stiffness'. In this case, the equations of motion can be written as

$$\begin{aligned} m_1 \ddot{y}_p + c_1 \dot{y}_p + (k_1 - \mu(v_r)k_c)y_p + \mu(v_r)k_c y_d &= 0 \\ m_2 \ddot{y}_d + c_2 \dot{y}_d + (k_2 - \mu(v_r)k_c)y_d + \mu(v_r)k_c y_p &= 0 \end{aligned} \tag{16}$$

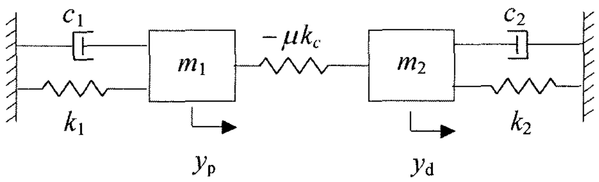

Figure 8. Modified two-degree-of-freedom out-of-plane model.

Similar to the in-plane model, the out-of-plane model can be depicted as in Figure 8. This shows the 'μk_c' term, which is the most important parameter acts as a 'negative stiffness'.

Equation (6) may be used for the stability analysis, where the new variables are:

$$c_{11} = \frac{c_1}{m_1}, \; c_{22} = \frac{c_2}{m_2}, \; k_{11} = \frac{k_1 - \mu k_c}{m_1}, \; k_{22} = \frac{k_2 - \mu k_c}{m_2},$$

$$k_{12} = \frac{\mu k_c}{m_1}, \; k_{21} = \frac{\mu k_c}{m_2}$$

and

$$a_1 = c_{11} + c_{22}, \; a_2 = c_{11}c_{22} + k_{11} + k_{22},$$
$$a_3 = k_{11}c_{22} + k_{22}c_{11}, \; a_4 = k_{11}k_{22} - k_{12}k_{21}.$$

Stability analysis is performed for various conditions paying particular attention to the term 'μk_c'. The results are shown in Figure 9, where the 'μk_c' term is normalized by 'k_1' (i.e., $\mu k_c/k_1$) for convenience. Then, the normalized 'μk_c' term versus other normalized system parameters (m_2/m_1, k_2/k_1 and c_2/c_1) are examined respectively.

First, the normalized 'μk_c' term is compared with the mass terms and damping parameters as shown in Figures 9(a)~(d). Unlike in the case of in-plane vibration, it is found that the stability of the system does not depend on the damping parameters or the masses. Consequently, the natural frequencies of the pad and the disc are not important, but only the magnitude of stiffness parameter influences on the stability of the system.

Next, the normalized 'μk_c' term is compared with the stiffness terms. The result is shown in Figure 9(e). This shows that the system is more stable as the stiffness of the system increases. It also shows that the stable area never exceeds the value of '1' in the vertical axis (corresponding to the normalized 'μk_c' term), and the normalized stiffness 'k_2/k_1' is never smaller than twice the normalized 'μk_c' term while maintaining its stability (see Figure 9(e) in the horizontal axis of '1' and the vertical axis of '0.5'). That is, it is guaranteed that the system is always stable if $2\mu k_c$ is smaller than both stiffness parameters (k_1 and k_2) of the pad and the disc. Thus, the criterion for the system to be stable is

$$\min(k_1, k_2) > 2\mu k_c \tag{17}$$

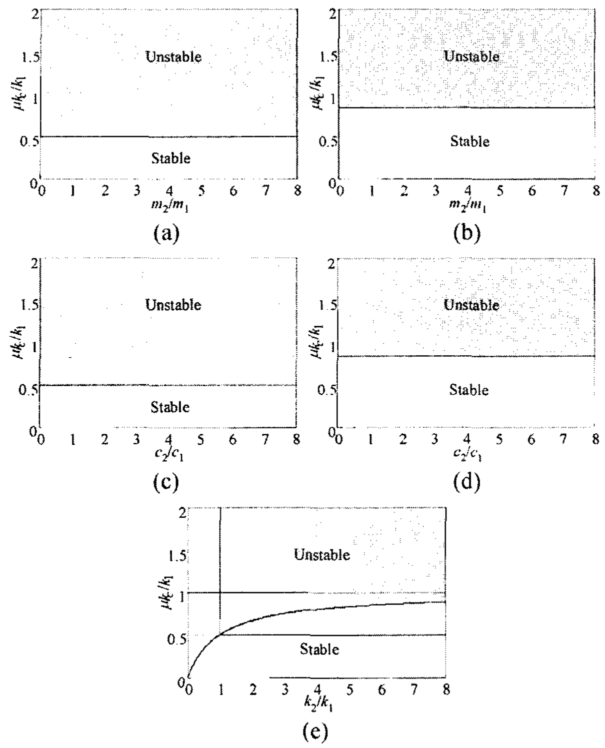

Figure 9. Stability area of the '$\mu k_c/k_1$' versus normalized system parameters: (a) $\mu k_c/k_1$ versus m_2/m_1, where $c_2 = c_1$, $k_2 = k_1$; (b) $\mu k_c/k_1$ versus m_2/m_1, where $c_2 = c_1$, $k_2 = 5k_1$; (c) $\mu k_c/k_1$ versus c_2/c_1, where $m_2 = m_1$, $k_2 = k_1$; (d) $\mu k_c/k_1$ versus c_2/c_1, where $m_2 = m_1$, $k_2 = 5k_1$; (e) $\mu k_c/k_1$ versus k_2/k_1, where $m_2 = m_1$, $c_2 = c_1$ or $m_2 \neq m_1$, $c_2 \neq c_1$.

This implies that both stiffness parameters of the pad and the disc are equally important, and they must be large enough to suppress the effect of the negative stiffness due to the friction interface.

4. CONCLUSION

Two simple two-degree-of-freedom models have been proposed to investigate the basic instability mechanisms of a disc brake for both in-plane and out-of-plane vibrations that may lead to squeal noise. The models describe the interacting dynamics between the pad and the disc through the friction interface between the pad and the disc.

From the stability analyses, following results have been established: First, for in-plane vibrations, the system is more prone to instability when the natural frequencies of pad and disc are close. It has also been shown that the damping of the system is the most important factor to be considered for reducing the in-plane vibrations, and that the damping of the pad and the disc are equally important.

Second, for the instability of the out-of-plane vibrations, the stiffness of both the pad and disc is important. They must be sufficiently large to suppress the effect of the negative stiffness due to the friction interface.

Finally, although the results of the stability analyses may not be considered comprehensive enough to understand details of the dynamics, they give important design criteria and facilitate some insight into the origins of brake squeal noise.

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