Bull. Korean Math. Soc. **41** (2004), No. 1, pp. 109–116

ON THE UNIQUENESS OF ENTIRE FUNCTIONS

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ABSTRACT. In this paper, we study the uniqueness of entire functions and prove the following result: Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 7$ a positive integer, and let a be a nonzero finite complex number. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share a CM, then $f(z) \equiv g(z)$. The result improves the theorem due to ref. [3].

1. Introduction and notations

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane. We use the following standard notation of value distribution theory,

$$T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \cdots$$

(see Hayman [1], Yang [2]). We denote by $S(r,f)$ any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \longrightarrow +\infty$, possibly outside of a set with finite measure.

Let a be a finite complex number. We denote by $N_{k)}(r, \frac{1}{f-a})$ the counting function for zeros of $f(z)-a$ with multiplicity at most k, and by $\overline{N}_{k)}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{f-a})$ be the counting function for zeros of $f(z)-a$ with multiplicity at least k and $\overline{N}_{(k}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Set $N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + \cdots + \overline{N}_{(k}(r, \frac{1}{f-a})$. We define

$$\delta_2(a,f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_2(r, \frac{1}{f-a})}{T(r,f)}.$$

Received January 23, 2002.
2000 Mathematics Subject Classification: 30D35.
Key words and phrases: entire function, sharing value, uniqueness.
The project supported by the NNSF of China.

Let $g(z)$ be a meromorphic function, a be a complex number. If $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share a CM.

In [3], Fang and Hong proved

THEOREM A. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 11$ be a positive integer, a be a nonzero finite complex number. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share a CM, then $f(z) \equiv g(z)$.*

In this paper, using different method from [3], we have proved that Theorem A remains valid for $n \geq 7$.

THEOREM 1. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 7$ be a positive integer, a be a nonzero finite complex number. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share a CM, then $f(z) \equiv g(z)$.*

2. Some lemmas

For the proof of Theorem 1 we need the following lemmas.

LEMMA 1 ([4]). *Let $f(z)$ be a meromorphic function. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r,f) + S(r,f).$$

Here $a_n(\neq 0)$, a_{n-1}, $\cdots$, a_0 are constants.

LEMMA 2 ([5, 6]). *Let $f_j(z)$ $(j=1,2,\cdots,p)$ be linearly independent meromorphic functions, p a positive integer. If*

$$\sum_{j=1}^{p} f_j(z) \equiv 1,$$

then for $1 \leq j \leq p$

$$T(r,f_j) \leq \sum_{i=1}^{p} N(r,\frac{1}{f_i}) + N(r,f_j) + N(r,W) - \sum_{i=1}^{p} N(r,f_i) - N(r,\frac{1}{W}) + S(r),$$

where $W(f_1, f_2, \cdots, f_p)$ is the Wronskian determinant of $f_j(z)$ $(j = 1,2,\cdots,p)$,

$$S(r) = o(T(r)), (r \to \infty, r \notin E).$$

Here

$$T(r) = \max_{1\le j\le p}\{T(r, f_j)\},$$

and E is a set of finite measure.

By Lemma 2 we can easily obtain

LEMMA 3. *Let $f_j(z)$ $(j = 1, 2, \cdots, p)$ be linearly independent transcendental entire functions, p a positive integer. If*

$$\sum_{j=1}^{p} f_j(z) \equiv 1,$$

then for $1 \le j \le p$

$$T(r, f_j) \le \sum_{i=1}^{p} N_{p-1}(r, \frac{1}{f_i}) + S(r).$$

Here $S(r)$ is the same as in Lemma 2.

LEMMA 4. *Let $f_j(z)(j = 1, 2, 3)$ be transcendental entire functions. If $f_1(z) + f_2(z) + f_3(z) \equiv 1$, then*

$$\delta_2(0, f_1) + \delta_2(0, f_2) + \delta_2(0, f_3) \le 2.$$

Here $\delta_2(0, f_j) = 1 - \overline{\lim}_{r\to\infty}\frac{N_2(r,\frac{1}{f_j})}{T(r,f_j)}$ $(j = 1, 2, 3)$.

Proof. We consider two cases.

Case 1. f_1, f_2, f_3 are linearly independent functions. Then by lemma 3 we have

$$T(r, f_j) \le \sum_{i=1}^{3} N_2(r, \frac{1}{f_i}) + S(r) \le \sum_{i=1}^{3} (1 - \delta_2(0, f_i))\, T(r, f_i) + S(r).$$

Thus we obtain

$$T(r) \le \sum_{i=1}^{3} (1 - \delta_2(0, f_i))\, T(r) + S(r).$$

That is

$$\left(\sum_{i=1}^{3} \delta_2(0, f_i) - 2\right) T(r) \le S(r).$$

Hence we get

$$\sum_{i=1}^{3} \delta_2(0, f_i) \le 2. \tag{2.1}$$

Case 2. f_1, f_2, f_3 are linearly dependent functions. Without loss of generality, we assume that f_1, f_2 are linearly independent functions and that $f_3 = c_1 f_1 + c_2 f_2$, where c_1, c_2 are constants. Hence we have

$$(1 + c_1)f_1(z) + (1 + c_2)f_2(z) \equiv 1.$$

Obviously $1 + c_1 \neq 0, 1 + c_2 \neq 0$. Then by the same argument as do in case 1 we obtain

$$\Theta(0, f_1) + \Theta(0, f_2) \leq 1.$$

Considering $\Theta(0, f_i) \geq \delta_2(0, f_i) \quad (i = 1, 2)$ we obtain

$$\sum_{i=1}^{3} \delta_2(0, f_i) \leq \Theta(0, f_1) + \Theta(0, f_2) + \delta_2(0, f_3) \leq 2. \tag{2.2}$$

The proof of the lemma is complete. □

3. Proof of Theorem 1

By the assumption of the theorem we know that either both f and g are two transcendental entire functions or both f and g are two polynomials.

We first assume that both f and g are transcendental entire functions. Then by the assumption of the theorem we have

$$\frac{f^n(f-1)f' - a}{g^n(g-1)g' - a} = e^{h(z)}, \tag{3.1}$$

where $h(z)$ is an entire function. Thus we obtain

$$\frac{f^n(f-1)f'}{a} - \frac{e^{h(z)}g^n(g-1)g'}{a} + e^{h(z)} \equiv 1. \tag{3.2}$$

We claim that either $\frac{e^{h(z)}g^n(g-1)g'}{a}$ or $e^{h(z)}$ is not a transcendental function. Suppose that both $\frac{e^{h(z)}g^n(g-1)g'}{a}$ and $e^{h(z)}$ are transcendental functions. Then we have

$$\begin{aligned} &N_2\left(r, \frac{1}{\frac{f^n(f-1)f'}{a}}\right) \\ &\leq N(r, \frac{1}{f}) + N(r, \frac{1}{f-1}) + N(r, \frac{1}{f'}) + S(r, f) \\ &\leq \frac{2}{n}\left(nN(r, \frac{1}{f}) + N(r, \frac{1}{f-1}) + N(r, \frac{1}{f'})\right) \end{aligned} \tag{3.3}$$

$$+ \ (1-\frac{2}{n})\left(N(r,\frac{1}{f-1})+N(r,\frac{1}{f'})\right)+S(r,f).$$

$$\begin{aligned} & N\left(r,\frac{a}{f^n(f-1)f'}\right) \\ (3.4) \qquad & = N(r,\frac{1}{f^n(f-1)f'})+S(r,f) \\ & = nN(r,\frac{1}{f})+N(r,\frac{1}{f-1})+N(r,\frac{1}{f'})+S(r,f). \end{aligned}$$

By Lemma 1 we have

$$\begin{aligned} (n+1)T(r,f) &= T(r,f^n(f-1))+S(r,f) \\ &\le T(r,\frac{f^n(f-1)f'}{a})+T(r,\frac{1}{f'})+S(r,f) \\ &\le T(r,\frac{f^n(f-1)f'}{a})+T(r,f)+S(r,f), \end{aligned}$$

thus we have

$$(3.5) \qquad nT(r,f) \le T(r,\frac{f^n(f-1)f'}{a})+S(r,f).$$

By Lemma 1 and (3.5) we have

$$\begin{aligned} & N(r,\frac{1}{f-1})+N(r,\frac{1}{f'}) \\ (3.6) \qquad & \le N(r,\frac{1}{f-1})+T(r,\frac{1}{f'})+S(r,f) \\ & \le 2T(r,f)+S(r,f) \le \frac{2}{n}T(r,\frac{f^n(f-1)f'}{a})+S(r,f). \end{aligned}$$

Hence by (3.3)-(3.6) we obtain

$$N_2(r,\frac{1}{\frac{f^n(f-1)f'}{a}}) \le \frac{4n-4}{n^2}T(r,\frac{f^n(f-1)f'}{a})+S(r,\frac{f^n(f-1)f'}{a}).$$

Considering $n \ge 7$ we get

$$\varlimsup_{r\to\infty}\frac{N_2(r,\frac{1}{\frac{f^n(f-1)f'}{a}})}{T(r,\frac{f^n(f-1)f'}{a})} \le \frac{4n-4}{n^2} \le \frac{24}{49}.$$

Thus we have

$$\delta_2(0,\frac{f^n(f-1)f'}{a}) \ge 1-\frac{24}{49}=\frac{25}{49}.$$

Likewise, we have

$$\delta_2(0, -\frac{e^{h(z)}g^n(g-1)g'}{a}) \geq \frac{25}{49}.$$

Obviously,

$$\delta_2(0, e^{h(z)}) = 1.$$

Thus we have

$$\delta_2(0, \frac{f^n(f-1)f'}{a}) + \delta_2(0, -\frac{e^{h(z)}g^n(g-1)g'}{a}) + \delta_2(0, e^{h(z)}) > 2.$$

On the other hand, by Lemma 4 we have

$$\delta_2(0, \frac{f^n(f-1)f'}{a}) + \delta_2(0, -\frac{e^{h(z)}g^n(g-1)g'}{a}) + \delta_2(0, e^{h(z)}) \leq 2.$$

Thus we get a contradiction. Hence we prove that either $\frac{e^{h(z)}g^n(g-1)g'}{a}$ or $e^{h(z)}$ is not a transcendental function. Next we consider two cases.

Case 1. $\frac{e^{h(z)}g^n(g-1)g'}{a}$ is not a transcendental function. In this case we can easily obtain that $-\frac{e^{h(z)}g^n(g-1)g'}{a} \equiv 1$. Hence we get $g^n(g-1)g' \equiv -\frac{a}{e^{h(z)}}$. Thus by (3.2) we deduce that $f^n(f-1)f' \equiv -ae^{h(z)}$,which is a contradiction.

Case 2. $e^{h(z)}$ is not a transcendental function. In this case we can also easily obtain that $e^{h(z)} \equiv 1$. By (3.2) we get $f^n(f-1)f' \equiv g^n(g-1)g'$, that is

$$(\frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1})' \equiv (\frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1})'.$$

Hence we obtain

$$\frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} \equiv \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1} + c, \tag{3.7}$$

where c is a constant.

We claim that $c = 0$. If $c \neq 0$, then by Lemma 1 and $n \geq 7$ we have

$$\begin{aligned}
&\Theta(0, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) + \Theta(c, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) \\
&= \Theta(0, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) + \Theta(0, \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1}) \\
&\geq 2(1 - \frac{2}{n+2}) = \frac{2n}{n+2} \geq \frac{14}{9} > 1,
\end{aligned}$$

which contradicts $\Theta(0, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) + \Theta(c, \frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1}) \leq 1$. Thus we deduce that

$$\frac{f^{n+2}}{n+2} - \frac{f^{n+1}}{n+1} \equiv \frac{g^{n+2}}{n+2} - \frac{g^{n+1}}{n+1}. \tag{3.8}$$

Let $f/g = h$. If $h \not\equiv 1$, then by (3.8) we have

$$g = \frac{(n+2)(1+h+\cdots+h^n)}{(n+1)(1+h+\cdots+h^{n+1})}.$$

Thus we deduce by Picard's theorem that $h(z)$ is a constant. Hence g is a constant, a contradiction. Therefore we deduce that $h(z) \equiv 1$, that is $f(z) \equiv g(z)$.

Next we assume that both f and g are two polynomials. Then by $f^n(f-1)f'$ and $g^n(g-1)g'$ share a CM we have

$$f^n(z)(f(z)-1)f'(z) - a \equiv k[g^n(z)(g(z)-1)g'(z) - a], \tag{3.9}$$

where k is a constant.

Thus by (3.9) and $n \geq 7$ we deduce that there exists z_0 such that $f(z_0) = g(z_0) = 0$. Substituting this into (3.9) we get $k = 1$, that is $f^n(z)(f(z)-1)f'(z) \equiv g^n(z)(g(z)-1)g'(z)$. In the following by using the same argument as do in case 2 we get $f(z) \equiv g(z)$. □

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