ANALYTIC PROPERTIES OF THE LIMITS OF THE EVEN AND ODD HYPERPOWER SEQUENCES

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Dedicated to the memory of the late professor Eulyong Pak.

Abstract. Let \( h_e(x) \) and \( h_o(x) \) denote the limits of the sequences \( \{2^n x\} \) and \( \{2^{n+1} x\} \), respectively. Asymptotic formulas for the functions \( h_e \) and \( h_o \) at the points \( e^{-e} \) and 0 are established.

1. Introduction

For \( x \geq 0 \) the hyperpowers of \( x \), denoted by \( 0^x, 1^x, 2^x, \ldots \), are defined inductively as follows:

\[
0^x = 1 \quad \text{and} \quad n+1^x = x^n x.
\]

Throughout this paper, we adopt the convention that \( 0^0 = 1 \) and \( 0^1 = 0 \), so that \( 2^n 0 = 1 \) and \( 2^n 1 = 0 \) for all non-negative integers \( n \): The even and odd hyperpower sequences \( \{2^n x\} \) and \( \{2^{n+1} x\} \) converge to 1 and 0 respectively when \( x = 0 \). Since \( n^1 = 1 \) for all \( n \), the hyperpower sequence \( \{n^x\} \) converges to 1 when \( x = 1 \).

From the definition, if one of the sequences \( \{2^n x\} \) and \( \{2^{n+1} x\} \) converges, then so does the other. In fact, it is well known that they converge if and only if \( x \in [0, e^{1/e}] \). (See [4] and [7].) We denote their limits by \( h_e(x) \) and \( h_o(x) \), respectively:

\[
h_e(x) = \lim_{n \to \infty} 2^n x \quad \text{and} \quad h_o(x) = \lim_{n \to \infty} 2^{n+1} x \quad (0 \leq x \leq e^{1/e}).
\]

It is clear that \( x^{h_e(x)} = h_o(x) \) and \( x^{h_o(x)} = h_e(x) \). Therefore if \( h_e(x) = y \) or \( h_o(x) = y \), then \( x^y = y \).

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Many authors have dealt with the hyperpower sequences, their limits and related objects. (See [1], [3], [4] and [7]. Especially, [4] and its references.) Among the results, the following are established in [4] and [7].

(1) They are continuous in \([0, e^{1/e}]\) and analytic in \((0, e^{-e}) \cup (e^{-e}, e^{1/e})\).

(2) \(h_e(0) = 1, h_o(0) = 0\), \(h_e\) is strictly decreasing but \(h_o\) is strictly increasing in \([0, e^{-e}]\), and \(h_o(e^{-e}) = h_o(e^{-e}) = e^{-1}\). In particular, \(h_o(x) < h_e(x)\) for \(x \in (0, e^{-e})\).

(3) If \(x \in [e^{-e}, e^{1/e}]\), then \(h_e(x) = h_o(x)\); and \(h_e(e^{1/e}) = h_o(e^{1/e}) = e\).

As a consequence, the sequence \(\{^n x\}\) converges if and only if \(x \in [e^{-e}, e^{1/e}]\). We denote the limit by \(h(x)\): If \(x \in [e^{-e}, e^{1/e}]\), then \(h_e(x) = h_o(x) = h(x)\) and \(x^{h(x)} = h(x)\). In particular, the function \(h : [e^{-e}, e^{1/e}] \rightarrow [e^{-1}, e]\) is the inverse of the strictly increasing function \([e^{-1}, e] \ni x \mapsto x^{1/x} \in [e^{-e}, e^{1/e}]\). Therefore the properties of \(h\) can be derived from those of \(x \mapsto x^{1/x}\). On the other hand, for \(a \in (0, e^{-e})\) the functions \(h_e\) and \(h_o\) can be approximated by their Taylor polynomials in a neighborhood of \(a\), because they are analytic at \(a\). (The general properties of analytic functions that are needed in this paper can be found in [5, Chapter 2] and [6, Chapter 10].) Since \(x^{h_e(x)} = h_e(x)\) and \(x^{h_o(x)} = h_o(x)\), we can calculate, at least theoretically, the Taylor polynomials by implicit differentiations. It seems, however, few results are known about the behavior of \(h_e\) and \(h_o\) at the points \(e^{-e}\) and 0.

In this paper, we describe the asymptotic behavior of \(h_e(x)\) and \(h_o(x)\) for \(x \rightarrow e^{-e}\) with \(x < e^{-e}\) and for \(x \rightarrow 0\) with \(x > 0\): We shall use the Landau \(O\)- and \(o\)-notation. (For the definition, see [2, Chapter 1].) The main results are stated and explained in Section 2. In Section 3, we briefly state some basic properties of the functions \(h_e\) and \(h_o\). Finally, in Sections 4 and 5, we prove the main results.

2. Main results

We start this section by explaining the speed of convergence of the sequences \(\{^{2n} x\}\) and \(\{^{2n+1} x\}\). For \(n = 0, 1, 2, \ldots\) we set \(U_n(x) =\)
$2^{n+2}x - 2^nx$ and $V_n(x) = 2^{n+3}x - 2^{n+1}x$, so that

$$h_e(x) = 2^nx + \sum_{k=n}^{\infty} U_k(x) \quad \text{and}$$

$$h_o(x) = 2^{n+1}x + \sum_{k=n}^{\infty} V_k(x) \quad (0 \leq x \leq e^{1/e})$$

for every $n$. In the next section, we will show that

$$\lim_{n \to \infty} \frac{U_n(x)}{U_{n-1}(x)} = \lim_{n \to \infty} \frac{V_n(x)}{V_{n-1}(x)} = \log h_e(x) \log h_o(x) \quad (0 < x \leq e^{1/e}, \ x \neq 1).$$

Since $h_e(e^{-e}) = h_o(e^{-e}) = e^{-1}$, we have $\log h_e(e^{-e}) \log h_o(e^{-e}) = 1$, and we will show that $\log h_e(x) \log h_o(x) \to 0$ as $x \to 0^+$. (See Proposition 4.5.) This implies that the sequences \{$2^n x$\} and \{$2^{n+1} x$\} converge very slowly when $x$ is near $e^{-e}$, but very fast when $x$ is near 0.

To describe the behavior of $h_e$ and $h_o$ at $e^{-e}$, we represent them without using the sequences \{$2^n x$\} and \{$2^{n+1} x$\}; and at 0, approximate them with the sequences. The following is proved in Section 4.

**Theorem 2.1.** There is a continuous and bijective function $\varphi : [-e^{-e/2}, e^{-e/2}] \to [0, 1]$ such that

1. $\varphi$ is analytic in $(-e^{-e/2}, e^{-e/2})$,
2. $\varphi'(s) > 0$ for $s \in (-e^{-e/2}, e^{-e/2})$,
3. $\varphi(0) = e^{-1}$,
4. $h_e(x) = \varphi(\sqrt{e^{-e} - x})$ for $x \in [0, e^{-e}]$, and
5. $h_o(x) = \varphi(-\sqrt{e^{-e} - x})$ for $x \in [0, e^{-e}]$.

For $k = 0, 1, 2, \ldots$ we set $a_k = \varphi^{(k)}(0) / k!$. Since $\varphi$ is analytic at 0, there is a positive constant $\delta$ such that $\sum_{k=0}^{\infty} a_k s^k$ converges absolutely to $\varphi(s)$ for every $s \in (-\delta, \delta)$. In particular, we obtain the following:

**Corollary.** Suppose $n$ is a non-negative integer. Then, for $x \to e^{-e}$ with $x < e^{-e}$, the following hold:

$$h_e(x) = \sum_{k=0}^{n} a_k (e^{-e} - x)^{k/2} + O \left((e^{-e} - x)^{(n+1)/2}\right)$$

and

$$h_o(x) = \sum_{k=0}^{n} (-1)^k a_k (e^{-e} - x)^{k/2} + O \left((e^{-e} - x)^{(n+1)/2}\right).$$
Since $\varphi(0) = e^{-1}$, we have $a_0 = e^{-1}$. To determine the coefficients $a_1, a_2, \ldots$, we put $t = e^{s/2} s$ and write

$$\varphi(s) = e^{-1} \left( 1 + A t \left( 1 + \sum_{n=1}^{\infty} b_n t^n \right) \right),$$

so that $a_1 = e^{-1} A e^{e/2}$ and $a_n = e^{-1} A e^{ne/2} b_{n-1}$ for $n = 2, 3, \ldots$. The right-hand side converges absolutely for all $t$ sufficiently close to 0. Since $h_e(x) \log x = \log h_o(x)$ and $h_o(x) \log x = \log h_e(x)$ for all $x \in (0, e^{-e}]$, Theorem 2.1 implies that

$$\varphi(-s) \log(e^{-e} - s^2) = \log \varphi(s) \quad (-e^{-e}/2 < s < e^{-e}/2).$$

In Section 4, we will show that an analytic function $\varphi$ is uniquely determined by this equation and the condition that $\varphi'(0) > 0$. (See Proposition 4.6.) From (2.3) and (2.4),

$$e^{-1} \left( 1 - A t \left( 1 + \sum_{n=1}^{\infty} (-1)^n b_n t^n \right) \right) \left( -e - \sum_{n=1}^{\infty} \frac{1}{n} t^{2n} \right)$$

$$= -1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} A^m t^m \left( 1 + \sum_{n=1}^{\infty} b_n t^n \right)^m$$

for all $t$ sufficiently close to 0; and we have $A > 0$, because $\varphi'(0) > 0$. Hence, by comparing the coefficients of both sides of this equation, one can determine $A, b_1, b_2, \ldots$ successively. For instance, $A = \sqrt{6/e}, b_1 = \frac{1}{6} A, b_2 = \frac{1}{4} - \frac{57}{360} A^2, b_3 = \frac{1}{12} A - \frac{2}{45} A^3, b_4 = \frac{13}{96} - \frac{19}{108} A^2 + \frac{10857}{605480} A^4$ and $b_5 = \frac{1}{18} A - \frac{2}{45} A^3 + \frac{134}{14178} A^5$. This result and the corollary to Theorem 2.1 describe the asymptotic behavior of $h_e(x)$ and $h_o(x)$ for $x \to e^{-e}$ with $x < e^{-e}$. For instance, we have

$$h_e(x) = e^{-1} + \sqrt{6} e^{(e-3)/2} \sqrt{e^{-e} - x} + O(e^{-e} - x) \quad \text{and}$$

$$h_o(x) = e^{-1} - \sqrt{6} e^{(e-3)/2} \sqrt{e^{-e} - x} + O(e^{-e} - x).$$

REMARKS 2.1. (i) The result shows that the curves $y = h_e(x)$ and $y = h_o(x)$ have a vertical tangent at $(e^{-e}, e^{-1})$, and hence $h_e$ and $h_o$ are not analytic at the point $e^{-e}$. (ii) It seems that $\sum_{k=1}^{\infty} |b_k| < \infty$. If it were true, we would have

$$h_e(x) = \sum_{k=0}^{\infty} a_k (e^{-e} - x)^{k/2} \quad \text{and}$$

$$h_o(x) = \sum_{k=0}^{\infty} (-1)^k a_k (e^{-e} - x)^{k/2} \quad (0 \leq x \leq e^{-e}),$$
and the series converge absolutely for every $x \in [0, e^{-\epsilon}]$. The authors do not know how to prove this.

To describe the behavior of $h_e$ and $h_o$ at 0, we introduce the polynomials $P_0, P_1, P_2, \ldots$ and $Q_0, Q_1, Q_2, \ldots$ that are defined inductively as follows: $P_0 = Q_0 = 1$, and

$$P_n(y) = \lim_{x \to 0} x^{-n} \left( \exp \left( \sum_{k=1}^{n} x^k y Q_{k-1}(y) \right) - \sum_{k=0}^{n-1} x^k P_k(y) \right),$$

$$Q_n(y) = \lim_{x \to 0} x^{-n} \left( \exp \left( \sum_{k=1}^{n} x^k y P_k(y) \right) - \sum_{k=0}^{n-1} x^k Q_k(y) \right).$$

In Section 5, we will prove that these polynomials are well defined, and that $\deg P_n = 2n - 1$ and $\deg Q_n = 2n$ for $n \geq 1$. (See Lemma 5.1, (5.4), (5.5) and Proposition 5.2.) A direct calculation shows that $P_1(y) = y$, $Q_1(y) = y^2$, $P_2(y) = \frac{1}{2}y^2 + y^3$, $Q_2(y) = \frac{1}{2}y^3 + \frac{3}{2}y^4$, $P_3(y) = \frac{1}{6}y^3 + \frac{3}{2}y^4 + \frac{3}{2}y^5$, $Q_3(y) = \frac{1}{6}y^4 + 2y^5 + \frac{8}{3}y^6$, and so on. In the same section, the following are proved:

**PROPOSITION 2.2.** Suppose $n$ is a non-negative integer. Then, for $x \to 0+$, the following hold:

$$2^n x = \sum_{k=0}^{n} x^k P_k(\log x) + O \left( x^{n+1} |\log x|^{2n+1} \right) \quad \text{and}$$

$$2^{n+1} x = x \sum_{k=0}^{n} x^k Q_k(\log x) + O \left( x^{n+2} |\log x|^{2n+2} \right).$$

**PROPOSITION 2.3.** For each $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < x \leq \delta$, then the inequalities

$$|h_e(x) - 2^n x| \leq (1 + \epsilon)^{n+1} x^{n+1} |\log x|^{2n+1}$$

and

$$|h_o(x) - 2^{n+1} x| \leq (1 + \epsilon)^{n+2} x^{n+2} |\log x|^{2n+2}$$

hold for all non-negative integers $n$.

As an immediate consequence of these propositions, we obtain the following:
THEOREM 2.4. Suppose $n$ is a non-negative integer. Then, for $x \to 0+$, the following hold:

$$h_e(x) = \sum_{k=0}^{n} x^k P_k(\log x) + O\left(x^{n+1} |\log x|^{2n+1}\right) \quad \text{and}$$

$$h_o(x) = x \sum_{k=0}^{n} x^k Q_k(\log x) + O\left(x^{n+2} |\log x|^{2n+2}\right).$$

This theorem describes the asymptotic behavior of $h_e(x)$ and $h_o(x)$ for $x \to 0$ with $x > 0$. For instance, we have

$$(2.5) \quad h_e(x) = 1 + x \log x + O\left(x^{2} |\log x|^3\right) \quad \text{and}$$

$$h_o(x) = x + x^2 (\log x)^2 + O\left(x^{3} |\log x|^4\right).$$

REMARKS 2.2. (i) The result shows that $h_e$ and $h_o$ cannot be extended to analytic functions in an open interval containing 0. (ii) The result also shows that the right-hand derivative of $h_e$ at 0 does not exist, but that of $h_o$ exists and is equal to 1. (iii) Theorem 2.4 gives no information about the convergence of the series

$$\sum_{k=0}^{\infty} x^k P_k(\log x) \quad \text{and} \quad \sum_{k=0}^{\infty} x^{k+1} Q_k(\log x).$$

It seems, however, that for every $x \in (0, e^{1/e}]$ these series converge to $h_e(x)$ and $h_o(x)$, respectively.

Finally, the following shows that $h_o$ can be extended to a $C^1$-function in an open interval containing 0, but not to a $C^2$-one.

PROPOSITION 2.5. $h'_o(x) \to 1$ and $x^{-1}(h'_o(x) - 1) \to \infty$ as $x \to 0+$.

This proposition also is proved in Section 5.

3. Preliminaries

In this short section, we state some basic properties of the functions $h_e$ and $h_o$. First of all, it is easy to see that if $x \in [0, 1]$, then

$$2^{n+1} x \leq 2^{n+2} x \leq 2^n x \quad \text{and} \quad 2^{n+2} x \geq 2^{n+3} x \geq 2^{n+1} x.$$
From this, it follows that \( 2^{n+1} x \leq h_0(x) \leq h_e(x) \leq 2^n x \) for all \( x \in [0, 1] \). On the other hand, it is not hard to see that if \( x \in [0, e^{-e}] \), then \( 2^{n+1} x \leq e^{-1} \leq 2^n x \). (See [4, p. 242] and [7, p. 14].) Consequently,

\[
2^{n+1} x \leq h_0(x) \leq e^{-1} \leq h_e(x) \leq 2^n x \quad (0 \leq x \leq e^{-e}, \ n = 0, 1, 2, \ldots).
\]

In particular,

\[
(3.1) \quad 0 \leq h_0(x) \leq e^{-1} \leq h_e(x) \leq 1 \quad (0 \leq x \leq e^{-e}).
\]

Since \( h_e(x) = x^{h_0(x)} \) and \( h_0(x) = x^{h_e(x)} \) whenever the sequences converge, and since \( h_e(x), h_0(x) > 0 \) for \( x \in (0, e^{1/e}] \), we have

\[
(3.2) \quad h_e(x)^{1/h_0(x)} = h_0(x)^{1/h_e(x)} = x \quad (0 < x \leq e^{1/e}),
\]

and hence

\[
(3.3) \quad h_e(x)^{h_0(x)} = h_0(x)^{h_e(x)} \quad (0 \leq x \leq e^{1/e}).
\]

We close this section by proving (2.2).

**Proof of (2.2).** Suppose that \( x \in (0, e^{1/e}] \setminus \{1\} \). Then the sequences \( \{U_n(x)\} \) and \( \{V_n(x)\} \) converge to zero. Since \( U_n(x) = 2^{n+2} x - 2^n x = 2^n x (\exp(V_{n-1}(x) \log x) - 1) \), this implies that

\[
\lim_{n \to \infty} \frac{U_n(x)}{V_{n-1}(x)} = \lim_{n \to \infty} 2^n x \frac{e^{V_{n-1}(x) \log x} - 1}{V_{n-1}(x)} = h_e(x) \log x = \log h_0(x).
\]

Similarly,

\[
\lim_{n \to \infty} \frac{V_n(x)}{U_n(x)} = \log h_e(x).
\]

Now the result is obvious. \( \square \)

**4. Proof of Theorem 2.1**

In this section, the following lemma will play a basic role. The proof is trivial.
Lemma 4.1. Let $f$ be a real analytic function defined in an open interval $(a, b)$, and suppose that $c \in (a, b)$, $f(x) > 0$ for $x \neq c$, $f(c) = f'(c) = 0$, and $f''(c) \neq 0$. If $\tilde{f}$ is defined by

$$
\tilde{f}(x) = \begin{cases} 
-\frac{f(x)}{f'(c)} & (a < x < c), \\
\frac{f(x)}{f'(c)} & (c \leq x < b),
\end{cases}
$$

then $\tilde{f}$ is analytic in $(a, b)$ and $\tilde{f}'(c) = \sqrt{f''(c)/2}$.

Note that we must have $f''(c) > 0$.

We need to introduce some functions and establish their properties. Let the function $F : [0, 1] \rightarrow [-\sqrt{1 - e^{-1/e}}, \sqrt{1 - e^{-1/e}}]$ be defined by

$$
F(x) = \begin{cases} 
-\frac{x^x - e^{-1/e}}{x^x - e^{-1/e}} & (0 \leq x \leq e^{-1}), \\
\frac{x^x - e^{-1/e}}{x^x - e^{-1/e}} & (e^{-1} \leq x \leq 1).
\end{cases}
$$

This function is well defined, continuous and bijective; Lemma 4.1 implies that $F$ is analytic in $(0, 1)$; and it is clear that $F'(x) > 0$ for all $x \in (0, 1)$.

Let $G$ denote the function

$$
[0, 1] \ni x \mapsto F^{-1}(-F(x)) \in [0, 1].
$$

Then $G(0) = 1$, $G(e^{-1}) = e^{-1}$, $G(1) = 0$, $G$ is continuous in $[0, 1]$, analytic in $(0, 1)$, and we have $G'(x) < 0$ for $x \in (0, 1)$. It is easy to see that $G(G(x)) = x$ and $G(x)^{G(x)} = x^x$ for $x \in [0, 1]$. From this, (3.1) and (3.3), we obtain

(4.1) \hspace{1cm} h_e(x) = G(h_o(x)) \quad \text{and} \quad h_o(x) = G(h_e(x)) \quad (0 \leq x \leq e^{-e})

and

(4.2) \hspace{1cm} G(x) \log G(x) = x \log x \quad (0 < x < 1).

Since $G(e^{-1}) = e^{-1}$, $G'(e^{-1}) < 0$ and $G(G(x)) = x$, we have $G'(e^{-1}) = -1$. In a neighborhood of $e^{-1}$, the analytic function $G$ is represented by an absolutely convergent power series:

$$
G(x) = \sum_{n=0}^{\infty} c_n (x - e^{-1})^n.
$$
We have $c_0 = e^{-1}$ and $c_1 = -1$, because $G(e^{-1}) = e^{-1}$ and $G'(e^{-1}) = -1$. Hence, using (4.2), one can determine the coefficients $c_2, c_3, \ldots$ successively. For instance, $c_2 = \frac{1}{3} e$, $c_3 = -\frac{1}{9} e^2$, $c_4 = \frac{17}{270} e^3$, and $c_5 = \frac{31}{810} e^4$. Note that an analytic function $G : (0, 1) \to \mathbb{R}$ is uniquely determined by (4.2) and the condition that $G'(e^{-1}) < 0$.

We can parameterize the curve $y = G(x)$ as follows: Put $x^{-1}G(x) = t$. As $x$ increases from $0$ to $1$, $t$ decreases from $\infty$ to $0$; and $t = 1$ if and only if $x = e^{-1}$. From (4.2) one can easily deduce that

\begin{equation}
\log x = \frac{t}{1-t} \log t \quad \text{and} \quad \log G(x) = \frac{1}{1-t} \log t \quad (0 < t < \infty, \ t \neq 1),
\end{equation}

and this is equivalent to

\begin{equation*}
x = t \frac{1}{1-t} \quad \text{and} \quad y = t \frac{1}{1-t} \quad (0 < t < \infty, \ t \neq 1).
\end{equation*}

We remark that an equivalent version of this parameterization is due to Goldbach. See [4, p. 237].

**Lemma 4.2.** The function $x \mapsto \log x \log G(x)$ is strictly increasing in $(0, e^{-1})$, has maximum value $1$ at $x = e^{-1}$, and is strictly decreasing in $(e^{-1}, 1)$; and

\[
\lim_{x \to 0^+} \log x \log G(x) = \lim_{x \to 1^-} \log x \log G(x) = 0.
\]

**Proof.** If we put $x^{-1}G(x) = t$, then $t$ decreases from $\infty$ to $0$ as $x$ increases from $0$ to $1$, $t = 1$ if and only if $x = e^{-1}$, and (4.3) implies that

\[
\log x \log G(x) = t \left( \frac{\log t}{t-1} \right)^2 \quad (0 < t < \infty, \ t \neq 1).
\]

Now, the result is proved by calculus. \qed

Define $H$ by

\[
H(x) = \begin{cases} 
onumber
0 & (x = 0), \\
G(x)^{1/x} & (0 < x \leq 1).
\end{cases}
\]

This function is analytic in $(0, 1)$, continuous at $1$, and $H(1) = 0$. Moreover, $H$ is continuous at $0$ too: Since $0 \leq G(x) \leq 1$ and $G(x)^{G(x)} = x^x$ for $x \in [0, 1]$, we have

\[
0 \leq H(x)^x = G(x) \leq G(x)^{G(x)} = x^x \quad (0 \leq x \leq 1),
\]

and hence

\[
0 \leq H(x) \leq x \quad (0 \leq x \leq 1).
\]
Proposition 4.3. If $x \in [0, e^{-e}]$, then $H(h_o(x)) = H(h_o(x)) = x$.

Proof. The result follows from (3.2), (4.1) and the definition of $H$. $\Box$

Proposition 4.4. $H'(x) > 0$ for $x \in (0, e^{-1})$, $H'(x) < 0$ for $x \in (e^{-1}, 1)$, $H(e^{-1}) = e^{-e}$, $H'(e^{-1}) = 0$, and $H''(e^{-1}) = -\frac{1}{3}e^{3-e}$.

Proof. First of all, the last three assertions are proved by straightforward calculation, because $G(e^{-1}) = e^{-1}$, $G''(e^{-1}) = 1$, $G''(e^{-1}) = 2c_2 = 2e$, and $H(x) = G(x)^{1/x}$ for $x \in (0, 1)$.

By differentiating both sides of (4.2), we obtain $G'(x)(1 + \log G(x)) = 1 + \log x$, which is valid for all $x \in (0, 1)$. On the other hand, $\log H(x) = x^{-1} \log G(x)$ for $x \in (0, 1)$. Hence, by straightforward calculation, we have

$$\frac{H'(x)}{H(x)} = \frac{-G(x) \log G(x) (1 + \log G(x)) + x(1 + \log x)}{x^2G(x) (1 + \log G(x))}$$

(0 < x < 1, x $\neq$ $e^{-1}$).

The right-hand side is simplified with the aid of (4.2):

$$\frac{H'(x)}{H(x)} = \frac{-x \log x (1 + \log G(x)) + x(1 + \log x)}{x^2G(x) (1 + \log G(x))}$$

(4.4)

$$= \frac{1 - \log x \log G(x)}{xG(x) (1 + \log G(x))}$$

(0 < x < 1, x $\neq$ $e^{-1}$).

Lemma 4.2 implies that $1 - \log x \log G(x) > 0$ for all $x \in (0, 1)$ with $x \neq e^{-1}$, and it is clear that $xG(x) > 0$ for all $x \in (0, 1)$. From this, the first two assertions follow, because $G$ is strictly decreasing and $\log G(e^{-1}) = -1$. $\Box$

Now, we can prove Theorem 2.1.

Proof of Theorem 2.1. Let the function $\tilde{H} : [0, 1] \rightarrow [-e^{-e/2}, e^{-e/2}]$ be defined by

$$\tilde{H}(x) = \begin{cases} 
-\sqrt{e^{-e} - H(x)} & (0 \leq x \leq e^{-1}), \\
\sqrt{e^{-e} - H(x)} & (e^{-1} \leq x \leq 1).
\end{cases}$$

From Lemma 4.1 and Proposition 4.4, we have the following: $\tilde{H}$ is well defined, continuous, bijective, $\tilde{H}(e^{-1}) = 0$, $\tilde{H}$ is analytic in $(0, 1)$ and
\( \tilde{H}'(x) > 0 \) for all \( x \in (0, 1) \). Moreover (3.1), Proposition 4.3, and the definition of \( \tilde{H} \) imply that
\[
(4.5) \quad \tilde{H}(h_e(x)) = \sqrt{e^{-e} - x} \quad \text{and} \quad \tilde{H}(h_o(x)) = -\sqrt{e^{-e} - x} \quad (0 \leq x \leq e^{-e}).
\]

If we denote the inverse of \( \tilde{H} \) by \( \varphi \), then \( \varphi \) is continuous in \([-e^{-e/2}, e^{-e/2}]\), analytic in \((-e^{-e/2}, e^{-e/2})\), \( \varphi'(s) > 0 \) for all \( s \in (-e^{-e/2}, e^{-e/2}) \), \( \varphi(0) = e^{-1} \), and (4.5) is equivalent to
\[
\begin{align*}
h_e(x) &= \varphi(\sqrt{e^{-e} - x}) \quad \text{and} \\
h_o(x) &= \varphi(-\sqrt{e^{-e} - x}) \quad (0 \leq x \leq e^{-e}).
\end{align*}
\]

This proves Theorem 2.1. \[ \Box \]

At this point, it should be remarked that (3.1) and (3.2) are the only properties of \( h_e \) and \( h_o \) that are used in our proof of Theorem 2.1: (3.3) is a consequence of (3.2).

It remains to prove the following two propositions.

**Proposition 4.5.** \( \log h_e(x) \log h_o(x) \to 0 \) as \( x \to 0^+ \).

**Proposition 4.6.** Let \( \tilde{\varphi} : (-e^{-e/2}, e^{-e/2}) \to \mathbb{R} \) be an analytic function. Suppose that \( \tilde{\varphi}'(0) > 0 \) and
\[
(4.6) \quad \tilde{\varphi}(-s) \log(e^{-e} - s^2) = \log \tilde{\varphi}(s) \quad (-e^{-e/2} < s < e^{-e/2}).
\]

Then \( \tilde{\varphi}(s) = \varphi(s) \) for all \( s \in (-e^{-e/2}, e^{-e/2}) \).

**Proof of Proposition 4.5.** Theorem 2.1 implies that \( h_e \) is continuous at 0. Hence \( h_e(x) \to h_e(0) = 1 \) as \( x \to 0^+ \). From (4.1), we have \( \log h_e(x) \log h_o(x) = \log h_e(x) \log G(h_e(x)) \). Therefore the result follows from Lemma 4.2. \[ \Box \]

**Proof of Proposition 4.6.** First of all, (4.6) implies that \( \tilde{\varphi}(0) = e^{-1} \).
Since \( \tilde{\varphi}'(0) > 0 \), there is a positive real number \( a \), with \( a < e^{-e/2} \), such that \( \tilde{\varphi} \) is increasing in the interval \((-a, a)\). Since \( 0 < \tilde{\varphi}(0) = e^{-1} < 1 \), we may assume, by taking \( a \) sufficiently small, that \( 0 < \tilde{\varphi}(-a) < e^{-1} < \tilde{\varphi}(a) < 1 \).

For \( x \in (e^{-e} - a^2, e^{-e}] \) define \( \tilde{h}_e(x) \) and \( \tilde{h}_o(x) \) by
\[
\tilde{h}_e(x) = \tilde{\varphi}(\sqrt{e^{-e} - x}) \quad \text{and} \quad \tilde{h}_o(x) = \tilde{\varphi}(-\sqrt{e^{-e} - x}).
\]
Since $\tilde{\varphi}$ is increasing in $(-a,a)$ and $0 < \tilde{\varphi}(-a) < \tilde{\varphi}(0) = e^{-1} < \tilde{\varphi}(a) < 1$, we have

$$0 < \tilde{h}_o(x) \leq e^{-1} \leq \tilde{h}_e(x) < 1 \quad (e^{-e} - a^2 < x \leq e^{-e}).$$

Moreover, (4.6) implies that

$$\tilde{h}_e(x)^{1/\tilde{h}_o(x)} = \tilde{h}_o(x)^{1/\tilde{h}_e(x)} = x \quad (e^{-e} - a^2 < x \leq e^{-e}).$$

Hence essentially the same argument as the proof of Theorem 2.1 shows that

$$\tilde{h}_e(x) = \varphi(\sqrt{e^{-e} - x}) \quad \text{and}$$

$$\tilde{h}_o(x) = \varphi(-\sqrt{e^{-e} - x}) \quad (e^{-e} - a^2 < x \leq e^{-e}).$$

Therefore $\tilde{\varphi}(s) = \varphi(s)$ for all $s \in (-a,a)$. From this, we obtain the desired result. \qed

5. Proofs of Propositions 2.2, 2.3 and 2.5

For $x,y \in \mathbb{R}$ we define $h_0(x,y), h_1(x,y), h_2(x,y), \ldots$ as follows: $h_0(x,y) = 1$, and

$$h_{2n+1}(x,y) = x \exp \left( yh_{2n}(x,y) - y \right),$$

$$h_{2n+2}(x,y) = \exp \left( yh_{2n+1}(x,y) \right).$$

For instance, $h_1(x,y) = x$, $h_2(x,y) = 1 + xy + \frac{1}{2}x^2y^2 + \cdots$, and so on. It is clear that $n^x = h_n(x, \log x)$ for $x > 0$ and $n = 0, 1, 2, \ldots$.

**Lemma 5.1.** For each non-negative integer $n$ there are polynomials $P_{(n,0)}, \ldots, P_{(n,n)}$ and $Q_{(n,0)}, \ldots, Q_{(n,n)}$, with $\deg P_{(n,k)} = \max\{0, 2k-1\}$ and $\deg Q_{(n,k)} = 2k$ for all $k$, such that

$$(5.1) \quad h_{2n}(x,y) = \sum_{k=0}^{n} x^k P_{(n,k)}(y) + O(x^{n+1}) \quad (x \to 0)$$

and

$$(5.2) \quad h_{2n+1}(x,y) = x \sum_{k=0}^{n} x^k Q_{(n,k)}(y) + O(x^{n+2}) \quad (x \to 0)$$

hold for each fixed $y$. 
Proof. First of all, we set \( P_{(n,0)} = Q_{(n,0)} = 1 \) for all non-negative integers \( n \): The lemma holds trivially when \( n = 0 \). For each positive integer \( n \) let \( P(n) \) denote the statement that there are polynomials \( P_{(n,1)}, \ldots, P_{(n,n)} \), with \( \deg P_{(n,k)} = 2k - 1 \) for all \( k \), such that (5.1) holds for each fixed \( y \); and \( Q(n) \) the statement that there are polynomials \( Q_{(n,1)}, \ldots, Q_{(n,n)} \), with \( \deg Q_{(n,k)} = 2k \) for all \( k \), such that (5.2) holds for each fixed \( y \). Since \( h_2(x,y) = 1 + xy + \frac{1}{2}x^2y^2 + \cdots \), \( P(1) \) is obvious, with \( P_{(1,1)}(y) = y \). Hence the lemma will follow once we show that \( P(n) \) implies \( Q(n) \) and \( Q(n) \) implies \( P(n + 1) \).

Let \( n \) be arbitrary. Suppose that \( P(n) \) is true. For convenience, we set \( \tilde{P}_{(n,k)}(y) = yP_{(n,k)}(y) \): It is clear that \( \deg \tilde{P}_{(n,k)} = 2k \). For each fixed \( y \) we have
\[
\begin{align*}
  h_{2n+1}(x,y) &= x \exp (yh_{2n}(x,y) - y) \\
  &= x \exp \left( \sum_{k=1}^{n} x^k \tilde{P}_{(n,k)}(y) + O(x^{n+1}) \right) \\
  &= x \exp \left( \sum_{k=1}^{n} x^k \tilde{P}_{(n,k)}(y) \right) + O(x^{n+2}) \quad (x \to 0).
\end{align*}
\]

Since \( \deg \tilde{P}_{(n,k)} = 2k \) for all \( k \), it follows that
\[
\exp \left( \sum_{k=1}^{n} x^k \tilde{P}_{(n,k)}(y) \right) = 1 + \sum_{k=1}^{n} x^k Q_{(n,k)}(y) + O(x^{n+1}) \quad (x \to 0)
\]
for some polynomials \( Q_{(n,1)}, \ldots, Q_{(n,n)} \), with \( \deg Q_{(n,k)} = 2k \) for all \( k \). From this, \( Q(n) \) follows. The statement that \( Q(n) \) implies \( P(n + 1) \) is proved similarly.

For each non-negative integer \( n \) we set \( P_n = P_{(n,n)} \) and \( Q_n = Q_{(n,n)} \). Then (5.3) implies that
\[
Q_n(y) = \lim_{x \to 0} x^{-n} \left( \exp \left( \sum_{k=1}^{n} x^k yP_{(n,k)}(y) \right) - \sum_{k=0}^{n-1} x^k Q_{(n,k)}(y) \right)
\]
for every positive integer \( n \); and similarly,
\[
P_n(y) = \lim_{x \to 0} x^{-n} \left( \exp \left( \sum_{k=1}^{n} x^k yQ_{(n-1,k-1)}(y) \right) - \sum_{k=0}^{n-1} x^k P_{(n,k)}(y) \right)
\]
for every positive integer \( n \).
PROPOSITION 5.2. Suppose $0 \leq k < n$. Then $P_{(n,k)} = P_k$ and $Q_{(n,k)} = Q_k$.

To prove this proposition as well as Propositions 2.2, 2.3 and 2.5, we need some lemmas.

LEMMA 5.3. Suppose $n$ is a non-negative integer. Then, for $x \to 0+$, the following hold:

$$2^n x = \sum_{k=0}^{n} x^k P_{(n,k)}(\log x) + O\left(x^{n+1} |\log x|^{2n+1}\right) \quad \text{and}$$

$$2^{n+1} x = \sum_{k=0}^{n} x^k Q_{(n,k)}(\log x) + O\left(x^{n+2} |\log x|^{2n+2}\right).$$

Proof. The proof is essentially the same as Lemma 5.1. \qed

LEMMA 5.4. For each non-negative integer $n$ the following hold:

$$|U_n(x)| \leq x h_o(x)^n |\log x|^{2n+1},$$

$$|V_n(x)| \leq x h_o(x)^{n+1} |\log x|^{2n+2} \quad (0 < x < 1).$$

Proof. It is easy to see that the inequality

$$|a - b| \leq \max\{a, b\} |\log a - \log b|$$

holds for all positive real numbers $a$ and $b$. Let $0 < x < 1$. Then the sequence $\{2^n x\}$ is decreasing and $\{2^{n+1} x\}$ increasing. Since $\{2^n x\}$ is decreasing, we have

$$|U_n(x)| = |2^{n+2} x - 2^n x|$$

$$\leq 2^n x |\log (2^{n+2} x) - \log (2^n x)|$$

$$\leq 2^n x |2^{n+1} x \log x - 2^{n-1} x \log x|$$

$$= 2^n x |\log x| |2^{n+1} x - 2^{n-1} x|$$

$$= 2^n x |\log x| |V_{n-1}(x)| \quad (n = 1, 2, \ldots),$$

and it is clear that $|U_0(x)| \leq x |\log x|$. Similarly,

$$|V_n(x)| \leq 2^{n+3} x |\log x| |U_n(x)| \quad (n = 0, 1, 2, \ldots).$$

Now, the result is proved by induction, because $2^n x \leq 1$ and $2^{n+1} x \leq h_o(x)$ for all $n$. \qed
Lemma 5.5. For each \( \epsilon > 0 \) there is a \( \delta \), with \( 0 < \delta \leq e^{-\epsilon} \), such that if \( 0 \leq x \leq \delta \), then \( h_o(x) \leq (1 + \epsilon)x \).

Proof. From Propositions 4.3 and 4.4, the function \( h_o : [0, e^{-\epsilon}] \to [0, e^{-1}] \) is the inverse of \( H : [0, e^{-1}] \to [0, e^{-\epsilon}] \). Hence the assertion will follow once we prove that \( x^{-1}H(x) \to 1 \) as \( x \to 0^+ \), and that \( \frac{d}{dx}(x^{-1}H(x)) < 0 \) for \( x \in (0, e^{-1}) \).

Since \( G(x) \to 1 \) as \( x \to 0^+ \) and \( x \log x = G(x) \log G(x) \) for \( x \in (0, 1) \), we have

\[
\lim_{x \to 0^+} \frac{1 - G(x)}{x \log x} = \lim_{x \to 0^+} \frac{1 - G(x)}{G(x) \log G(x)} = \lim_{s \to 1} \frac{1 - s}{s \log s} = -1;
\]

and since \( H(x) = G(x)^{1/x} = x^{1/G(x)} \) for \( x \in (0, 1) \),

\[
\log x^{-1}H(x) = \log H(x) - \log x = \frac{1}{G(x)} \log x - \log x
\]

\[
= \frac{1}{G(x)} (1 - G(x)) \log x = \frac{1}{G(x)} \frac{1 - G(x)}{x \log x} (\log x)^2.
\]

Thus \( \log x^{-1}H(x) \to 0 \) as \( x \to 0^+ \), that is, \( x^{-1}H(x) \to 1 \) as \( x \to 0^+ \).

It remains to show that \( \frac{d}{dx}(x^{-1}H(x)) < 0 \) for \( x \in (0, e^{-1}) \). From (4.4), we have

\[
\frac{d}{dx} \log x^{-1}H(x) = \frac{H'(x)}{H(x)} - \frac{1}{x}
\]

\[
= \frac{1}{xG(x)} \left( \log G(x) + G(x) (1 + \log G(x)) \right) - \frac{1 - \log x \log G(x) - G(x) (1 + \log G(x))}{xG(x) (1 + \log G(x))}
\]

Hence we need only to show that

\[(5.6) \quad G(x) (1 + \log G(x)) + \log x \log G(x) - 1 > 0 \quad (0 < x < e^{-1}),\]

because \( x^{-1}H(x) > 0 \) and \( xG(x) (1 + \log G(x)) > 0 \) for \( x \in (0, e^{-1}) \).

Suppose that \( 0 < x < e^{-1} \). Since \( e^s \geq 1 + s \) for all \( s \in \mathbb{R} \), we have \( G(x) (1 + \log G(x)) \geq (1 + \log G(x))^2 \); and hence the left-hand side of (5.6) is greater than or equal to

\[(5.7) \quad (1 + \log G(x))^2 + \log x \log G(x) - 1.
\]

If we put \( t = x^{-1}G(x) \), then \( t > 1 \); and (4.3) implies that (5.7) is equal to \( (1 - t)^{-2} (2 - 2t + \log t + t \log t) \log t \). From this, (5.6) follows, because \( 2 - 2t + \log t + t \log t > 0 \) for all \( t > 1 \).

Now, we can prove Propositions 5.2, 2.2, 2.3 and 2.5.
Proof of Proposition 5.2. The proposition will follow once we show that for each non-negative integer \( n \) we have

\[ P_{(n,k)} = P_{(n+1,k)} \quad \text{and} \quad Q_{(n,k)} = Q_{(n+1,k)} \quad (k = 0, 1, \ldots, n). \]

Let \( n \) be arbitrary. For \( k = 0, 1, \ldots, n \) we set \( P_k^* = P_{(n+1,k)} - P_{(n,k)} \). Lemma 5.3 implies that

\[
U_n(x) = 2^{n+2}x - 2^n x
\]

\[
= \sum_{k=0}^{n} x^k P_k^*(\log x) + x^{n+1} P_{(n+1,n+1)}(\log x)
\]

\[
+ O\left(x^{n+1}\log x|^{2n+1}\right)
\]

for \( x \to 0^+ \). From Lemma 5.1, \( \deg P_{(n+1,n+1)} = 2n + 1 \); and Lemmas 5.4 and 5.5 imply that \( U_n(x) = O\left(x^{n+1}\log x|^{2n+1}\right) \) for \( x \to 0^+ \). Hence

\[
\sum_{k=0}^{n} x^k P_k^*(\log x) = O\left(x^{n+1}\log x|^{2n+1}\right) = o(x^n) \quad (x \to 0^+).
\]

From this, we obtain \( P_0^* = 0 \), \( P_1^* = 0, \ldots, \) and \( P_n^* = 0 \) successively. Therefore \( P_{(n,k)} = P_{(n+1,k)} \) for all \( k \). The statement that \( Q_{(n,k)} = Q_{(n+1,k)} \) for all \( k \) is proved similarly.

\[ \square \]

Proof of Proposition 2.2. The proposition is an immediate consequence of Proposition 5.2 and Lemma 5.3.

\[ \square \]

Proof of Proposition 2.3. Let \( \epsilon > 0 \) be arbitrary. From Lemma 5.5, there is a \( \delta > 0 \) such that if \( 0 < x \leq \delta \), then \( h_o(x) \leq (1 + \epsilon)x \). Suppose that \( 0 < x \leq \delta \). By Lemma 5.4, the inequalities

\[
|U_n(x)| \leq (1 + \epsilon)^n x^{n+1} |\log x|^{2n+1}
\]

and

\[
|V_n(x)| \leq (1 + \epsilon)^{n+1} x^{n+2} |\log x|^{2n+2}
\]

hold for all non-negative integers \( n \). Since \( x(\log x)^2 \to 0 \) as \( x \to 0^+ \), we may assume, by replacing \( \delta \) with a smaller one, that

\[ (1 - (1 + \epsilon)x(\log x)^2)^{-1} \leq 1 + \epsilon. \]

Hence the desired result follows from (2.1).

\[ \square \]
Proof of Proposition 2.5. From Proposition 4.3, \( h'_o(x) = 1/H'(h_o(x)) \) for \( x \in (0, e^{-e}) \). Hence we obtain, by (4.1) and (4.4),

\[
(5.8) \quad h'_o(x) = \frac{h_o(x)h_e(x) (1 + \log h_e(x))}{x (1 - \log h_o(x) \log h_e(x))} \quad (0 < x < e^{-e}).
\]

Therefore \( h'_o(x) \to 1 \) as \( x \to 0^+ \), by (2.5) and Proposition 4.5.

It remains to show that \( x^{-1}(h'_o(x) - 1) \to \infty \) as \( x \to 0^+ \). From (5.8),

\[
\frac{h'_o(x) - 1}{x} = \frac{h_o(x)h_e(x) (1 + \log h_e(x)) - x (1 - \log h_o(x) \log h_e(x))}{x^2 (1 - \log h_o(x) \log h_e(x))}
\]

for \( x \in (0, e^{-e}) \); and (2.5) implies that

\[
\frac{h_o(x)h_e(x) (1 + \log h_e(x)) - x (1 - \log h_o(x) \log h_e(x))}{x^2 (1 - \log h_o(x) \log h_e(x))} = 2x^2 \left((\log x)^2 + \log x\right) + O(x^3|\log x|^4)
\]

for \( x \to 0^+ \). Hence the desired result follows from Proposition 4.5. \( \Box \)

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References


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