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ANALYTIC PROPERTIES OF THE LIMITS OF THE EVEN AND ODD HYPERPOWER SEQUENCES

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Dedicated to the memory of the late professor Eulyong Pak.

ABSTRACT. Let $h_e(x)$ and $h_o(x)$ denote the limits of the sequences $\{{}^{2n}x\}$ and $\{{}^{2n+1}x\}$, respectively. Asymptotic formulas for the functions h_e and h_o at the points e^{-e} and 0 are established.

1. Introduction

For $x \geq 0$ the hyperpowers of x, denoted by ${}^0x, {}^1x, {}^2x, \ldots$, are defined inductively as follows:

$$ {}^0x = 1 \quad \text{and} \quad {}^{n+1}x = x^{({}^nx)}. $$

Throughout this paper, we adopt the convention that $0^0 = 1$ and $0^1 = 0$, so that ${}^{2n}0 = 1$ and ${}^{2n+1}0 = 0$ for all non-negative integers n: The even and odd hyperpower sequences $\{{}^{2n}x\}$ and $\{{}^{2n+1}x\}$ converge to 1 and 0 respectively when $x = 0$. Since ${}^n1 = 1$ for all n, the hyperpower sequence $\{{}^nx\}$ converges to 1 when $x = 1$.

From the definition, if one of the sequences $\{{}^{2n}x\}$ and $\{{}^{2n+1}x\}$ converges, then so does the other. In fact, it is well known that they converge if and only if $x \in [0, e^{1/e}]$. (See [4] and [7].) We denote their limits by $h_e(x)$ and $h_o(x)$, respectively:

$$ h_e(x) = \lim_{n\to\infty} {}^{2n}x \quad \text{and} \quad h_o(x) = \lim_{n\to\infty} {}^{2n+1}x \qquad (0 \leq x \leq e^{1/e}). $$

It is clear that $x^{h_e(x)} = h_o(x)$ and $x^{h_o(x)} = h_e(x)$. Therefore if $h_e(x) = y$ or $h_o(x) = y$, then $x^{x^y} = y$.

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Many authors have dealt with the hyperpower sequences, their limits and related objects. (See [1], [3], [4] and [7]. Especially, [4] and its references.) Among the results, the following are established in [4] and [7].

(1) They are continuous in $[0, e^{1/e}]$ and analytic in $(0, e^{-e}) \cup (e^{-e}, e^{1/e})$.
(2) $h_e(0) = 1$, $h_o(0) = 0$, h_e is strictly decreasing but h_o is strictly increasing in $[0, e^{-e}]$, and $h_e(e^{-e}) = h_o(e^{-e}) = e^{-1}$. In particular, $h_o(x) < h_e(x)$ for $x \in [0, e^{-e})$.
(3) If $x \in [e^{-e}, e^{1/e}]$, then $h_e(x) = h_o(x)$; and $h_e(e^{1/e}) = h_o(e^{1/e}) = e$.

As a consequence, the sequence $\{{}^n x\}$ converges if and only if $x \in [e^{-e}, e^{1/e}]$. We denote the limit by $h(x)$: If $x \in [e^{-e}, e^{1/e}]$, then $h_e(x) = h_o(x) = h(x)$ and $x^{h(x)} = h(x)$. In particular, the function $h : [e^{-e}, e^{1/e}] \to [e^{-1}, e]$ is the inverse of the strictly increasing function $[e^{-1}, e] \ni x \mapsto x^{1/x} \in [e^{-e}, e^{1/e}]$. Therefore the properties of h can be derived from those of $x \mapsto x^{1/x}$. On the other hand, for $a \in (0, e^{-e})$ the functions h_e and h_o can be approximated by their Taylor polynomials in a neighborhood of a, because they are analytic at a. (The general properties of analytic functions that are needed in this paper can be found in [5, Chapter 2] and [6, Chapter 10].) Since $x^{x^{h_e(x)}} = h_e(x)$ and $x^{x^{h_o(x)}} = h_o(x)$, we can calculate, at least theoretically, the Taylor polynomials by implicit differentiations. It seems, however, few results are known about the behavior of h_e and h_o at the points e^{-e} and 0.

In this paper, we describe the asymptotic behavior of $h_e(x)$ and $h_o(x)$ for $x \to e^{-e}$ with $x < e^{-e}$ and for $x \to 0$ with $x > 0$: We shall use the Landau O- and o-notation. (For the definition, see [2, Chapter 1].) The main results are stated and explained in Section 2. In Section 3, we briefly state some basic properties of the functions h_e and h_o. Finally, in Sections 4 and 5, we prove the main results.

2. Main results

We start this section by explaining the speed of convergence of the sequences $\{{}^{2n} x\}$ and $\{{}^{2n+1} x\}$. For $n = 0, 1, 2, \ldots$ we set $U_n(x) =$

${}^{2n+2}x - {}^{2n}x$ and $V_n(x) = {}^{2n+3}x - {}^{2n+1}x$, so that

$$(2.1)\qquad \begin{aligned} h_e(x) &= {}^{2n}x + \sum_{k=n}^{\infty} U_k(x) \quad \text{and} \\ h_o(x) &= {}^{2n+1}x + \sum_{k=n}^{\infty} V_k(x) \qquad (0 \le x \le e^{1/e}) \end{aligned}$$

for every n. In the next section, we will show that

$$(2.2)\qquad \begin{aligned} \lim_{n\to\infty} \frac{U_n(x)}{U_{n-1}(x)} &= \lim_{n\to\infty} \frac{V_n(x)}{V_{n-1}(x)} \\ &= \log h_e(x) \log h_o(x) \qquad (0 < x \le e^{1/e},\ x \ne 1). \end{aligned}$$

Since $h_e(e^{-e}) = h_o(e^{-e}) = e^{-1}$, we have $\log h_e(e^{-e}) \log h_o(e^{-e}) = 1$, and we will show that $\log h_e(x) \log h_o(x) \to 0$ as $x \to 0+$. (See Proposition 4.5.) This implies that the sequences $\{{}^{2n}x\}$ and $\{{}^{2n+1}x\}$ converge very slowly when x is near e^{-e}, but very fast when x is near 0.

To describe the behavior of h_e and h_o at e^{-e}, we represent them without using the sequences $\{{}^{2n}x\}$ and $\{{}^{2n+1}x\}$; and at 0, approximate them with the sequences. The following is proved in Section 4.

THEOREM 2.1. *There is a continuous and bijective function* $\varphi : [-e^{-e/2}, e^{-e/2}] \to [0,1]$ *such that*

(1) φ *is analytic in* $(-e^{-e/2}, e^{-e/2})$,
(2) $\varphi'(s) > 0$ *for* $s \in (-e^{-e/2}, e^{-e/2})$,
(3) $\varphi(0) = e^{-1}$,
(4) $h_e(x) = \varphi(\sqrt{e^{-e} - x})$ *for* $x \in [0, e^{-e}]$, *and*
(5) $h_o(x) = \varphi(-\sqrt{e^{-e} - x})$ *for* $x \in [0, e^{-e}]$.

For $k = 0, 1, 2, \ldots$ we set $a_k = \varphi^{(k)}(0)/k!$. Since φ is analytic at 0, there is a positive constant δ such that $\sum_{k=0}^{\infty} a_k s^k$ converges absolutely to $\varphi(s)$ for every $s \in (-\delta, \delta)$. In particular, we obtain the following:

COROLLARY. *Suppose* n *is a non-negative integer. Then, for* $x \to e^{-e}$ *with* $x < e^{-e}$, *the following hold:*

$$h_e(x) = \sum_{k=0}^{n} a_k (e^{-e} - x)^{k/2} + O\left((e^{-e} - x)^{(n+1)/2}\right) \quad \textit{and}$$

$$h_o(x) = \sum_{k=0}^{n} (-1)^k a_k (e^{-e} - x)^{k/2} + O\left((e^{-e} - x)^{(n+1)/2}\right).$$

Since $\varphi(0) = e^{-1}$, we have $a_0 = e^{-1}$. To determine the coefficients $a_1, a_2, \ldots$, we put $t = e^{e/2}s$ and write

$$\varphi(s) = e^{-1}\left(1 + At\left(1 + \sum_{n=1}^{\infty} b_n t^n\right)\right), \tag{2.3}$$

so that $a_1 = e^{-1}Ae^{e/2}$ and $a_n = e^{-1}Ae^{ne/2}b_{n-1}$ for $n = 2, 3, \ldots$. The right-hand side converges absolutely for all t sufficiently close to 0. Since $h_e(x)\log x = \log h_o(x)$ and $h_o(x)\log x = \log h_e(x)$ for all $x \in (0, e^{-e}]$, Theorem 2.1 implies that

$$\varphi(-s)\log(e^{-e} - s^2) = \log\varphi(s) \qquad (-e^{-e/2} < s < e^{-e/2}). \tag{2.4}$$

In Section 4, we will show that an analytic function φ is uniquely determined by this equation and the condition that $\varphi'(0) > 0$. (See Proposition 4.6.) From (2.3) and (2.4),

$$\begin{aligned} & e^{-1}\left(1 - At\left(1 + \sum_{n=1}^{\infty}(-1)^n b_n t^n\right)\right)\left(-e - \sum_{n=1}^{\infty}\frac{1}{n}t^{2n}\right) \\ = & -1 + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}A^m t^m\left(1 + \sum_{n=1}^{\infty} b_n t^n\right)^m \end{aligned}$$

for all t sufficiently close to 0; and we have $A > 0$, because $\varphi'(0) > 0$. Hence, by comparing the coefficients of both sides of this equation, one can determine $A, b_1, b_2, \ldots$ successively. For instance, $A = \sqrt{6/e}$, $b_1 = \frac{1}{6}A$, $b_2 = \frac{1}{4} - \frac{57}{360}A^2$, $b_3 = \frac{1}{12}A - \frac{2}{45}A^3$, $b_4 = \frac{13}{96} - \frac{19}{160}A^2 + \frac{16547}{604800}A^4$ and $b_5 = \frac{1}{18}A - \frac{2}{45}A^3 + \frac{134}{14175}A^5$. This result and the corollary to Theorem 2.1 describe the asymptotic behavior of $h_e(x)$ and $h_o(x)$ for $x \to e^{-e}$ with $x < e^{-e}$. For instance, we have

$$h_e(x) = e^{-1} + \sqrt{6}e^{(e-3)/2}\sqrt{e^{-e} - x} + O\left(e^{-e} - x\right) \quad \text{and}$$
$$h_o(x) = e^{-1} - \sqrt{6}e^{(e-3)/2}\sqrt{e^{-e} - x} + O\left(e^{-e} - x\right).$$

REMARKS 2.1. (i) The result shows that the curves $y = h_e(x)$ and $y = h_o(x)$ have a vertical tangent at (e^{-e}, e^{-1}), and hence h_e and h_o are not analytic at the point e^{-e}. (ii) It seems that $\sum_{k=1}^{\infty}|b_k| < \infty$. If it were true, we would have

$$h_e(x) = \sum_{k=0}^{\infty} a_k(e^{-e} - x)^{k/2} \quad \text{and}$$
$$h_o(x) = \sum_{k=0}^{\infty}(-1)^k a_k(e^{-e} - x)^{k/2} \qquad (0 \le x \le e^{-e}),$$

and the series converge absolutely for every $x \in [0, e^{-e}]$. The authors do not know how to prove this.

To describe the behavior of h_e and h_o at 0, we introduce the polynomials $P_0, P_1, P_2, \ldots$ and $Q_0, Q_1, Q_2, \ldots$ that are defined inductively as follows: $P_0 = Q_0 = 1$, and

$$P_n(y) = \lim_{x\to 0} x^{-n}\left(\exp\left(\sum_{k=1}^{n} x^k y Q_{k-1}(y)\right) - \sum_{k=0}^{n-1} x^k P_k(y)\right),$$
$$Q_n(y) = \lim_{x\to 0} x^{-n}\left(\exp\left(\sum_{k=1}^{n} x^k y P_k(y)\right) - \sum_{k=0}^{n-1} x^k Q_k(y)\right).$$

In Section 5, we will prove that these polynomials are well defined, and that $\deg P_n = 2n-1$ and $\deg Q_n = 2n$ for $n \geq 1$. (See Lemma 5.1, (5.4), (5.5) and Proposition 5.2.) A direct calculation shows that $P_1(y) = y$, $Q_1(y) = y^2$, $P_2(y) = \frac{1}{2}y^2 + y^3$, $Q_2(y) = \frac{1}{2}y^3 + \frac{3}{2}y^4$, $P_3(y) = \frac{1}{6}y^3 + \frac{3}{2}y^4 + \frac{3}{2}y^5$, $Q_3(y) = \frac{1}{6}y^4 + 2y^5 + \frac{8}{3}y^6$, and so on. In the same section, the following are proved:

PROPOSITION 2.2. *Suppose n is a non-negative integer. Then, for $x \to 0+$, the following hold:*

$$^{2n}x = \sum_{k=0}^{n} x^k P_k(\log x) + O\left(x^{n+1}|\log x|^{2n+1}\right) \quad \textit{and}$$
$$^{2n+1}x = x\sum_{k=0}^{n} x^k Q_k(\log x) + O\left(x^{n+2}|\log x|^{2n+2}\right).$$

PROPOSITION 2.3. *For each $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < x \leq \delta$, then the inequalities*

$$|h_e(x) - {}^{2n}x| \leq (1+\epsilon)^{n+1} x^{n+1} |\log x|^{2n+1}$$

and

$$|h_o(x) - {}^{2n+1}x| \leq (1+\epsilon)^{n+2} x^{n+2} |\log x|^{2n+2}$$

hold for all non-negative integers n.

As an immediate consequence of these propositions, we obtain the following:

THEOREM 2.4. *Suppose n is a non-negative integer. Then, for $x \to 0+$, the following hold:*

$$\begin{aligned} h_e(x) =& \sum_{k=0}^{n} x^k P_k(\log x) + O\left(x^{n+1} |\log x|^{2n+1}\right) \quad \textit{and} \\ h_o(x) =& x \sum_{k=0}^{n} x^k Q_k(\log x) + O\left(x^{n+2} |\log x|^{2n+2}\right). \end{aligned}$$

This theorem describes the asymptotic behavior of $h_e(x)$ and $h_o(x)$ for $x \to 0$ with $x > 0$. For instance, we have

$$\begin{aligned} h_e(x) &= 1 + x \log x + O\left(x^2 |\log x|^3\right) \quad \text{and} \\ h_o(x) &= x + x^2 (\log x)^2 + O\left(x^3 |\log x|^4\right). \end{aligned} \tag{2.5}$$

REMARKS 2.2. (i) The result shows that h_e and h_o cannot be extended to analytic functions in an open interval containing 0. (ii) The result also shows that the right-hand derivative of h_e at 0 does not exist, but that of h_o exists and is equal to 1. (iii) Theorem 2.4 gives no information about the convergence of the series

$$\sum_{k=0}^{\infty} x^k P_k(\log x) \quad \text{and} \quad \sum_{k=0}^{\infty} x^{k+1} Q_k(\log x).$$

It seems, however, that for every $x \in (0, e^{1/e}]$ these series converge to $h_e(x)$ and $h_o(x)$, respectively.

Finally, the following shows that h_o can be extended to a C^1-function in an open interval containing 0, but not to a C^2-one.

PROPOSITION 2.5. *$h_o'(x) \to 1$ and $x^{-1}(h_o'(x) - 1) \to \infty$ as $x \to 0+$.*

This proposition also is proved in Section 5.

3. Preliminaries

In this short section, we state some basic properties of the functions h_e and h_o. First of all, it is easy to see that if $x \in [0, 1]$, then

$${}^{2n+1}x \leq {}^{2n+2}x \leq {}^{2n}x \text{ and } {}^{2n+2}x \geq {}^{2n+3}x \geq {}^{2n+1}x.$$

From this, it follows that ${}^{2n+1}x \le h_o(x) \le h_e(x) \le {}^{2n}x$ for all $x \in [0,1]$. On the other hand, it is not hard to see that if $x \in [0, e^{-e}]$, then ${}^{2n+1}x \le e^{-1} \le {}^{2n}x$. (See [4, p. 242] and [7, p. 14].) Consequently,

$${}^{2n+1}x \le h_o(x) \le e^{-1} \le h_e(x) \le {}^{2n}x \qquad (0 \le x \le e^{-e},\ n = 0, 1, 2, \dots).$$

In particular,

$$0 \le h_o(x) \le e^{-1} \le h_e(x) \le 1 \qquad (0 \le x \le e^{-e}). \tag{3.1}$$

Since $h_e(x) = x^{h_o(x)}$ and $h_o(x) = x^{h_e(x)}$ whenever the sequences converge, and since $h_e(x), h_o(x) > 0$ for $x \in (0, e^{1/e}]$, we have

$$h_e(x)^{1/h_o(x)} = h_o(x)^{1/h_e(x)} = x \qquad (0 < x \le e^{1/e}), \tag{3.2}$$

and hence

$$h_e(x)^{h_e(x)} = h_o(x)^{h_o(x)} \qquad (0 \le x \le e^{1/e}). \tag{3.3}$$

We close this section by proving (2.2).

Proof of (2.2). Suppose that $x \in (0, e^{1/e}] \setminus \{1\}$. Then the sequences $\{U_n(x)\}$ and $\{V_n(x)\}$ converge to zero. Since $U_n(x) = {}^{2n+2}x - {}^{2n}x = {}^{2n}x\,(\exp(V_{n-1}(x)\log x) - 1)$, this implies that

$$\lim_{n\to\infty} \frac{U_n(x)}{V_{n-1}(x)} = \lim_{n\to\infty} {}^{2n}x \frac{e^{V_{n-1}(x)\log x} - 1}{V_{n-1}(x)} = h_e(x)\log x = \log h_o(x).$$

Similarly,

$$\lim_{n\to\infty} \frac{V_n(x)}{U_n(x)} = \log h_e(x).$$

Now the result is obvious. □

4. Proof of Theorem 2.1

In this section, the following lemma will play a basic role. The proof is trivial.

LEMMA 4.1. *Let f be a real analytic function defined in an open interval (a,b), and suppose that $c \in (a,b)$, $f(x) > 0$ for $x \neq c$, $f(c) = f'(c) = 0$, and $f''(c) \neq 0$. If $\tilde{f}$ is defined by*

$$\tilde{f}(x) = \begin{cases} -\sqrt{f(x)} & (a < x < c), \\ \sqrt{f(x)} & (c \leq x < b), \end{cases}$$

then $\tilde{f}$ is analytic in (a,b) and $\tilde{f}'(c) = \sqrt{f''(c)/2}$.

Note that we must have $f''(c) > 0$.

We need to introduce some functions and establish their properties. Let the function $F : [0,1] \to [-\sqrt{1-e^{-1/e}}, \sqrt{1-e^{-1/e}}]$ be defined by

$$F(x) = \begin{cases} -\sqrt{x^x - e^{-1/e}} & (0 \leq x \leq e^{-1}), \\ \sqrt{x^x - e^{-1/e}} & (e^{-1} \leq x \leq 1). \end{cases}$$

This function is well defined, continuous and bijective; Lemma 4.1 implies that F is analytic in $(0,1)$; and it is clear that $F'(x) > 0$ for all $x \in (0,1)$.

Let G denote the function

$$[0,1] \ni x \mapsto F^{-1}(-F(x)) \in [0,1].$$

Then $G(0) = 1$, $G(e^{-1}) = e^{-1}$, $G(1) = 0$, G is continuous in $[0,1]$, analytic in $(0,1)$, and we have $G'(x) < 0$ for $x \in (0,1)$. It is easy to see that $G(G(x)) = x$ and $G(x)^{G(x)} = x^x$ for $x \in [0,1]$. From this, (3.1) and (3.3), we obtain

$$h_e(x) = G(h_o(x)) \quad \text{and} \quad h_o(x) = G(h_e(x)) \qquad (0 \leq x \leq e^{-e}), \tag{4.1}$$

and

$$G(x) \log G(x) = x \log x \qquad (0 < x < 1). \tag{4.2}$$

Since $G(e^{-1}) = e^{-1}$, $G'(e^{-1}) < 0$ and $G(G(x)) = x$, we have $G'(e^{-1}) = -1$. In a neighborhood of e^{-1}, the analytic function G is represented by an absolutely convergent power series:

$$G(x) = \sum_{n=0}^{\infty} c_n (x - e^{-1})^n.$$

We have $c_0 = e^{-1}$ and $c_1 = -1$, because $G(e^{-1}) = e^{-1}$ and $G'(e^{-1}) = -1$. Hence, using (4.2), one can determine the coefficients $c_2, c_3, \dots$ successively. For instance, $c_2 = \frac{1}{3}e$, $c_3 = -\frac{1}{9}e^2$, $c_4 = \frac{17}{270}e^3$, and $c_5 = -\frac{31}{810}e^4$. Note that an analytic function $G : (0,1) \to \mathbb{R}$ is uniquely determined by (4.2) and the condition that $G'(e^{-1}) < 0$.

We can parameterize the curve $y = G(x)$ as follows: Put $x^{-1}G(x) = t$. As x increases from 0 to 1, t decreases from ∞ to 0; and $t = 1$ if and only if $x = e^{-1}$. From (4.2) one can easily deduce that

(4.3)
$$\log x = \frac{t}{1-t}\log t \quad \text{and} \quad \log G(x) = \frac{1}{1-t}\log t \qquad (0 < t < \infty,\ t \neq 1),$$

and this is equivalent to

$$x = t^{\frac{t}{1-t}} \quad \text{and} \quad y = t^{\frac{1}{1-t}} \qquad (0 < t < \infty,\ t \neq 1).$$

We remark that an equivalent version of this parameterization is due to Goldbach. See [4, p. 237].

LEMMA 4.2. *The function* $x \mapsto \log x \log G(x)$ *is strictly increasing in* $(0, e^{-1})$, *has maximum value* 1 *at* $x = e^{-1}$, *and is strictly decreasing in* $(e^{-1}, 1)$; *and*

$$\lim_{x\to 0+} \log x \log G(x) = \lim_{x \to 1-} \log x \log G(x) = 0.$$

Proof. If we put $x^{-1}G(x) = t$, then t decreases from ∞ to 0 as x increases from 0 to 1, $t = 1$ if and only if $x = e^{-1}$, and (4.3) implies that

$$\log x \log G(x) = t\left(\frac{\log t}{t-1}\right)^2 \qquad (0 < t < \infty,\ t \neq 1).$$

Now, the result is proved by calculus. □

Define H by

$$H(x) = \begin{cases} 0 & (x = 0), \\ G(x)^{1/x} & (0 < x \leq 1). \end{cases}$$

This function is analytic in $(0,1)$, continuous at 1, and $H(1) = 0$. Moreover, H is continuous at 0 too: Since $0 \leq G(x) \leq 1$ and $G(x)^{G(x)} = x^x$ for $x \in [0,1]$, we have

$$0 \leq H(x)^x = G(x) \leq G(x)^{G(x)} = x^x \qquad (0 \leq x \leq 1),$$

and hence

$$0 \leq H(x) \leq x \qquad (0 \leq x \leq 1).$$

PROPOSITION 4.3. *If* $x \in [0, e^{-e}]$, *then* $H(h_e(x)) = H(h_o(x)) = x$.

Proof. The result follows from (3.2), (4.1) and the definition of H.□

PROPOSITION 4.4. $H'(x) > 0$ *for* $x \in (0, e^{-1})$, $H'(x) < 0$ *for* $x \in (e^{-1}, 1)$, $H(e^{-1}) = e^{-e}$, $H'(e^{-1}) = 0$, *and* $H''(e^{-1}) = -\frac{1}{3}e^{3-e}$.

Proof. First of all, the last three assertions are proved by straightforward calculation, because $G(e^{-1}) = e^{-1}$, $G'(e^{-1}) = -1$, $G''(e^{-1}) = 2c_2 = \frac{2}{3}e$, and $H(x) = G(x)^{1/x}$ for $x \in (0, 1)$.

By differentiating both sides of (4.2), we obtain $G'(x)(1+\log G(x)) = 1+\log x$, which is valid for all $x \in (0, 1)$. On the other hand, $\log H(x) = x^{-1}\log G(x)$ for $x \in (0, 1)$. Hence, by straightforward calculation, we have

$$\frac{H'(x)}{H(x)} = \frac{-G(x)\log G(x)\,(1+\log G(x)) + x(1+\log x)}{x^2G(x)\,(1+\log G(x))} \qquad (0 < x < 1,\ x \neq e^{-1}).$$

The right-hand side is simplified with the aid of (4.2):

$$\begin{aligned} \frac{H'(x)}{H(x)} &= \frac{-x\log x\,(1+\log G(x)) + x(1+\log x)}{x^2G(x)\,(1+\log G(x))} \\ &= \frac{1-\log x\log G(x)}{xG(x)\,(1+\log G(x))} \qquad (0 < x < 1,\ x \neq e^{-1}). \end{aligned} \tag{4.4}$$

Lemma 4.2 implies that $1 - \log x \log G(x) > 0$ for all $x \in (0, 1)$ with $x \neq e^{-1}$, and it is clear that $xG(x) > 0$ for all $x \in (0, 1)$. From this, the first two assertions follow, because G is strictly decreasing and $\log G(e^{-1}) = -1$. □

Now, we can prove Theorem 2.1.

Proof of Theorem 2.1. Let the function $\tilde{H} : [0, 1] \to [-e^{-e/2}, e^{-e/2}]$ be defined by

$$\tilde{H}(x) = \begin{cases} -\sqrt{e^{-e} - H(x)} & (0 \le x \le e^{-1}), \\ \sqrt{e^{-e} - H(x)} & (e^{-1} \le x \le 1). \end{cases}$$

From Lemma 4.1 and Proposition 4.4, we have the following: $\tilde{H}$ is well defined, continuous, bijective, $\tilde{H}(e^{-1}) = 0$, $\tilde{H}$ is analytic in $(0, 1)$ and

$\tilde{H}'(x) > 0$ for all $x \in (0,1)$. Moreover (3.1), Proposition 4.3, and the definition of $\tilde{H}$ imply that
(4.5)
$$\tilde{H}(h_e(x)) = \sqrt{e^{-e} - x} \text{ and } \tilde{H}(h_o(x)) = -\sqrt{e^{-e} - x} \quad (0 \le x \le e^{-e}).$$

If we denote the inverse of $\tilde{H}$ by φ, then φ is continuous in $[-e^{-e/2}, e^{-e/2}]$, analytic in $(-e^{-e/2}, e^{-e/2})$, $\varphi'(s) > 0$ for all $s \in (-e^{-e/2}, e^{-e/2})$, $\varphi(0) = e^{-1}$, and (4.5) is equivalent to

$$h_e(x) = \varphi(\sqrt{e^{-e} - x}) \quad \text{and}$$
$$h_o(x) = \varphi(-\sqrt{e^{-e} - x}) \quad (0 \le x \le e^{-e}).$$

This proves Theorem 2.1. □

At this point, it should be remarked that (3.1) and (3.2) are the only properties of h_e and h_o that are used in our proof of Theorem 2.1: (3.3) is a consequence of (3.2).

It remains to prove the following two propositions.

PROPOSITION 4.5. *$\log h_e(x) \log h_o(x) \to 0$ as $x \to 0+$.*

PROPOSITION 4.6. *Let $\tilde{\varphi} : (-e^{-e/2}, e^{-e/2}) \to \mathbb{R}$ be an analytic function. Suppose that $\tilde{\varphi}'(0) > 0$ and*

$$\tilde{\varphi}(-s) \log(e^{-e} - s^2) = \log \tilde{\varphi}(s) \qquad (-e^{-e/2} < s < e^{-e/2}). \tag{4.6}$$

Then $\tilde{\varphi}(s) = \varphi(s)$ for all $s \in (-e^{-e/2}, e^{-e/2})$.

Proof of Proposition 4.5. Theorem 2.1 implies that h_e is continuous at 0. Hence $h_e(x) \to h_e(0) = 1$ as $x \to 0+$. From (4.1), we have $\log h_e(x) \log h_o(x) = \log h_e(x) \log G(h_e(x))$. Therefore the result follows from Lemma 4.2. □

Proof of Proposition 4.6. First of all, (4.6) implies that $\tilde{\varphi}(0) = e^{-1}$. Since $\tilde{\varphi}'(0) > 0$, there is a positive real number a, with $a < e^{-e/2}$, such that $\tilde{\varphi}$ is increasing in the interval $(-a, a)$. Since $0 < \tilde{\varphi}(0) = e^{-1} < 1$, we may assume, by taking a sufficiently small, that $0 < \tilde{\varphi}(-a) < e^{-1} < \tilde{\varphi}(a) < 1$.

For $x \in (e^{-e} - a^2, e^{-e}]$ define $\tilde{h}_e(x)$ and $\tilde{h}_o(x)$ by

$$\tilde{h}_e(x) = \tilde{\varphi}(\sqrt{e^{-e} - x}) \quad \text{and} \quad \tilde{h}_o(x) = \tilde{\varphi}(-\sqrt{e^{-e} - x}).$$

Since $\tilde{\varphi}$ is increasing in $(-a, a)$ and $0 < \tilde{\varphi}(-a) < \tilde{\varphi}(0) = e^{-1} < \tilde{\varphi}(a) < 1$, we have

$$0 < \tilde{h}_o(x) \leq e^{-1} \leq \tilde{h}_e(x) < 1 \qquad (e^{-e} - a^2 < x \leq e^{-e}).$$

Moreover, (4.6) implies that

$$\tilde{h}_e(x)^{1/\tilde{h}_o(x)} = \tilde{h}_o(x)^{1/\tilde{h}_e(x)} = x \qquad (e^{-e} - a^2 < x \leq e^{-e}).$$

Hence essentially the same argument as the proof of Theorem 2.1 shows that

$$\tilde{h}_e(x) = \varphi(\sqrt{e^{-e} - x}) \quad \text{and}$$
$$\tilde{h}_o(x) = \varphi(-\sqrt{e^{-e} - x}) \qquad (e^{-e} - a^2 < x \leq e^{-e}).$$

Therefore $\tilde{\varphi}(s) = \varphi(s)$ for all $s \in (-a, a)$. From this, we obtain the desired result. □

5. Proofs of Propositions 2.2, 2.3 and 2.5

For $x, y \in \mathbb{R}$ we define $h_0(x, y), h_1(x, y), h_2(x, y), \ldots$ as follows: $h_0(x, y) = 1$, and

$$h_{2n+1}(x, y) = x \exp\left(y h_{2n}(x, y) - y\right),$$
$$h_{2n+2}(x, y) = \exp\left(y h_{2n+1}(x, y)\right).$$

For instance, $h_1(x, y) = x$, $h_2(x, y) = 1 + xy + \frac{1}{2}x^2y^2 + \cdots$, and so on. It is clear that ${}^n x = h_n(x, \log x)$ for $x > 0$ and $n = 0, 1, 2, \ldots$.

LEMMA 5.1. *For each non-negative integer n there are polynomials $P_{(n,0)}, \ldots, P_{(n,n)}$ and $Q_{(n,0)}, \ldots, Q_{(n,n)}$, with $\deg P_{(n,k)} = \max\{0, 2k-1\}$ and $\deg Q_{(n,k)} = 2k$ for all k, such that*

$$h_{2n}(x, y) = \sum_{k=0}^{n} x^k P_{(n,k)}(y) + O\left(x^{n+1}\right) \qquad (x \to 0) \tag{5.1}$$

and

$$h_{2n+1}(x, y) = x \sum_{k=0}^{n} x^k Q_{(n,k)}(y) + O\left(x^{n+2}\right) \qquad (x \to 0) \tag{5.2}$$

hold for each fixed y.

Proof. First of all, we set $P_{(n,0)} = Q_{(n,0)} = 1$ for all non-negative integers n: The lemma holds trivially when $n = 0$. For each positive integer n let P(n) denote the statement that there are polynomials $P_{(n,1)}, \ldots, P_{(n,n)}$, with $\deg P_{(n,k)} = 2k - 1$ for all k, such that (5.1) holds for each fixed y; and Q(n) the statement that there are polynomials $Q_{(n,1)}, \ldots, Q_{(n,n)}$, with $\deg Q_{(n,k)} = 2k$ for all k, such that (5.2) holds for each fixed y. Since $h_2(x,y) = 1 + xy + \frac{1}{2}x^2y^2 + \cdots$, P(1) is obvious, with $P_{(1,1)}(y) = y$. Hence the lemma will follow once we show that P(n) implies Q(n) and Q(n) implies P($n+1$).

Let n be arbitrary. Suppose that P(n) is true. For convenience, we set $\tilde{P}_{(n,k)}(y) = yP_{(n,k)}(y)$: It is clear that $\deg \tilde{P}_{(n,k)} = 2k$. For each fixed y we have

$$\begin{aligned} h_{2n+1}(x,y) &= x \exp\left(yh_{2n}(x,y) - y\right) \\ &= x \exp\left(\sum_{k=1}^{n} x^k \tilde{P}_{(n,k)}(y) + O\left(x^{n+1}\right)\right) \\ &= x \exp\left(\sum_{k=1}^{n} x^k \tilde{P}_{(n,k)}(y)\right) + O\left(x^{n+2}\right) \qquad (x \to 0). \end{aligned}$$

Since $\deg \tilde{P}_{(n,k)} = 2k$ for all k, it follows that

(5.3)
$$\exp\left(\sum_{k=1}^{n} x^k \tilde{P}_{(n,k)}(y)\right) = 1 + \sum_{k=1}^{n} x^k Q_{(n,k)}(y) + O\left(x^{n+1}\right) \qquad (x \to 0)$$

for some polynomials $Q_{(n,1)}, \ldots, Q_{(n,n)}$, with $\deg Q_{(n,k)} = 2k$ for all k. From this, Q(n) follows. The statement that Q(n) implies P($n+1$) is proved similarly. □

For each non-negative integer n we set $P_n = P_{(n,n)}$ and $Q_n = Q_{(n,n)}$. Then (5.3) implies that

(5.4)
$$Q_n(y) = \lim_{x \to 0} x^{-n} \left(\exp\left(\sum_{k=1}^{n} x^k y P_{(n,k)}(y)\right) - \sum_{k=0}^{n-1} x^k Q_{(n,k)}(y)\right)$$

for every positive integer n; and similarly,

(5.5)
$$P_n(y) = \lim_{x \to 0} x^{-n} \left(\exp\left(\sum_{k=1}^{n} x^k y Q_{(n-1,k-1)}(y)\right) - \sum_{k=0}^{n-1} x^k P_{(n,k)}(y)\right)$$

for every positive integer n.

PROPOSITION 5.2. *Suppose* $0 \le k < n$. *Then* $P_{(n,k)} = P_k$ *and* $Q_{(n,k)} = Q_k$.

To prove this proposition as well as Propositions 2.2, 2.3 and 2.5, we need some lemmas.

LEMMA 5.3. *Suppose* n *is a non-negative integer. Then, for* $x \to 0+$, *the following hold:*

$$\begin{aligned} {}^{2n}x &= \sum_{k=0}^{n} x^k P_{(n,k)}(\log x) + O\left(x^{n+1}|\log x|^{2n+1}\right) \quad \textit{and} \\ {}^{2n+1}x &= x\sum_{k=0}^{n} x^k Q_{(n,k)}(\log x) + O\left(x^{n+2}|\log x|^{2n+2}\right). \end{aligned}$$

Proof. The proof is essentially the same as Lemma 5.1. □

LEMMA 5.4. *For each non-negative integer* n *the following hold:*

$$\begin{aligned} &|U_n(x)| \le x h_o(x)^n |\log x|^{2n+1}, \\ &|V_n(x)| \le x h_o(x)^{n+1} |\log x|^{2n+2} \quad (0 < x < 1). \end{aligned}$$

Proof. It is easy to see that the inequality

$$|a - b| \le \max\{a, b\} |\log a - \log b|$$

holds for all positive real numbers a and b. Let $0 < x < 1$. Then the sequence $\{{}^{2n}x\}$ is decreasing and $\{{}^{2n+1}x\}$ increasing. Since $\{{}^{2n}x\}$ is decreasing, we have

$$\begin{aligned} |U_n(x)| &= |{}^{2n+2}x - {}^{2n}x| \\ &\le {}^{2n}x \,|\log({}^{2n+2}x) - \log({}^{2n}x)| \\ &\le {}^{2n}x \,|{}^{2n+1}x \log x - {}^{2n-1}x \log x| \\ &= {}^{2n}x \,|\log x|\, |{}^{2n+1}x - {}^{2n-1}x| \\ &= {}^{2n}x \,|\log x|\, |V_{n-1}(x)| \qquad (n = 1, 2, \dots), \end{aligned}$$

and it is clear that $|U_0(x)| \le x|\log x|$. Similarly,

$$|V_n(x)| \le {}^{2n+3}x \,|\log x|\, |U_n(x)| \qquad (n = 0, 1, 2, \dots).$$

Now, the result is proved by induction, because ${}^{2n}x \le 1$ and ${}^{2n+1}x \le h_o(x)$ for all n. □

LEMMA 5.5. *For each $\epsilon > 0$ there is a δ, with $0 < \delta \le e^{-e}$, such that if $0 \le x \le \delta$, then $h_o(x) \le (1+\epsilon)x$.*

Proof. From Propositions 4.3 and 4.4, the function $h_o : [0, e^{-e}] \to [0, e^{-1}]$ is the inverse of $H : [0, e^{-1}] \to [0, e^{-e}]$. Hence the assertion will follow once we prove that $x^{-1}H(x) \to 1$ as $x \to 0+$, and that $\frac{d}{dx}(x^{-1}H(x)) < 0$ for $x \in (0, e^{-1})$.

Since $G(x) \to 1$ as $x \to 0+$ and $x \log x = G(x) \log G(x)$ for $x \in (0,1)$, we have

$$\lim_{x\to 0+} \frac{1-G(x)}{x\log x} = \lim_{x\to 0+} \frac{1-G(x)}{G(x)\log G(x)} = \lim_{s\to 1} \frac{1-s}{s\log s} = -1;$$

and since $H(x) = G(x)^{1/x} = x^{1/G(x)}$ for $x \in (0,1)$,

$$\begin{aligned}\log x^{-1}H(x) &= \log H(x) - \log x = \frac{1}{G(x)}\log x - \log x \\ &= \frac{1}{G(x)}(1-G(x))\log x = \frac{1}{G(x)}\frac{1-G(x)}{x\log x}x(\log x)^2.\end{aligned}$$

Thus $\log x^{-1}H(x) \to 0$ as $x \to 0+$, that is, $x^{-1}H(x) \to 1$ as $x \to 0+$.

It remains to show that $\frac{d}{dx}(x^{-1}H(x)) < 0$ for $x \in (0, e^{-1})$. From (4.4), we have

$$\begin{aligned}\frac{d}{dx}\log x^{-1}H(x) &= \frac{H'(x)}{H(x)} - \frac{1}{x} \\ &= \frac{1 - \log x \log G(x) - G(x)\,(1+\log G(x))}{xG(x)\,(1+\log G(x))}.\end{aligned}$$

Hence we need only to show that

$$G(x)\,(1+\log G(x)) + \log x \log G(x) - 1 > 0 \qquad (0 < x < e^{-1}), \tag{5.6}$$

because $x^{-1}H(x) > 0$ and $xG(x)\,(1+\log G(x)) > 0$ for $x \in (0, e^{-1})$.

Suppose that $0 < x < e^{-1}$. Since $e^s \ge 1+s$ for all $s \in \mathbb{R}$, we have $G(x)\,(1+\log G(x)) \ge (1+\log G(x))^2$; and hence the left-hand side of (5.6) is greater than or equal to

$$(1+\log G(x))^2 + \log x \log G(x) - 1. \tag{5.7}$$

If we put $t = x^{-1}G(x)$, then $t > 1$; and (4.3) implies that (5.7) is equal to $(1-t)^{-2}(2-2t+\log t + t\log t)\log t$. From this, (5.6) follows, because $2 - 2t + \log t + t\log t > 0$ for all $t > 1$. □

Now, we can prove Propositions 5.2, 2.2, 2.3 and 2.5.

Proof of Proposition 5.2. The proposition will follow once we show that for each non-negative integer n we have

$$P_{(n,k)} = P_{(n+1,k)} \quad \text{and} \quad Q_{(n,k)} = Q_{(n+1,k)} \qquad (k = 0, 1, \ldots, n).$$

Let n be arbitrary. For $k = 0, 1, \ldots, n$ we set $P_k^* = P_{(n+1,k)} - P_{(n,k)}$. Lemma 5.3 implies that

$$\begin{aligned} U_n(x) &= {}^{2n+2}x - {}^{2n}x \\ &= \sum_{k=0}^{n} x^k P_k^*(\log x) + x^{n+1} P_{(n+1,n+1)}(\log x) \\ &\quad + O\left(x^{n+1} |\log x|^{2n+1}\right) \end{aligned}$$

for $x \to 0+$. From Lemma 5.1, $\deg P_{(n+1,n+1)} = 2n + 1$; and Lemmas 5.4 and 5.5 imply that $U_n(x) = O\left(x^{n+1}|\log x|^{2n+1}\right)$ for $x \to 0+$. Hence

$$\sum_{k=0}^{n} x^k P_k^*(\log x) = O\left(x^{n+1}|\log x|^{2n+1}\right) = o(x^n) \qquad (x \to 0+).$$

From this, we obtain $P_0^* = 0$, $P_1^* = 0, \ldots,$ and $P_n^* = 0$ successively. Therefore $P_{(n,k)} = P_{(n+1,k)}$ for all k. The statement that $Q_{(n,k)} = Q_{(n+1,k)}$ for all k is proved similarly. □

Proof of Proposition 2.2. The proposition is an immediate consequence of Proposition 5.2 and Lemma 5.3. □

Proof of Proposition 2.3. Let $\epsilon > 0$ be arbitrary. From Lemma 5.5, there is a $\delta > 0$ such that if $0 < x \leq \delta$, then $h_o(x) \leq (1 + \epsilon)x$. Suppose that $0 < x \leq \delta$. By Lemma 5.4, the inequalities

$$\begin{aligned} |U_n(x)| &\leq (1+\epsilon)^n x^{n+1} |\log x|^{2n+1} \quad \text{and} \\ |V_n(x)| &\leq (1+\epsilon)^{n+1} x^{n+2} |\log x|^{2n+2} \end{aligned}$$

hold for all non-negative integers n. Since $x(\log x)^2 \to 0$ as $x \to 0+$, we may assume, by replacing δ with a smaller one, that

$$(1 - (1+\epsilon)x(\log x)^2)^{-1} \leq 1 + \epsilon.$$

Hence the desired result follows from (2.1). □

Proof of Proposition 2.5. From Proposition 4.3, $h_o'(x) = 1/H'(h_o(x))$ for $x \in (0, e^{-e})$. Hence we obtain, by (4.1) and (4.4),

$$h_o'(x) = \frac{h_o(x)h_e(x)\,(1 + \log h_e(x))}{x\,(1 - \log h_o(x) \log h_e(x))} \qquad (0 < x < e^{-e}). \tag{5.8}$$

Therefore $h_o'(x) \to 1$ as $x \to 0+$, by (2.5) and Proposition 4.5.

It remains to show that $x^{-1}(h_o'(x) - 1) \to \infty$ as $x \to 0+$. From (5.8),

$$\frac{h_o'(x) - 1}{x} = \frac{h_o(x)h_e(x)\,(1 + \log h_e(x)) - x\,(1 - \log h_o(x) \log h_e(x))}{x^2\,(1 - \log h_o(x) \log h_e(x))}$$

for $x \in (0, e^{-e})$; and (2.5) implies that

$$\begin{aligned} h_o(x)h_e(x)\,&(1 + \log h_e(x)) - x\,(1 - \log h_o(x) \log h_e(x)) \\ &= 2x^2\left((\log x)^2 + \log x\right) + O\left(x^3 |\log x|^4\right) \end{aligned}$$

for $x \to 0+$. Hence the desired result follows from Proposition 4.5. ☐

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