

## CONDITIONAL GENERALIZED FOURIER-FEYNMAN TRANSFORM AND CONDITIONAL CONVOLUTION PRODUCT ON A BANACH ALGEBRA

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ABSTRACT. In [10], Chang and Skoug used a generalized Brownian motion process to define a generalized analytic Feynman integral and a generalized analytic Fourier-Feynman transform. In this paper we define the conditional generalized Fourier-Feynman transform and conditional generalized convolution product on function space. We then establish some relationships between the conditional generalized Fourier-Feynman transform and conditional generalized convolution product for functionals on function space that belonging to a Banach algebra.

### 1. Introduction

The concept of  $L_1$  analytic Fourier-Feynman transform (FFT) was introduced by Brue in [1]. In [2], Cameron and Storvick introduced the concept of an  $L_2$  analytic FFT on Wiener space. In [19], Johnson and Skoug developed an  $L_p$  analytic FFT theory for  $1 \leq p \leq 2$  which extended the results in [1, 2] and gave various relationships between the  $L_1$  and  $L_2$  theories. In [15], Huffman, Park and Skoug defined a convolution product(CP) for functionals on Wiener space and in [16, 17] obtained various results involving and relating the FFT and CP. For further work of the conditional FFT(CFFT) and the conditional CP(CCP), see the references [4, 5, 9, 12-14, 21, 22].

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In this paper, we study functionals on function space but with  $x$  in a general function space  $C_{a,b}[0, T]$  rather than in the Wiener space  $C_0[0, T]$ . The Wiener process used in [4, 9, 12-22] is free of drift and is stationary in time while the stochastic process used in this paper is nonstationary in time and is subject to a drift  $a(t)$ .

The class of functionals on function space that we study with throughout this paper is the Banach algebra  $\mathcal{S}(L_{a,b}^2[0, T])$  introduced by Chang and Skoug in [10]. Results in [6-8, 10] show that  $\mathcal{S}(L_{a,b}^2[0, T])$  contains many interests in connection with generalized Feynman transform and quantum mechanics.

In this paper we define the concepts of a conditional generalized FFT(CGFFT) and a conditional generalized CP(CGCP) and obtain several interesting relationships between them.

## 2. Definitions and preliminaries

Let  $D = [0, T]$  and let  $(\Omega, \mathcal{B}, P)$  be a probability measure space. A real valued stochastic process  $Y$  on  $(\Omega, \mathcal{B}, P)$  and  $D$  is called a *generalized Brownian motion process* if  $Y(0, \omega) = 0$  almost everywhere and for  $0 = t_0 < t_1 < \cdots < t_n \leq T$ , the  $n$ -dimensional random vector  $(Y(t_1, \omega), \cdots, Y(t_n, \omega))$  is normally distributed with the density function

$$(2.1) \quad K(\vec{t}, \vec{\eta}) = \left( (2\pi)^n \prod_{j=1}^n (b(t_j) - b(t_{j-1})) \right)^{-1/2} \cdot \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})} \right\}$$

where  $\vec{\eta} = (\eta_1, \cdots, \eta_n)$ ,  $\eta_0 = 0$ ,  $\vec{t} = (t_1, \cdots, t_n)$ ,  $a(t)$  is an absolutely continuous real-valued function on  $[0, T]$  with  $a(0) = 0$ ,  $a'(t) \in L^2[0, T]$ , and  $b(t)$  is a strictly increasing, continuously differentiable real-valued function with  $b(0) = 0$  and  $b'(t) > 0$  for each  $t \in [0, T]$ .

As explained in [23, pp.18–20],  $Y$  induces a probability measure  $\mu$  on the measurable space  $(\mathbb{R}^D, \mathcal{B}^D)$  where  $\mathbb{R}^D$  is the space of all real valued functions  $x(t)$ ,  $t \in D$ , and  $\mathcal{B}^D$  is the smallest  $\sigma$ -algebra of subsets of  $\mathbb{R}^D$  with respect to which all the coordinate evaluation maps  $e_t(x) = x(t)$  defined on  $\mathbb{R}^D$  are measurable. The triple  $(\mathbb{R}^D, \mathcal{B}^D, \mu)$  is a probability measure space. This measure space is called the function space induced

by the generalized Brownian motion process  $Y$  determined by  $a(\cdot)$  and  $b(\cdot)$ .

We note that the generalized Brownian motion process  $Y$  determined by  $a(\cdot)$  and  $b(\cdot)$  is a Gaussian process with mean function  $a(t)$  and covariance function  $r(s, t) = \min\{b(s), b(t)\}$ . By Theorem 14.2 [23, p.187], the probability measure  $\mu$  induced by  $Y$ , taking a separable version, is supported by  $C_{a,b}[0, T]$  (which is equivalent to the Banach space of continuous functions  $x$  on  $[0, T]$  with  $x(0) = 0$  under the sup norm). Hence  $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$  is the function space induced by  $Y$  where  $\mathcal{B}(C_{a,b}[0, T])$  is the Borel  $\sigma$ -algebra of  $C_{a,b}[0, T]$ .

A subset  $B$  of  $C_{a,b}[0, T]$  is said to be scale-invariant measurable [11, 20] provided  $\rho B$  is  $\mathcal{B}(C_{a,b}[0, T])$ -measurable for all  $\rho > 0$ , and a scale-invariant measurable set  $N$  is said to be scale-invariant null set provided  $\mu(\rho N) = 0$  for all  $\rho > 0$ . A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.).

Let  $L_{a,b}^2[0, T]$  be the Hilbert space of functions on  $[0, T]$  which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on  $[0, T]$  induced by  $a(\cdot)$  and  $b(\cdot)$ : i.e.,

(2.2)

$$L_{a,b}^2[0, T] = \left\{ v : \int_0^T v^2(s)db(s) < \infty \text{ and } \int_0^T v^2(s)d|a|(s) < \infty \right\}$$

where  $|a|(t)$  denotes the total variation of the function  $a$  on the interval  $[0, t]$ .

For  $u, v \in L_{a,b}^2[0, T]$ , let

$$(2.3) \quad (u, v)_{a,b} = \int_0^T u(t)v(t)d[b(t) + |a|(t)].$$

Then  $(\cdot, \cdot)_{a,b}$  is an inner product on  $L_{a,b}^2[0, T]$  and  $\|u\|_{a,b} = \sqrt{(u, u)_{a,b}}$  is a norm on  $L_{a,b}^2[0, T]$ . In particular note that  $\|u\|_{a,b} = 0$  if and only if  $u(t) = 0$  a.e. on  $[0, T]$ . Furthermore  $(L_{a,b}^2[0, T], \|\cdot\|_{a,b})$  is a separable Hilbert space.

Let  $\{\phi_j\}_{j=1}^\infty$  be a complete orthogonal set of real-valued functions of bounded variation on  $[0, T]$  such that

$$(\phi_j, \phi_k)_{a,b} = \begin{cases} 0, & j \neq k \\ 1, & j = k, \end{cases}$$

and for each  $v \in L^2_{a,b}[0, T]$ , let

$$(2.4) \quad v_n(t) = \sum_{j=1}^n (v, \phi_j)_{a,b} \phi_j(t)$$

for  $n = 1, 2, \dots$ . Then for each  $v \in L^2_{a,b}[0, T]$ , the Paley-Wiener-Zygmund(PWZ) stochastic integral  $\langle v, x \rangle$  is defined by the formula

$$(2.5) \quad \langle v, x \rangle = \lim_{n \rightarrow \infty} \int_0^T v_n(t) dx(t)$$

for all  $x \in C_{a,b}[0, T]$  for which the limit exists.; one can show that for each  $v \in L^2_{a,b}[0, T]$ , the PWZ integral  $\langle v, x \rangle$  exists for  $\mu$ -a.e.  $x \in C_{a,b}[0, T]$ .

We denote the function space integral of a  $\mathcal{B}(C_{a,b}[0, T])$ -measurable functional  $F$  by

$$(2.6) \quad E[F] = \int_{C_{a,b}[0, T]} F(x) d\mu(x)$$

whenever the integral exists.

We are now ready to state the definition of the generalized analytic Feynman integral.

**DEFINITION 2.1.** Let  $\mathbb{C}$  denote the complex numbers. Let  $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$  and  $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \operatorname{Re}\lambda \geq 0\}$ . Let  $F : C_{a,b}[0, T] \rightarrow \mathbb{C}$  be such that for each  $\lambda > 0$ , the function space integral

$$J(\lambda) = \int_{C_{a,b}[0, T]} F(\lambda^{-\frac{1}{2}}x) d\mu(x)$$

exists for all  $\lambda > 0$ . If there exists a function  $J^*(\lambda)$  analytic in  $\mathbb{C}_+$  such that  $J^*(\lambda) = J(\lambda)$  for all  $\lambda > 0$ , then  $J^*(\lambda)$  is defined to be the analytic function space integral of  $F$  over  $C_{a,b}[0, T]$  with parameter  $\lambda$ , and for  $\lambda \in \mathbb{C}_+$  we write

$$(2.7) \quad E^{\operatorname{an}\lambda}[F] \equiv E_x^{\operatorname{an}\lambda}[F(x)] = J^*(\lambda).$$

Let  $q \neq 0$  be a real number and let  $F$  be a functional such that  $E^{\operatorname{an}\lambda}[F]$  exists for all  $\lambda \in \mathbb{C}_+$ . If the following limit exists, we call it the generalized analytic Feynman integral of  $F$  with parameter  $q$  and we write

$$(2.8) \quad E^{\operatorname{anf}q}[F] = \lim_{\lambda \rightarrow -iq} E^{\operatorname{an}\lambda}[F]$$

where  $\lambda$  approaches  $-iq$  through  $\mathbb{C}_+$ .

Next we state the definitions of the generalized analytic Fourier-Feynman transform (GFFT) and the generalized convolution product (GCP).

DEFINITION 2.2. For  $\lambda \in \mathbb{C}_+$  and  $y \in C_{a,b}[0, T]$ , let

$$(2.9) \quad T_\lambda(F)(y) = E_x^{\text{an}\lambda}[F(y+x)].$$

In the standard Fourier theory the integrals involved are often interpreted in the mean; a similar concept is useful in the FFT theory [19]. Let  $p \in (1, 2]$  and let  $p$  and  $p'$  be related by  $1/p+1/p' = 1$ . Let  $\{H_n\}$  and  $H$  be scale-invariant measurable functionals such that for each  $\rho > 0$ ,

$$\lim_{n \rightarrow \infty} E[|H_n(\rho y) - H(\rho y)|^{p'}] = 0.$$

Then we write

$$H \approx \text{l.i.m.}_{n \rightarrow \infty} H_n$$

and we call  $H$  the scale-invariant limit in the mean of order  $p'$ . A similar definition is understood when  $n$  is replaced by the continuously varying parameter  $\lambda$ .

We are ready to state the definition of the  $L_p$  analytic GFFT.

DEFINITION 2.3. Let  $q$  be a nonzero real number and let  $F$  be a measurable functional. For  $p \in (1, 2]$ , we define the  $L_p$  analytic GFFT,  $T_q^{(p)}(F)$  of  $F$ , by the formula ( $\lambda \in \mathbb{C}_+$ )

$$(2.10) \quad T_q^{(p)}(F)(y) = \text{l.i.m.}_{\lambda \rightarrow -iq} T_\lambda(F)(y)$$

if it exists. We define the  $L_1$  analytic GFFT,  $T_q^{(1)}(F)$  of  $F$ , by the formula ( $\lambda \in \mathbb{C}_+$ )

$$(2.11) \quad T_q^{(1)}(F)(y) = \lim_{\lambda \rightarrow -iq} T_\lambda(F)(y)$$

if it exists.

We note that for  $1 \leq p \leq 2$ ,  $T_q^{(p)}(F)$  is defined only s-a.e.. We also note that if  $T_q^{(p)}(F)$  exists and if  $F \approx G$ , then  $T_q^{(p)}(G)$  exists and  $T_q^{(p)}(G) \approx T_q^{(p)}(F)$ .

DEFINITION 2.4. Let  $F$  and  $G$  be measurable functionals on  $C_{a,b}[0, T]$ .

For  $\lambda \in \tilde{\mathbb{C}}_+$ , we define their GCP  $(F * G)_\lambda$  (if it exists) by (2.12)

$$(F * G)_\lambda(y) = \begin{cases} E_x^{\text{an}\lambda} [F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})], & \lambda \in \mathbb{C}_+ \\ E_x^{\text{anf}_q} [F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})], & \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases}$$

REMARK 2.1. (1) When  $\lambda = -iq$ , we denote  $(F * G)_\lambda$  by  $(F * G)_q$ .

(2) For any real  $q \neq 0$ , we briefly describe  $F_q^*$  and  $*F_q$  of a functional  $F$  on  $C_{a,b}[0, T]$  as follows :

$$(2.13) \quad F_q^* = (F * 1)_q \quad \text{and} \quad *F_q = (1 * F)_q.$$

The following generalized analytic Feynman integral formula is used several times in this paper.

$$(2.14) \quad E_x[\exp\{i\lambda^{-\frac{1}{2}}\langle v, x \rangle\}] = \exp\left\{-\frac{1}{2\lambda}(v^2, b') + i\lambda^{-\frac{1}{2}}(v, a')\right\}$$

for all  $\lambda \in \tilde{\mathbb{C}}_+$  and  $v \in L_{a,b}^2[0, T]$  where

$$(2.15) \quad (v, a') = \int_0^T v(t)a'(t)dt = \int_0^T v(t)da(t)$$

and

$$(2.16) \quad (v^2, b') = \int_0^T v^2(t)b'(t)dt = \int_0^T v^2(t)db(t).$$

In this paper for each  $\lambda \in \tilde{\mathbb{C}}_+$ ,  $\lambda^{\frac{1}{2}}$  (or  $\lambda^{-\frac{1}{2}}$ ) is always chosen to have positive real part.

Now we introduce the Banach algebra  $\mathcal{S}(L_{a,b}^2[0, T])$  referred to in Section 1.

DEFINITION 2.5. Let  $M(L_{a,b}^2[0, T])$  be the space of complex-valued, countably additive (and hence finite) Borel measures on  $L_{a,b}^2[0, T]$ . The Banach algebra  $\mathcal{S}(L_{a,b}^2[0, T])$  consists of those functionals  $F$  on  $C_{a,b}[0, T]$  expressible in the form

$$(2.17) \quad F(x) = \int_{L_{a,b}^2[0, T]} \exp\{i\langle u, x \rangle\}df(u)$$

for s-a.e.  $x \in C_{a,b}[0, T]$  where the associated measure  $f$  is an element of  $M(L_{a,b}^2[0, T])$ .

REMARK 2.2. (i) When  $a(t) \equiv 0$  and  $b(t) = t$  on  $[0, T]$ ,  $\mathcal{S}(L_{a,b}^2[0, T])$  reduces to a Banach algebra  $\mathcal{S}$  introduced by Cameron and Storvick in [3]. For further work on  $\mathcal{S}$ , see the references referred to in Section 20.1 of [18].

(ii)  $M(L_{a,b}^2[0, T])$  is a Banach algebra under the total variation norm where convolution is taken as the multiplication.

(iii) One can show that the correspondence  $f \rightarrow F$  is injective, carries convolution into pointwise multiplication and that  $\mathcal{S}(L_{a,b}^2[0, T])$  is a Banach algebra with norm

$$\|F\| = \|f\| = \int_{L_{a,b}^2[0, T]} |df(u)|.$$

In [3], Cameron and Storvick carry out these arguments in detail for the Banach algebra  $\mathcal{S}$ .

REMARK 2.3. If  $a(t) \equiv 0$  on  $[0, T]$ , then for all  $F \in \mathcal{S}(L_{a,b}^2[0, T])$  associated with measure  $f$ , the generalized analytic Feynman integral  $E^{\text{anf}_q}[F]$  will always exist for all real  $q \neq 0$  and be given by the formula

$$(2.18) \quad E^{\text{anf}_q}[F] = \int_{L_{a,b}^2[0, T]} \exp \left\{ -\frac{i(v^2, b')}{2q} \right\} df(v).$$

However for  $a(t)$  as in this section, and proceeding formally using equation (2.14), we see that  $E^{\text{anf}_q}[F]$  will be given by the formula

$$(2.19) \quad E^{\text{anf}_q}[F] = \int_{L_{a,b}^2[0, T]} \exp \left\{ -\frac{i(v^2, b')}{2q} + i \left( \frac{i}{q} \right)^{\frac{1}{2}} (v, a') \right\} df(v)$$

if it exists. But the integral on the right hand-side of (2.19) might not exist if the real part of

$$-\frac{i(v^2, b')}{2q} + i \left( \frac{i}{q} \right)^{\frac{1}{2}} (v, a')$$

is positive. However

$$\left| \exp \left\{ -\frac{i(v^2, b')}{2q} + i \left( \frac{i}{q} \right)^{\frac{1}{2}} (v, a') \right\} \right| = \begin{cases} \exp\{-(2q)^{-1/2}(v, a')\}, & q > 0 \\ \exp\{(-2q)^{-1/2}(v, a')\}, & q < 0, \end{cases}$$

and so the generalized analytic Feynman integral  $E^{\text{anf}_q}[F]$  exist provided the associated measure  $f$  satisfies the condition

$$(2.20) \quad \int_{L_{a,b}^2[0, T]} \exp \left\{ \frac{1}{\sqrt{|2q|}} \int_0^T |u(s)| |d|a|(s)| \right\} |df(u)| < \infty.$$

REMARK 2.4. (i) Let  $q_0$  be a non-zero real number and let  $F$  be an element of  $\mathcal{S}(L_{a,b}^2[0, T])$  whose associated measure  $f$  satisfies the condition

$$(2.21) \quad \int_{L_{a,b}^2[0, T]} \exp\left\{\frac{1}{\sqrt{|2q_0|}} \int_0^T |u(s)|d|a|(s)\right\} |df(u)| < \infty.$$

Then for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$ , the GFFT of  $F$ ,  $T_q^{(p)}(F)$  exists and is given by the formula

$$(2.22) \quad T_q^{(p)}(F)(y) = \int_{L_{a,b}^2[0, T]} \exp\left\{i\langle u, y \rangle - \frac{i}{2q}(u^2, b') + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u, a')\right\} df(u)$$

for s-a.e.  $y \in C_{a,b}[0, T]$ .

(ii) Let  $F$  and  $G$  be elements of  $\mathcal{S}(L_{a,b}^2[0, T])$  whose associated measures  $f$  and  $g$  satisfy the condition

$$(2.23) \quad \int_{L_{a,b}^2[0, T]} \exp\left\{\frac{1}{\sqrt{|4q_0|}} \int_0^T |u(s)|d|a|(s)\right\} [|df(u)| + |dg(u)|] < \infty.$$

Then their GCP  $(F * G)_q$  exists for all real  $q$  with  $|q| \geq |q_0|$  and is given by the formula

$$(2.24) \quad (F * G)_q(y) = \int_{L_{a,b}^2[0, T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u + v, y \rangle - \frac{i}{4q}((u - v)^2, b') + i\left(\frac{i}{2q}\right)^{\frac{1}{2}}(u - v, a')\right\} df(u)dg(v)$$

for s-a.e.  $y \in C_{a,b}[0, T]$ . Furthermore  $(F * G)_q$  is an element of  $\mathcal{S}(L_{a,b}^2[0, T])$ .

### 3. Conditional transforms and conditional convolutions

In this section we first obtain the CGFFT and CGCP. We then establish several relationships between CGFFT and CGCP.

Throughout this section we will condition by the function

$$X : C_{a,b}[0, T] \rightarrow \mathbb{R}$$

given by

$$(3.1) \quad X(x) = x(T).$$

Now, we state the definition of the conditional function space integral.



DEFINITION 3.1. Let  $X$  be a real-valued measurable function on  $C_{a,b}[0, T]$  whose probability distribution  $\mu_X$  is absolutely continuous with respect to Lebesgue measure on  $\mathbb{R}$ . Let  $F$  be a complex-valued  $\mu$ -integrable function on  $C_{a,b}[0, T]$ . Then the conditional integral of  $F$  given  $X$ , denoted by  $E[F|X](\eta)$ , is a Lebesgue measurable function of  $\eta$ , unique up to null sets in  $\mathbb{R}$ , satisfying the equation

$$(3.2) \quad \int_{X^{-1}(B)} F(x) d\mu(x) = \int_B E[F|X](\eta) d\mu_X(\eta)$$

for all Borel sets  $B$  in  $\mathbb{R}$ .

DEFINITION 3.2. Let  $F : C_{a,b}[0, T] \rightarrow \mathbb{C}$  be such that for each  $\lambda > 0$ ,

$$(3.3) \quad \int_{C_{a,b}[0, T]} |F(\lambda^{-1/2}x)| d\mu(x) < \infty.$$

Let  $X : C_{a,b}[0, T] \rightarrow \mathbb{R}$  be such that for each  $\lambda > 0$  and a.e.  $\eta \in \mathbb{R}$ ,  $X(\lambda^{-1/2}x + \eta)$  is a  $\mu$ -integrable function of  $x$  on  $C_{a,b}[0, T]$ , i.e., for a.e.  $\eta \in \mathbb{R}$ ,  $Y(x) = X(\lambda^{-1/2}x + \eta)$  is scale-invariant measurable on  $C_{a,b}[0, T]$ . For  $\lambda > 0$  and  $\eta \in \mathbb{R}$ , let

$$(3.4) \quad J_\lambda(\eta) = E_x[F(\lambda^{-1/2}x)|X(\lambda^{-1/2}x)](\eta)$$

denote the conditional function space integral of  $F(\lambda^{-1/2}x)$  given  $\lambda^{-1/2}x(T)$ . If for a.e.  $\eta \in \mathbb{R}$ , there exists a function  $J_\lambda^*(\eta)$  analytic in  $\lambda$  on  $\mathbb{C}_+$  such that  $J_\lambda^*(\eta) = J_\lambda(\eta)$  for all  $\lambda > 0$ , then  $J_\lambda^*(\cdot)$  is defined to be the conditional analytic function space integral of  $F$  given  $x(T)$  with parameter  $\lambda$  and for  $\lambda \in \mathbb{C}_+$  we write

$$(3.5) \quad E^{\text{an}\lambda}[F|X](\eta) \equiv E_x^{\text{an}\lambda}[F(x)|X(x)](\eta) = J_\lambda^*(\eta).$$

If for fixed real  $q \neq 0$  the limit

$$(3.6) \quad \lim_{\lambda \rightarrow -iq} E^{\text{an}\lambda}[F|X](\eta)$$

exists for a.e.  $\eta \in \mathbb{R}$ , where  $\lambda \rightarrow -iq$  through  $\mathbb{C}_+$ , we will denote the value of this limit by  $E^{\text{anf}_q}[F|X](\eta)$  and we call it the conditional generalized analytic Feynman integral of  $F$  given  $X$  with parameter  $q$ .

REMARK 3.1. In [21], Park and Skoug gave a formula for expressing conditional Wiener integrals in terms of ordinary Wiener integrals; namely that for  $\lambda > 0$ ,

$$(3.7) \quad \begin{aligned} & E_x[F(\lambda^{-1/2}x)|\lambda^{-1/2}x(T)](\eta) \\ &= E_x\left[F\left(\lambda^{-1/2}x(\cdot) - \frac{\dot{\cdot}}{T}\lambda^{-1/2}x(T) + \frac{\dot{\cdot}}{T}\eta\right)\right]. \end{aligned}$$

Thus we have that

$$(3.8) \quad E_x^{\text{an}\lambda}[F(x)|x(T)](\eta) = E_x^{\text{an}\lambda}\left[F\left(x(\cdot) - \frac{b(\cdot)}{b(T)}x(T) + \frac{b(\cdot)}{b(T)}\eta\right)\right]$$

and

$$(3.9) \quad E_x^{\text{anf}_q}[F(x)|x(T)](\eta) = E_x^{\text{anf}_q}\left[F\left(x(\cdot) - \frac{b(\cdot)}{b(T)}x(T) + \frac{b(\cdot)}{b(T)}\eta\right)\right]$$

where in (3.8) and (3.9) the existence of either side implies the existence of the other side and their equality.

Next we define the CGFFT and the CGCP.

DEFINITION 3.3. For  $\lambda \in \mathbb{C}_+$ ,  $\eta \in \mathbb{R}$  and  $y \in C_{a,b}[0, T]$ , let  $T_\lambda(F|X)(y, \eta)$  denote the conditional analytic function space integral of  $F(y+x)$  given  $X(x) = x(T)$ ; that is to say

$$(3.10) \quad \begin{aligned} T_\lambda(F|X)(y, \eta) &= E_x^{\text{an}\lambda}[F(y+x)|x(T)](\eta) \\ &= E_x^{\text{an}\lambda}\left[F\left(y(\cdot) + x(\cdot) - \frac{b(\cdot)}{b(T)}x(T) + \frac{b(\cdot)}{b(T)}\eta\right)\right]. \end{aligned}$$

Then for  $p \in [1, 2]$  we define the CGFFT,  $T_q^{(p)}(F|X)$  of  $F$ , by the formula ( $\lambda \in \mathbb{C}_+$ ),

$$(3.11) \quad T_q^{(p)}(F|X)(y, \eta) = \begin{cases} \text{l.i.m.}_{\lambda \rightarrow -iq} T_\lambda(F|X)(y, \eta), & 1 < p \leq 2 \\ \lim_{\lambda \rightarrow -iq} T_\lambda(F|X)(y, \eta), & p = 1 \end{cases}$$

if it exists. Note that for  $p = 1$

$$(3.12) \quad T_q^{(1)}(F|X)(y, \eta) = E_x^{\text{anf}_q}\left[F\left(y(\cdot) + x(\cdot) - \frac{b(\cdot)}{b(T)}x(T) + \frac{b(\cdot)}{b(T)}\eta\right)\right].$$

And we define the CGCP  $((F * G)_\lambda | X)(y, \eta)$  (if it exists) by the formula

$$(3.13) \quad \begin{aligned} & ((F * G)_\lambda | X)(y, \eta) \\ &= \begin{cases} E_x^{\text{an}\lambda} [F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})|x(T)](\eta), & \lambda \in \mathbb{C}_+ \\ E_x^{\text{anf}_q} [F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})|x(T)](\eta), & \lambda = -iq, q \in \mathbb{R}, q \neq 0 \end{cases} \\ &= \begin{cases} E_x^{\text{an}\lambda} [F(\frac{y(\cdot)+x(\cdot)}{\sqrt{2}} - \frac{b(\cdot)x(T)}{b(T)\sqrt{2}} + \frac{b(\cdot)\eta}{b(T)\sqrt{2}}) \\ \quad \cdot G(\frac{y(\cdot)-x(\cdot)}{\sqrt{2}} + \frac{b(\cdot)x(T)}{b(T)\sqrt{2}} - \frac{b(\cdot)\eta}{b(T)\sqrt{2}})], & \lambda \in \mathbb{C}_+ \\ E_x^{\text{anf}_q} [F(\frac{y(\cdot)+x(\cdot)}{\sqrt{2}} - \frac{b(\cdot)x(T)}{b(T)\sqrt{2}} + \frac{b(\cdot)\eta}{b(T)\sqrt{2}}) \\ \quad \cdot G(\frac{y(\cdot)-x(\cdot)}{\sqrt{2}} + \frac{b(\cdot)x(T)}{b(T)\sqrt{2}} - \frac{b(\cdot)\eta}{b(T)\sqrt{2}})], & \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases} \end{aligned}$$

Again if  $\lambda = -iq$ , we will denote  $((F * G)_\lambda | X)(y, \eta)$  by  $((F * G)_q | X)(y, \eta)$ .

REMARK 3.2. By using (3.9) and (2.14) we see that the conditional generalized analytic Feynman integral  $E_x^{\text{anf}_q} [F(x)|x(T)](\eta)$  is given by the formula

$$(3.14) \quad \begin{aligned} E_x^{\text{anf}_q} [F(x)|x(T)](\eta) &= \int_{L_{a,b}^2[0,T]} \exp \left\{ -\frac{i}{2q}(u^2, b) + \frac{i}{2q}u_1^2 b(T) \right. \\ &\quad \left. + i \left( \frac{i}{q} \right)^{\frac{1}{2}} (u, a') - i \left( \frac{i}{q} \right)^{\frac{1}{2}} u_1 a(T) + iu_1 \eta \right\} df(u) \end{aligned}$$

where  $u_1 = (u, b')/b(T)$  if it exists. But the integral on the right-hand side of (3.14) might not exist if the real part of

$$-\frac{i}{2q}(u^2, b) + \frac{i}{2q}u_1^2 b(T) + i \left( \frac{i}{q} \right)^{\frac{1}{2}} (u, a') - i \left( \frac{i}{q} \right)^{\frac{1}{2}} u_1 a(T) + iu_1 \eta$$

is positive. However

$$\begin{aligned} & \left| \exp \left\{ -\frac{i}{2q}(u^2, b) + \frac{i}{2q}u_1^2 b(T) \right. \right. \\ & \quad \left. \left. + i \left( \frac{i}{q} \right)^{\frac{1}{2}} (u, a') - i \left( \frac{i}{q} \right)^{\frac{1}{2}} u_1 a(T) + iu_1 \eta \right\} \right| \\ &= \begin{cases} \exp \left\{ -\frac{1}{\sqrt{2q}}(v, a') + \frac{1}{\sqrt{2q}}(v, b') \frac{a(T)}{b(T)} \right\}, & q > 0 \\ \exp \left\{ \frac{1}{\sqrt{-2q}}(v, a') - \frac{1}{\sqrt{-2q}}(v, b') \frac{a(T)}{b(T)} \right\}, & q < 0, \end{cases} \end{aligned}$$

and so the conditional generalized analytic Feynman integral

$$E_x^{anf_q}[F(x)|x(T)](\eta)$$

exist provided the associated measure  $f$  satisfies the condition

$$(3.15) \quad \int_{L_{a,b}^2[0,T]} \exp \left\{ \frac{1}{\sqrt{|2q|}} \int_0^T |u(s)| d[|a|(s) + b(s)] \right\} |df(u)| < \infty.$$

In our next theorem, we obtain the CGFFT of a functional in

$$\mathcal{S}(L_{a,b}^2[0, T]).$$

**THEOREM 3.1.** Let  $q_0$  be a nonzero real number and let  $F$  be an element of  $\mathcal{S}(L_{a,b}^2[0, T])$  whose associated measure  $f$  satisfies the condition (3.15) with  $q$  replaced with  $q_0$ . Then for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$ , the CGFFT of  $F$ ,  $T_q^{(p)}(F|X)$  exists and is given by the formula

$$(3.16) \quad \begin{aligned} T_q^{(p)}(F|X)(y, \eta) &= \int_{L_{a,b}^2[0,T]} \exp \left\{ i\langle u, y \rangle + iu_1\eta - \frac{i}{2q}(u^2, b') + \frac{i}{2q}u_1^2b(T) \right. \\ &\quad \left. + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u, a') - i\left(\frac{i}{q}\right)^{\frac{1}{2}}u_1a(T) \right\} df(u) \end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$  where  $u_1 = (u, b')/b(T)$ .

*Proof.* First of all, using (3.10), the Fubini theorem, and (2.14), we obtain, for all  $\lambda > 0$ .

$$(3.17) \quad \begin{aligned} &T_\lambda(F|X)(y, \eta) \\ &= \int_{L_{a,b}^2[0,T]} \exp \left\{ i\langle u, y \rangle + i\frac{(u, b')}{b(T)}\eta \right\} \\ &\quad \cdot E_x \left[ \exp \left\{ \frac{i}{\sqrt{\lambda}}\langle u, x \rangle - \frac{i(u, b')}{\sqrt{\lambda}b(T)}x(T) \right\} \right] df(u) \\ &= \int_{L_{a,b}^2[0,T]} \exp \left\{ i\langle u, y \rangle + iu_1\eta \right\} E_x \left[ \exp \left\{ \frac{i}{\sqrt{\lambda}}\langle u - u_1, x \rangle \right\} \right] df(u) \end{aligned}$$

$$\begin{aligned}
 &= \int_{L^2_{a,b}[0,T]} \exp \left\{ i \langle u, y \rangle + i u_1 \eta \right. \\
 &\quad \left. - \frac{1}{2\lambda} ((u - u_1)^2, b') + \frac{i}{\sqrt{\lambda}} (u - u_1, a') \right\} df(u) \\
 &= \int_{L^2_{a,b}[0,T]} \exp \left\{ i \langle u, y \rangle + i u_1 \eta - \frac{1}{2\lambda} (u^2, b') + \frac{1}{2\lambda} u_1^2 b(T) \right. \\
 &\quad \left. + \frac{i}{\sqrt{\lambda}} (u, a') - \frac{i}{\sqrt{\lambda}} u_1 a(T) \right\} df(u)
 \end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$ . But the last expression above is analytic throughout  $\mathbb{C}_+$ , and is continuous in  $\tilde{\mathbb{C}}_+$ . Thus we have the equation (3.16). In addition, by using the condition (3.15) above, we see that for all real  $q$  with  $|q| \geq |q_0|$

$$\begin{aligned}
 &|T_q^{(p)}(F|X)(y, \eta)| \\
 &\leq \int_{L^2_{a,b}[0,T]} \exp \left\{ \frac{1}{\sqrt{|2q_0|}} \int_0^T |u(s)| d \left[ |a|(s) + \frac{|a(T)|}{b(T)} b(s) \right] \right\} |df(u)| < \infty.
 \end{aligned}$$

Hence we have the desired result.  $\square$

In our next theorem, we obtain the CGCP of functionals in

$$\mathcal{S}(L^2_{a,b}[0, T]).$$

**THEOREM 3.2.** *Let  $q_0$  be a nonzero real number and let  $F$  and  $G$  be elements of  $\mathcal{S}(L^2_{a,b}[0, T])$  whose associated measures  $f$  and  $g$  satisfy the condition*

$$(3.18) \quad \int_{L^2_{a,b}[0,T]} \exp \left\{ \frac{1}{\sqrt{|4q_0|}} \int_0^T |u(s)| d[|a|(s) + b(s)] \right\} [|df(u)| + |dg(u)|] < \infty.$$

*Then their CGCP  $((F * G)_q|X)$  exists for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$  and is given by the formula*

$$\begin{aligned}
 &((F * G)_q|X)(y, \eta) \\
 &= \int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle u + v, y \rangle + \frac{i}{\sqrt{2}} (u_1 - v_1) \eta \right. \\
 (3.19) \quad &\quad \left. - \frac{i}{4q} ((u - v)^2, b') + \frac{i}{4q} (u_1 - v_1)^2 b(T) \right. \\
 &\quad \left. + i \left( \frac{i}{2q} \right)^{\frac{1}{2}} (u - v, a') - i \left( \frac{i}{2q} \right)^{\frac{1}{2}} (u_1 - v_1) a(T) \right\} df(u) dg(v)
 \end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$  where  $u_1 = (u, b')/b(T)$  and  $v_1 = (v, b')/b(T)$ .

*Proof.* By using (3.13), the Fubini theorem, and (2.14), we have that for all  $\lambda > 0$ ,

$$\begin{aligned}
& ((F * G)_\lambda | X)(y, \eta) \\
&= \int_{L_{a,b}^2[0, T]} \int_{L_{a,b}^2[0, T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle u + v, y \rangle + \frac{i(u - v, b')}{\sqrt{2}b(T)} \eta \right\} \\
&\quad \cdot E_x \left[ \exp \left\{ \frac{i}{\sqrt{2\lambda}} \langle u - v, x \rangle - \frac{i(u - v, b')}{\sqrt{2\lambda}b(T)} x(T) \right\} \right] df(u) dg(v) \\
(3.20) \quad &= \int_{L_{a,b}^2[0, T]} \int_{L_{a,b}^2[0, T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle u + v, y \rangle + \frac{i}{\sqrt{2}} (u_1 - v_1) \eta \right\} \\
&\quad \cdot E_x \left[ \exp \left\{ \frac{i}{\sqrt{2\lambda}} \langle (u - v) - (u_1 - v_1), x \rangle \right\} \right] df(u) dg(v) \\
&= \int_{L_{a,b}^2[0, T]} \int_{L_{a,b}^2[0, T]} \exp \left\{ \frac{i}{\sqrt{2\lambda}} \langle u + v, y \rangle + \frac{i}{\sqrt{2}} (u_1 - v_1) \eta \right. \\
&\quad \left. - \frac{1}{4\lambda} ((u - v)^2, b') + \frac{1}{4\lambda} (u_1 - v_1)^2 b(T) \right. \\
&\quad \left. + \frac{i}{\sqrt{2\lambda}} (u - v, a') - \frac{i}{\sqrt{2\lambda}} (u_1 - v_1) a(T) \right\} df(u) dg(v)
\end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$ . But the last expression above is analytic through out  $\mathbb{C}_+$  and is continuous on  $\tilde{\mathbb{C}}_+$ . Thus we have the equation (3.19) above. In addition, the condition (3.18) will imply the existence of the equation (3.19).  $\square$

**REMARK 3.3.** Let  $F, G, f, g,$  and  $q_0$  be as in Theorem 3.2. Then if  $G \equiv 1$ , then for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$ ,

$$\begin{aligned}
(3.21) \quad (F_q^* | X)(y, \eta) &= \int_{L_{a,b}^2[0, T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle u, y \rangle + \frac{i}{\sqrt{2}} u_1 \eta - \frac{i}{4q} (u^2, b') \right. \\
&\quad \left. + \frac{i}{4q} u_1^2 b(T) + i \left( \frac{i}{2q} \right)^{\frac{1}{2}} (u, a') - i \left( \frac{i}{2q} \right)^{\frac{1}{2}} u_1 a(T) \right\} df(u)
\end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$ . Similarly, if  $F \equiv 1$ , then for all  $p \in [1, 2]$  and

all real  $q$  with  $|q| \geq |q_0|$ ,

(3.22)

$$(*G_q|X)(y, \eta) = \int_{L^2_{a,b}[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle v, y \rangle - \frac{i}{\sqrt{2}}v_1\eta - \frac{i}{4q}(v^2, b')\right. \\ \left. + \frac{i}{4q}v_1^2b(T) - i\left(\frac{i}{2q}\right)^{\frac{1}{2}}(v, a') + i\left(\frac{i}{2q}\right)^{\frac{1}{2}}v_1a(T)\right\} dg(v)$$

for s-a.e.  $y \in C_{a,b}[0, T]$  where  $u_1$  and  $v_1$  are as in Theorem 3.2 above.

In our next theorem, we obtain the CGFFT of the CGCP of functional in  $\mathcal{S}(L^2_{a,b}[0, T])$ .

**THEOREM 3.3.** *Let  $q_0$  be a nonzero real number and let  $F$  and  $G$  be elements of  $\mathcal{S}(L^2_{a,b}[0, T])$  whose associated measures  $f$  and  $g$  satisfy the condition*

(3.23)

$$\int_{L^2_{a,b}[0,T]} \exp\left\{\frac{2}{\sqrt{|4q_0|}} \int_0^T |u(s)|d[|a|(s) + b(s)]\right\} [|df(u)| + |dg(u)|] < \infty.$$

Then for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$ ,

$$T_q^{(p)}(((F * G)_q|X)(\cdot, \eta_1)|X)(y, \eta_2), T_q^{(p)}((F_q^*|X)(\cdot, \eta_1)|X)(y, \eta_2)$$

and  $T_q^{(p)}((*G_q|X)(\cdot, \eta_1)|X)(y, \eta_2)$  all exist and

$$(3.24) \quad T_q^{(p)}(((F * G)_q|X)(\cdot, \eta_1)|X)(y, \eta_2) \\ = T_q^{(p)}((F_q^*|X)(\cdot, \eta_1)|X)(y, \eta_2)T_q^{(p)}((*G_q|X)(\cdot, \eta_1)|X)(y, \eta_2)$$

for s-a.e.  $y \in C_{a,b}[0, T]$  where  $F_q^*$  and  $*G_q$  are given as in equation (2.13).

Also both of the expressions in (3.24) are given by

(3.25)

$$\int_{L^2_{a,b}[0,T]} \int_{L^2_{a,b}[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u + v, y \rangle + iu_1\left(\frac{\eta_2 + \eta_1}{\sqrt{2}}\right) + iv_1\left(\frac{\eta_2 - \eta_1}{\sqrt{2}}\right)\right. \\ \left. - \frac{i}{2q}(u^2 + v^2, b') + \frac{i}{2q}(u_1^2 + v_1^2)b(T)\right. \\ \left. + 2i\left(\frac{i}{2q}\right)^{\frac{1}{2}}(u, a') - 2i\left(\frac{i}{2q}\right)^{\frac{1}{2}}u_1a(T)\right\} df(u)dg(v)$$

for s-a.e.  $y \in C_{a,b}[0, T]$ .

*Proof.* By using (3.10), (3.13), the Fubini theorem, and (2.14), we have that for all  $\lambda > 0$ ,

$$(3.26) \quad \begin{aligned} & T_\lambda(((F * G)_\lambda|X)(\cdot, \eta_1)|X)(y, \eta_2) \\ & = T_\lambda((F_\lambda^*|X)(\cdot, \eta_1)|X)(y, \eta_2) T_\lambda((*G_\lambda|X)(\cdot, \eta_1)|X)(y, \eta_2) \end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$ . But both of the expressions on the right-hand side of equation (3.26) are analytic functions of  $\lambda$  through  $\mathbb{C}_+$ , and are continuous functions of  $\lambda$  on  $\tilde{\mathbb{C}}_+$  for all  $y \in C_{a,b}[0, T]$ . Thus equation (3.24) is established. Moreover the condition (3.23) will ensure the existence of the both sides of equation (3.24) above. Hence we have the desired result.  $\square$

REMARK 3.4. Let  $F, G, f, g$ , and  $q_0$  be as in Theorem 3.3 above. Then we have the followings:

(i) By using the expression in (3.25), we see that

$$T_q^{(p)}((F_q^*|X)(\cdot, \eta_1)|X)(y, \eta_2) = T_q^{(p)}((F_q^*|X)(\cdot, \eta_2)|X)(y, \eta_1)$$

and

$$T_q^{(p)}((*G_q|X)(\cdot, \eta_1)|X)(y, \eta_2) = T_q^{(p)}((*G_q|X)(\cdot, -\eta_2)|X)(y, -\eta_1).$$

Using these above, we obtain the following alternative form

$$(3.27) \quad \begin{aligned} & T_q^{(p)}(((F * G)_q|X)(\cdot, \eta_1)|X)(y, \eta_2) \\ & = T_q^{(p)}((F_q^*|X)(\cdot, \eta_2)|X)(y, \eta_1) T_q^{(p)}((*G_q|X)(\cdot, -\eta_2)|X)(y, -\eta_1) \end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$ .

(ii) If  $a(t) \equiv 0$ , then by using equations (3.21), (3.22) and (3.11), we have

$$T_q^{(p)}((F_q^*|X)(\cdot, \eta_1)|X)(y, \eta_2) = T_q^{(p)}(F|X)\left(\frac{y}{\sqrt{2}}, \frac{\eta_2 + \eta_1}{\sqrt{2}}\right)$$

and

$$T_q^{(p)}((*G_q|X)(\cdot, \eta_1)|X)(y, \eta_2) = T_q^{(p)}(G|X)\left(\frac{y}{\sqrt{2}}, \frac{\eta_2 - \eta_1}{\sqrt{2}}\right).$$



Hence we have

$$\begin{aligned} & T_q^{(p)}(((F * G)_q|X)(\cdot, \eta_1)|X)(y, \eta_2) \\ &= T_q^{(p)}(F|X)\left(\frac{y}{\sqrt{2}}, \frac{\eta_2 + \eta_1}{\sqrt{2}}\right) T_q^{(p)}(G|X)\left(\frac{y}{\sqrt{2}}, \frac{\eta_2 - \eta_1}{\sqrt{2}}\right) \end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$ .

In [22], Park and Skoug established this equation for a Banach algebra  $S$ .

In our next theorem we obtain an expression for the CGCP of CGFFT's of functionals in  $\mathcal{S}(L_{a,b}^2[0, T])$ .

**THEOREM 3.4.** *Let  $q_0$  be a nonzero real number and let  $F$  and  $G$  be elements of  $\mathcal{S}(L_{a,b}^2[0, T])$  whose associated measures  $f$  and  $g$  satisfy the condition*

$$(3.28) \quad \begin{aligned} & \int_{L_{a,b}^2[0, T]} \exp\left\{\frac{(1 + \sqrt{2})}{\sqrt{|4q_0|}} \int_0^T |u(s)|d[|a|(s) + b(s)]\right\} \\ & \cdot [|df(u)| + |dg(u)|] < \infty. \end{aligned}$$

Then for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$ , the following CGCP exists and

$$(3.29) \quad \begin{aligned} & ((T_q^{(p)}(F|X)(\cdot, \eta_1) * T_q^{(p)}(G|X)(\cdot, \eta_2))_{-q}|X)(y, \eta_3) \\ &= \int_{L_{a,b}^2[0, T]} \int_{L_{a,b}^2[0, T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u + v, y \rangle + iu_1\left(\eta_1 + \frac{\eta_3}{\sqrt{2}}\right)\right. \\ & \quad + iv_1\left(\eta_2 - \frac{\eta_3}{\sqrt{2}}\right) - \frac{i}{4q}((u + v)^2, b') + \frac{i}{4q}(u_1 + v_1)^2 b(T) \\ & \quad + i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}(u - v, a') - i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}(u_1 - v_1)a(T) \\ & \quad \left. + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u + v, a') - i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u_1 + v_1)a(T)\right\} df(u)dg(v) \end{aligned}$$

for s-a.e.  $y \in C_{a,b}[0, T]$ .

*Proof.* By using equations (3.19) and (3.16), we can easily obtain the equation (3.29) above. Moreover, the condition (3.28) will imply the existence of the equation (3.29).  $\square$

REMARK 3.5. Let  $F, G, f, g,$  and  $q_0$  be as in Theorem 3.4. Then by using (3.11), (3.19), the Fubini theorem, and (2.14), we have that for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$

$$\begin{aligned}
& T_{q/2}^{(p)}(((F * G)_{-q}|X)(\cdot, \eta_3)|X)(y, \eta_4) \\
&= \int_{L_{a,b}^2[0,T]} \int_{L_{a,b}^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y \rangle \right. \\
&\quad + iu_1\left(\frac{\eta_4 + \eta_3}{\sqrt{2}}\right) + iv_1\left(\frac{\eta_4 - \eta_3}{\sqrt{2}}\right) \\
(3.30) \quad & \left. - \frac{i}{4q}(u^2 + 6uv + v^2, b') + \frac{i}{4q}(u_1^2 + 6u_1v_1 + v_1^2)b(T) \right. \\
&\quad + i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}(u-v, a') - i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}(u_1 - v_1)a(T) \\
&\quad \left. + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u+v, a') - i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u_1 + v_1)a(T)\right\} df(u)dg(v).
\end{aligned}$$

In particular, by using (3.21), (3.22) and (3.30) above, we obtain

$$\begin{aligned}
(3.31) \quad & T_{q/2}^{(p)}((F_{-q}^*|X)(\cdot, \eta_3)|X)(y, \eta_4) \\
&= \int_{L_{a,b}^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u, y \rangle + iu_1\left(\frac{\eta_4 + \eta_3}{\sqrt{2}}\right) - \frac{i}{4q}(u^2, b') + \frac{i}{4q}u_1^2b(T) \right. \\
&\quad + i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}(u, a') - i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}u_1a(T) \\
&\quad \left. + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(u, a') - i\left(\frac{i}{q}\right)^{\frac{1}{2}}u_1a(T)\right\} df(u)
\end{aligned}$$

and

$$\begin{aligned}
(3.32) \quad & T_{q/2}^{(p)}((*G_{-q}|X)(\cdot, \eta_3)|X)(y, \eta_4) \\
&= \int_{L_{a,b}^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle v, y \rangle + iv_1\left(\frac{\eta_4 - \eta_3}{\sqrt{2}}\right) - \frac{i}{4q}(v^2, b') + \frac{i}{4q}v_1^2b(T) \right. \\
&\quad - i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}(v, a') + i\left(\frac{-i}{2q}\right)^{\frac{1}{2}}v_1a(T) \\
&\quad \left. + i\left(\frac{i}{q}\right)^{\frac{1}{2}}(v, a') - i\left(\frac{i}{q}\right)^{\frac{1}{2}}v_1a(T)\right\} dg(v).
\end{aligned}$$

Furthermore, we also using (3.21), (3.22) and (3.29), we obtain

$$\begin{aligned}
 (3.33) \quad & ((T_q^{(p)}(F|X)(\cdot, \eta_1))_{-q}^*|X)(y, \eta_2) \\
 &= \int_{L_{a,b}^2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}} \langle u, y \rangle + iu_1 \left( \eta_1 + \frac{\eta_2}{\sqrt{2}} \right) - \frac{i}{4q} (u^2, b') + \frac{i}{4q} u_1^2 b(T) \right. \\
 &\quad + i \left( \frac{-i}{2q} \right)^{\frac{1}{2}} (u, a') - i \left( \frac{-i}{2q} \right)^{\frac{1}{2}} u_1 a(T) \\
 &\quad \left. + i \left( \frac{i}{q} \right)^{\frac{1}{2}} (u, a') - i \left( \frac{i}{q} \right)^{\frac{1}{2}} u_1 a(T) \right\} df(u)
 \end{aligned}$$

and

$$\begin{aligned}
 (3.34) \quad & (*T_q^{(p)}(G|X)(\cdot, \eta_1))_{-q}|X)(y, \eta_2) \\
 &= \int_{L_{a,b}^2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}} \langle v, y \rangle + iv_1 \left( \eta_1 - \frac{\eta_2}{\sqrt{2}} \right) - \frac{i}{4q} (v^2, b') + \frac{i}{4q} v_1^2 b(T) \right. \\
 &\quad - i \left( \frac{-i}{2q} \right)^{\frac{1}{2}} (v, a') + i \left( \frac{-i}{2q} \right)^{\frac{1}{2}} v_1 a(T) \\
 &\quad \left. + i \left( \frac{i}{q} \right)^{\frac{1}{2}} (v, a') - i \left( \frac{i}{q} \right)^{\frac{1}{2}} v_1 a(T) \right\} dg(v).
 \end{aligned}$$

A close examination of the right-hand sides of (3.31) and (3.33) shows that they are equal if  $\{\eta_1, \eta_2, \eta_3, \eta_4\}$  is in the solution set of the equation

$$(3.35) \quad \sqrt{2}\eta_1 + \eta_2 = \eta_3 + \eta_4.$$

Also, a close examination of the right-hand sides of (3.32) and (3.34) shows that they are equal if  $\{\eta_1, \eta_2, \eta_3, \eta_4\}$  is in the solution set of the equation

$$(3.36) \quad \sqrt{2}\eta_1 - \eta_2 = \eta_3 - \eta_4.$$

**THEOREM 3.5.** *Let  $F, G, f, g,$  and  $q_0$  be as in Theorem 3.4. Then we have the followings:*

(i) *If  $\{\eta_1, \eta_2, \eta_3, \eta_4\}$  satisfy the equation (3.35), then for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$ ,*

$$(3.37) \quad ((T_q^{(p)}(F|X)(\cdot, \eta_1))_{-q}^*|X)(y, \eta_2) = T_{q/2}^{(p)}((F_{-q}^*|X)(\cdot, \eta_3)|X)(y, \eta_4)$$

for *s*-a.e.  $y \in C_{a,b}[0, T]$ .

(ii) If  $\{\eta_1, \eta_2, \eta_3, \eta_4\}$  satisfy the equation (3.36), then for all  $p \in [1, 2]$  and all real  $q$  with  $|q| \geq |q_0|$ ,

(3.38)

$$(*T_q^{(p)}(G|X)(\cdot, \eta_1))_{-q}|X)(y, \eta_2) = T_{q/2}^{(p)}((*G_{-q}|X)(\cdot, \eta_3)|X)(y, \eta_4)$$

for *s*-a.e.  $y \in C_{a,b}[0, T]$ .

Following are some interesting special cases of (3.37) and (3.38), respectively.

$$\begin{aligned} & ((T_q^{(p)}(F|X)(\cdot, \eta_1))_{-q}^*|X)(y, \eta_2) \\ &= ((T_q^{(p)}(F|X)(\cdot, \eta_2/\sqrt{2}))_{-q}^*|X)(y, \sqrt{2}\eta_1) \\ (3.39) \quad &= T_{q/2}^{(p)}((F_{-q}^*|X)(\cdot, \eta_2)|X)(y, \sqrt{2}\eta_1) \\ &= T_{q/2}^{(p)}((F_{-q}^*|X)(\cdot, \sqrt{2}\eta_1)|X)(y, \eta_2) \end{aligned}$$

and

$$\begin{aligned} & (*T_q^{(p)}(G|X)(\cdot, \eta_1))_{-q}|X)(y, \eta_2) \\ (3.40) \quad &= (*T_q^{(p)}(G|X)(\cdot, -\eta_2/\sqrt{2}))_{-q}|X)(y, -\sqrt{2}\eta_1) \\ &= T_{q/2}^{(p)}((*G_{-q}|X)(\cdot, \eta_2)|X)(y, \sqrt{2}\eta_1) \\ &= T_{q/2}^{(p)}((*G_{-q}|X)(\cdot, -\sqrt{2}\eta_1)|X)(y, -\eta_2). \end{aligned}$$

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