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THREE STEP ITERATIVE ALGORITHMS FOR GENERALIZED QUASIVARIATIONAL INCLUSIONS

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ABSTRACT. In this paper, we suggest and analyze some new classes of three step iterative algorithms for solving generalized quasivariational inclusions by using the properties of proximal maps. Our results include the Ishikawa, Mann iterations for solving variational inclusions(inequalities) as special cases. The results obtained in this paper represent an improvement and significant refinement of previously known results [3, 5-8, 10, 14-18].

1. Introduction

In recent years, variational inequality theory has emerged one of the main branch of mathematical and engineering sciences. This theory provides us with a simple, natural, unified, and general framework to study a wide class of unrelated problems arising in fluid flow through porous media, elasticity, transportation, economics, optimization, regional, physical, structural, and applied sciences, etc.. The ideas and techniques of variational inequalities are being used to interpret the basic principles of pure and applied sciences in the form of simplicity and elegance. Variational inequalities have been extended and generalized in different directions using novel and innovative techniques both for its own sake and for its applications. A useful and an important generalization of variational inequalities is a mixed variational inequality containing the nonlinear term. Due to the presence of the nonlinear term, the projection method and its variant forms including the Wiener-Hopf equations technique cannot be used to study the existence of a solution of the mixed variational inequalities. These facts motivated us to

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develop another technique. This technique is related to the resolvent of the maximal monotone operator. Using the concept of the resolvent operator technique, Noor [9, 10] introduced and studied a new system of equations which is called the resolvent equations. Noor [9, 10] has established the equivalence between the mixed variational inequalities and the resolvent equations.

In recent years, considerable interest has been shown in developing various extensions and generalizations of variational inequalities related to multivalued operators, nonconvex optimization, nonmonotone operators, and structural analysis. And three step forward backward splitting methods have been developed by Glowinski and Le Tallee [4] and Noor [11, 12] for solving various classes of variational inequalities by using the Lagrangian multiplier, updating the solution and the auxiliary principle techniques. It has been shown in [13] that the three step schemes give better numerical results than the two step and one step approximation iterations.

In this paper, we suggest and analyze some new classes of three step iterative algorithms for solving generalized quasivariational inclusions by using the properties of proximal maps. Our results include the Ishikawa, Mann iterations for solving variational inclusions(inequalities) as special cases. The results obtained in this paper represent an improvement and a significant refinement of previously known results [3, 5-8, 10, 14-18].

2. Preliminaries

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $C(H)$ be a family of nonempty compact subset of H. Let $U, V : H \to C(H)$ be the multivalued operators and $f, g, m : H \to H$ be the single valued operators. Suppose that $M : H \times H \to 2^H$ such that fixed $y \in H$, $M(\cdot, y) : H \to 2^H$ is a maximal monotone mapping and $\text{Range}(g - m) \cap \text{dom}(M(\cdot, y)) \neq \phi$ for each $y \in H$. For a given nonlinear operator $N(\cdot,\cdot) : H \times H \to H$, we consider the problem of finding $x \in H$, $u \in U(x)$, and $v \in V(x)$ such that

$$0 \in N(u, v) - f(x) + M((g - m)(x), x), \tag{2.1}$$

where $g - m$ is defined as $(g - m)(x) = g(x) - m(x)$ for each $x \in H$. The problem (2.1) is called a generalized quasivariational inclusion.

Since $M(\cdot, x)$ is maximal monotone, $x \in H$ is a solution of the problem (2.1) if and only if $x \in H$, $u \in U(x)$, and $v \in V(x)$ such that $(g-m)(x) \cap \text{dom}(M(\cdot, x)) \neq \phi$ and $f(x) - N(u, v) \in M((g - m)(x), x)$.

(I) If $m = 0$ and $M(x, y) = M(x)$ for all $y \in H$, where $M : H \to 2^H$ is a maximal monotone mapping, then the problem (2.1) is equivalent to finding $x \in H$, $u \in U(x)$, and $v \in V(x)$ such that $g(x) \cap \mathrm{dom}(M(x)) \neq \phi$ and

$$0 \in N(u, v) - f(x) + M(g(x)). \tag{2.2}$$

(II) If $M(\cdot, y) = \partial\phi(\cdot, y)$ for each $y \in H$, where $\phi(\cdot, y) : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H and $\mathrm{Range}(g - m) \cap \mathrm{dom}(\partial\phi(\cdot, y)) \neq \phi$ for each $y \in H$ and $\partial\phi(\cdot, y)$ denote the subdifferential of function $\phi(\cdot, y)$, then the problem (2.1) is equivalent to finding $x \in H$, $u \in U(x)$, and $v \in V(x)$ such that $(g - m)(x) \cap \mathrm{dom}(\partial\phi(\cdot, x)) \neq \phi$ and

$$\langle N(u, v) - f(x), y - (g - m)(x)\rangle \geq \phi((g - m)(x), x) - \phi(y, x), \quad \forall y \in H. \tag{2.3}$$

(III) If $M(\cdot, y) = \partial\phi$, for all $y \in H$, where $\partial\phi$ denotes the subdifferential of a proper, convex, and lower semicontinuous function $\phi : H \to R \cup \{+\infty\}$, then the problem (2.1) is equivalent to finding $x \in H$, $u \in U(x)$, and $v \in V(x)$ such that $(g - m)(x) \cap \mathrm{dom}(\partial\phi) \neq \phi$ and

$$\langle N(u, v) - f(x), y - (g - m)(x)\rangle \geq \phi((g - m)(x)) - \phi(y), \quad \forall y \in H. \tag{2.4}$$

It is clear that the generalized quasivariational inclusion (2.1) includes of many known variational inequalities and quasivariational inequalities [5, 7-10, 14-17].

3. Main results

First of all, we prove the following lemma.

LEMMA 3.1. *$x \in H$, $u \in U(x)$, and $v \in V(x)$ is a solution of (2.1) if and only if for some given $\rho > 0$, the mapping $F : H \to 2^H$ defined by*

$$\begin{aligned} F(x) = \cup_{u \in U(x)} \cup_{v \in V(x)} \{x - (g - m)(x) \\ + J_\rho^{M(\cdot, x)}[(g - m)(x) - \rho(N(u, v) - f(x))]\} \end{aligned} \tag{3.1}$$

has a fixed point, where $J_\rho^{M(\cdot, x)} = (I + \rho M(\cdot, x))^{-1}$ is the so-called proximal mapping on H.

Proof. Let x be a fixed point of F, i.e., there exist $u \in U(x)$ and $v \in V(x)$ such that

$$x = x - (g-m)(x) + J_\rho^{M(\cdot,x)}[(g-m)(x) - \rho(N(u,v) - f(x))].$$

Then we have

$$(g-m)(x) = J_\rho^{M(\cdot,x)}[(g-m)(x) - \rho(N(u,v) - f(x))].$$

From the definition of the proximal mapping $J_\rho^{M(\cdot,x)}$, we get

$$f(x) - N(u,v) \in M((g-m)(x), x).$$

Hence $x \in H$ is a solution of the problem (2.1).

Conversely, if $x \in H$ is a solution of the problem (2.1), then there exist $x \in H$, $u \in U(x)$, and $v \in V(x)$ such that $(g-m)(x) \cap \mathrm{dom}(M(\cdot,x)) \neq \phi$ and

$$0 \in N(u,v) - f(x) + M((g-m)(x), x).$$

Hence, we have

$$(g-m)(x) - \rho(N(u,v) - f(x)) \in (g-m)(x) + \rho M((g-m)(x), x).$$

From the definition of the proximal mapping $J_\rho^{M(\cdot,x)}$, we get

$$(g-m)(x) = J_\rho^{M(\cdot,x)}[(g-m)(x) - \rho(N(u,v) - f(x))].$$

From this we obtain

$$x = x - (g-m)(x) + J_\rho^{M(\cdot,x)}[(g-m)(x) - \rho(N(u,v) - f(x))].$$

This means that x is a fixed point of F. □

THEOREM 3.1. *Let the operator $N(\cdot,\cdot)$ be α-strongly monotone and β-Lipschitz continuous with respect to the first argument. Let $f, g, m : H \to H$ be Lipschitz continuous with Lipschitz constants δ, ξ, and ε respectively, and $g - m$ be strongly monotone with constant σ. Assume that*

$$\langle m(x) - m(y), g(x) - g(y) \rangle \geq \lambda \|x - y\|^2, \quad {}^\forall x, y \in H \tag{3.2}$$

for some constant λ *such that* $\lambda_0 \le \lambda \le \xi\varepsilon$, *where*

$$\lambda_0 = \inf\{s : \langle m(x) - m(y), g(x) - g(y)\rangle \le s\|x - y\|^2, {}^{\forall}x, y \in H\}.$$

Moreover, the operator $N(\cdot,\cdot) : H \times H \to H$ *is* γ*-Lipschitz continuous with respect to the second argument. Let* $U, V : H \to C(H)$ *be* μ*-*H*-Lipschitz, and* ν*-*H*-Lipschitz respectively. Suppose there exists a constant* $\eta > 0$ *such that for each* $x, y, z \in H$,

$$\|J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z)\| \le \eta\|x - y\|. \tag{3.3}$$

If the following conditions hold

$$\begin{aligned} &|\rho - \frac{\alpha - (1-k)(\delta + \gamma\nu)}{\beta^2\mu^2 - (\delta+\gamma\nu)^2}| \\ &< \frac{\sqrt{[\alpha - (1-k)(\delta+\gamma\nu)]^2 - k(\beta^2\mu^2 - (\delta+\gamma\nu)^2)(2-k)}}{\beta^2\mu^2 - (\delta+\gamma\nu)^2}, \end{aligned} \tag{3.4}$$

$$\alpha > (1-k)(\delta+\gamma\nu) + \sqrt{k\beta^2\mu^2 - (\delta+\gamma\nu)^2(2-k)},$$

$$\beta\mu > \delta + \gamma\nu, \quad \rho(\delta+\gamma\nu) < 1 - k,$$

$$k = 2\sqrt{1 - 2\sigma + \varepsilon^2 + \xi^2 - 2\lambda} + \eta, \quad k < 1,$$

then the generalized quasivariational inequality problem (2.1) *has a solution.*

Proof. From Lemma 3.1, it is enough to show that the mapping

$$\begin{aligned} F(x) = \cup_{u\in U(x)} \cup_{v\in V(x)} \{x - (g-m)(x) + J_\rho^{M(\cdot,x)}[(g-m)(x) \\ - \rho(N(u,v) - f(x))]\} \end{aligned}$$

has a fixed point. For any $x_1, x_2 \in H$, $p \in F(x_1)$, and $q \in F(x_2)$, there exist $u_1 \in U(x_1)$, $v_1 \in V(x_1)$, $u_2 \in U(x_2)$, and $v_2 \in V(x_2)$ such that

$$\begin{aligned} p &= x_1 - (g-m)(x_1) + J_\rho^{M(\cdot,x_1)}[(g-m)(x_1) \\ &\quad - \rho(N(u_1,v_1) - f(x_1))], \\ q &= x_2 - (g-m)(x_2) + J_\rho^{M(\cdot,x_2)}[(g-m)(x_2) \\ &\quad - \rho(N(u_2,v_2) - f(x_2))]. \end{aligned}$$

Then, by the condition (3.3) we have

$$
\begin{aligned}
&\|p-q\| \\
&\leq \|x_1 - x_2 - [(g-m)(x_1) - (g-m)(x_2)]\| \\
&\quad + \|J_\rho^{M(\cdot,x_1)}[(g-m)(x_1) - \rho(N(u_1,v_1) - f(x_1))] \\
&\quad - J_\rho^{M(\cdot,x_2)}[(g-m)(x_2) - \rho(N(u_2,v_2) - f(x_2))]\| \\
&\leq \|x_1 - x_2 - [(g-m)(x_1) - (g-m)(x_2)]\| \\
&\quad + \|J_\rho^{M(\cdot,x_1)}[(g-m)(x_1) - \rho(N(u_1,v_1) - f(x_1))] \\
&\quad - J_\rho^{M(\cdot,x_1)}[(g-m)(x_2) - \rho(N(u_2,v_2) - f(x_2))]\| \\
&\quad + \|J_\rho^{M(\cdot,x_1)}[(g-m)(x_2) - \rho(N(u_2,v_2) - f(x_2))] \\
&\quad - J_\rho^{M(\cdot,x_2)}[(g-m)(x_2) - \rho(N(u_2,v_2) - f(x_2))]\| \\
&\leq 2\|x_1 - x_2 - [(g-m)(x_1) - (g-m)(x_2)]\| \\
&\quad + \rho\|f(x_1) - f(x_2)\| + \|x_1 - x_2 - \rho(N(u_1,v_1) - N(u_2,v_2))\| \\
&\quad + \eta\|x_1 - x_2\| \\
&\leq 2\|x_1 - x_2 - [(g-m)(x_1) - (g-m)(x_2)]\| \\
&\quad + \rho\|f(x_1) - f(x_2)\| + \|x_1 - x_2 - \rho(N(u_1,v_1) - N(u_2,v_1))\| \\
&\quad + \rho\|N(u_2,v_1) - N(u_2,v_2)\| + \eta\|x_1 - x_2\|.
\end{aligned} \tag{3.5}
$$

By the Lipschitz continuity of f, g, m, the strong monotonicity of $g-m$ and the condition (3.2), we have

$$
\begin{aligned}
&\|x_1 - x_2 - [(g-m)(x_1) - (g-m)(x_2)]\|^2 \\
&\leq \|x_1 - x_2\|^2 - 2\langle x_1 - x_2, (g-m)(x_1) - (g-m)(x_2)\rangle \\
&\quad + \|(g-m)(x_1) - (g-m)(x_2)\|^2 \\
&\leq \|x_1 - x_2\|^2 - 2\langle x_1 - x_2, (g-m)(x_1) - (g-m)(x_2)\rangle \\
&\quad + \|g(x_1) - g(x_2)\|^2 - 2\langle g(x_1) - g(x_2), m(x_1) - m(x_2)\rangle \\
&\quad + \|m(x_1) - m(x_2)\|^2 \\
&\leq (1 - 2\sigma + \xi^2 - 2\lambda + \varepsilon^2)\|x_1 - x_2\|^2.
\end{aligned} \tag{3.6}
$$

Since $N(\cdot,\cdot)$ is α-strongly monotone and β-Lipschitz continuous with respect to the first argument and μ-H-Lipschitz continuity of U, we have that

$$
\begin{aligned}
&\|x_1 - x_2 - \rho(N(u_1, v_1) - N(u_2, v_1))\|^2 \\
&= \|x_1 - x_2\|^2 - 2\rho\langle N(u_1, v_1) - N(u_2, v_1), x_1 - x_2\rangle \\
&\quad + \rho^2\|N(u_1, v_1) - N(u_2, v_1)\|^2 \\
&\le \|x_1 - x_2\|^2 - 2\rho\langle N(u_1, v_1) - N(u_2, v_1), x_1 - x_2\rangle \\
&\quad + \rho^2\beta^2\|u_1 - u_2\|^2 \\
&\le \|x_1 - x_2\|^2 - 2\rho\langle N(u_1, v_1) - N(u_2, v_1), x_1 - x_2\rangle \\
&\quad + \rho^2\beta^2[H(U(x_1), U(x_2))]^2 \\
&\le (1 - 2\rho\alpha + \rho^2\beta^2\mu^2)\|x_1 - x_2\|^2,
\end{aligned} \tag{3.7}
$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $C(H)$. Using the γ-Lipschitz continuity of the operator $N(\cdot, \cdot)$ with respect to the second argument and the ν-H-Lipschitz continuity of V, we get

$$
\begin{aligned}
\|N(u_2, v_1) - N(u_2, v_2)\| &\le \gamma\|v_1 - v_2\| \\
&\le \gamma H(V(x_1), V(x_2)) \\
&\le \gamma\nu\|x_1 - x_2\|.
\end{aligned} \tag{3.8}
$$

From (3.5)-(3.8), we have

$$
\begin{aligned}
D(F(x_1), F(x_2)) &\le [2\sqrt{1 - 2\sigma + \xi^2 - 2\lambda + \varepsilon^2} + \eta + \rho\delta \\
&\quad + \rho\gamma\nu + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}]\|x_1 - x_2\| \\
&= [k + (\delta + \gamma\nu)\rho + t(\rho)]\|x_1 - x_2\| \\
&= \theta\|x_1 - x_2\|,
\end{aligned}
$$

where $\theta = k + (\delta + \gamma\nu)\rho + t(\rho)$, $k = 2\sqrt{1 - 2\sigma + \xi^2 - 2\lambda + \varepsilon^2} + \eta$, $t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}$, and $D(A, B) = \sup\{\|a - b\| : a \in A, b \in B\}$ for any $A, B \in 2^H$. From (3.4), it follows that $\theta < 1$. Thus the map F defined by (3.1) has a fixed point $x \in H$ such that $u \in U(x)$, $v \in V(x)$ satisfying the generalized quasivariational inequality (2.1). This completes the proof. □

The relation (3.1) can be written as

$$
x = x - (g - m)(x) + J_\rho^{M(\cdot, x)}[(g - m)(x) - \rho(N(u, v) - f(x))],
$$

where $\rho > 0$ is a constant.

This fixed point formulation allows us to suggest the following unified three step iterative algorithm.

Algorithm 3.1

Assume that $U, V : H \to C(H)$, $f, g, m : H \to H$, and $N(\cdot,\cdot) : H \times H \to H$ are operators. For a given $x_0 \in H$, $u_0 \in U(x_0)$, $v_0 \in V(x_0)$, compute the sequences $\{z_n\}$, $\{y_n\}$, $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{\bar{u}_n\}$, $\{\bar{v}_n\}$, $\{u_n^*\}$, and $\{v_n^*\}$ by the iterative schemes

$$u_n \in U(x_n) : \|u_{n+1} - u_n\| \le H(U(x_{n+1}), U(x_n)),$$

$$v_n \in V(x_n) : \|v_{n+1} - v_n\| \le H(V(x_{n+1}), V(x_n)),$$

$$\bar{u}_n \in U(y_n) : \|\bar{u}_{n+1} - \bar{u}_n\| \le H(U(y_{n+1}), U(y_n)),$$

$$\bar{v}_n \in V(y_n) : \|\bar{v}_{n+1} - \bar{v}_n\| \le H(V(y_{n+1}), V(y_n)),$$

$$u_n^* \in U(z_n) : \|u_{n+1}^* - u_n^*\| \le H(U(z_{n+1}), U(z_n)),$$

$$v_n^* \in V(z_n) : \|v_{n+1}^* - v_n^*\| \le H(V(z_{n+1}), V(z_n)),$$

$$\begin{aligned} y_n = {} & (1-\gamma_n)x_n + \gamma_n\{x_n - (g-m)(x_n) \\ & + J_\rho^{M(\cdot,x_n)}[(g-m)(x_n) - \rho(N(u_n,v_n) - f(x_n))]\}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} z_n = {} & (1-\beta_n)x_n + \beta_n\{y_n - (g-m)(y_n) \\ & + J_\rho^{M(\cdot,y_n)}[(g-m)(y_n) - \rho(N(\bar{u}_n,\bar{v}_n) - f(y_n))]\}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} x_{n+1} = {} & (1-\alpha_n)x_n + \alpha_n\{z_n - (g-m)(z_n) \\ & + J_\rho^{M(\cdot,z_n)}[(g-m)(z_n) - \rho(N(u_n^*,v_n^*) - f(z_n))]\}, \end{aligned} \tag{3.11}$$

for $n = 0, 1, 2, \cdots$, where $0 \le \alpha_n, \beta_n, \gamma_n \le 1$ for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges. For $\gamma_n = 0$, Algorithm 3.1 is the Ishikawa iterative scheme.

THEOREM 3.2. *Let the operator $N(\cdot,\cdot) : H \times H \to H$, $f, g, m : H \to H$ satisfy all the assumptions of Theorem 3.1. If the conditions (3.2)-(3.4) are hold, then the approximate solution $\{x_n\}$ obtained from Algorithm 3.1 converges to the exact solution x of the generalized quasivariational inequality (2.1).*

Proof. From Theorem 3.1, we see that there exists a unique solution $x \in H$ such that $u \in U(x)$, $v \in V(x)$ satisfying the generalized quasi-variational inequality (2.1). Then by Lemma 3.1, there exist $x \in H$, $u \in U(x)$, and $v \in V(x)$ such that

$$\begin{aligned} x &= x - (g-m)x + J_\rho^{M(\cdot,x)}[(g-m)(x) - \rho(N(u,v) - f(x))] \\ &= (1-\alpha_n)x + \alpha_n\{x - (g-m)(x) + J_\rho^{M(\cdot,x)}[(g-m)(x) \\ &\quad - \rho(N(u,v) - f(x))]\} \\ &= (1-\beta_n)x + \beta_n\{x - (g-m)(x) + J_\rho^{M(\cdot,x)}[(g-m)(x) \\ &\quad - \rho(N(u,v) - f(x))]\}. \end{aligned} \tag{3.12}$$

Then we have

$$\begin{aligned} &\|x_{n+1} - x\| \\ &\le (1-\alpha_n)\|x_n - x\| + \alpha_n\|z_n - x - [(g-m)(z_n) - (g-m)(x)]\| \\ &+ \alpha_n\|J_\rho^{M(\cdot,z_n)}[(g-m)(z_n) - \rho(N(u_n^*,v_n^*) - f(z_n))] \\ &- J_\rho^{M(\cdot,x)}[(g-m)(x) - \rho(N(u,v) - f(x))]\| \\ &\le (1-\alpha_n)\|x_n - x\| + \alpha_n\|z_n - x - [(g-m)(z_n) - (g-m)(x)\| \\ &+ \alpha_n\|J_\rho^{M(\cdot,z_n)}[(g-m)(z_n) - \rho(N(u_n^*,v_n^*) - f(z_n))] \\ &- J_\rho^{M(\cdot,z_n)}[(g-m)(x) - \rho(N(u,v) - f(x))]\| \\ &+ \alpha_n\|J_\rho^{M(\cdot,z_n)}[(g-m)(x) - \rho(N(u,v) - f(x))] \\ &- J_\rho^{M(\cdot,x)}[(g-m)(x) - \rho(N(u,v) - f(x))]\| \\ &\le (1-\alpha_n)\|x_n - x\| + 2\alpha_n\|z_n - x - [(g-m)(z_n) - (g-m)(x)]\| \\ &+ \alpha_n\rho\|f(z_n) - f(x)\| + \alpha_n\|z_n - x - \rho(N(u_n^*,v_n^*) - N(u,v))\| \\ &+ \alpha_n\eta\|z_n - x\| \\ &\le (1-\alpha_n)\|x_n - x\| + 2\alpha_n\|z_n - x - [(g-m)(z_n) - (g-m)(x)]\| \\ &+ \alpha_n\rho\|f(z_n) - f(x)\| + \alpha_n\|z_n - x - \rho[N(u_n^*,v_n^*) - N(u,v_n^*)]\| \\ &+ \alpha_n\rho\|N(u,v_n^*) - N(u,v)\| + \alpha_n\eta\|z_n - x\|. \end{aligned}$$

By using of (3.6)-(3.8), we obtain

$$\begin{aligned} &\|x_{n+1} - x\| \\ &\le (1-\alpha_n)\|x_n - x\| + \alpha_n[2\sqrt{1 - 2\sigma + \xi^2 - 2\lambda + \varepsilon^2} \\ &\quad + \eta + \rho\delta + \rho\gamma\nu + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}]\|z_n - x\| \\ &\le (1-\alpha_n)\|x_n - x\| + \alpha_n[k + (\delta + \gamma\nu)\rho + t(\rho)]\|z_n - x\| \\ &\le (1-\alpha_n)\|x_n - x\| + \alpha_n\theta\|z_n - x\|, \end{aligned} \tag{3.13}$$

where $\theta = k + (\delta + \gamma\nu)\rho + t(\rho)$, $k = 2\sqrt{1 - 2\sigma + \xi^2 - 2\lambda + \varepsilon^2} + \eta$, and $t(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\mu^2}$. From (3.4), we see that $\theta < 1$. In a similar way, from (3.10) and (3.12), we have

$$\|z_n - x\| \le (1 - \beta_n)\|x_n - x\| + \beta_n\theta\|y_n - x\| \tag{3.14}$$

and from (3.9) and (3.12), we obtain

$$\begin{aligned} \|y_n - z\| &\le (1 - \gamma_n)\|x_n - x\| + \gamma_n\theta\|x_n - x\| \\ &\le \|x_n - x\|. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we see that

$$\begin{aligned} \|z_n - x\| &\le (1 - \beta_n)\|x_n - x\| + \beta_n\theta\|x_n - x\| \\ &\le \|x_n - x\|. \end{aligned} \tag{3.16}$$

Combining (3.13) and (3.16), we have

$$\begin{aligned} \|x_{n+1} - x\| &\le (1 - \alpha_n)\|x_n - x\| + \alpha_n\theta\|x_n - x\| \\ &= [1 - (1 - \theta)\alpha_n]\|x_n - x\| \\ &\le \Pi_{i=0}^{n}[1 - (1 - \theta)\alpha_i]\|x_0 - x\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1 - \theta > 0$, we see that $\lim_{n\to\infty}\Pi_{i=0}^{n}[1 - (1 - \theta)\alpha_i] = 0$. Consequently, the sequence $\{x_n\}$ converges strongly to x. From (3.15) and (3.16), it follows that the sequences $\{y_n\}$ and $\{z_n\}$ also converge to x in H. □

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