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CRITICAL POINTS AND WARPED PRODUCT METRICS

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Abstract. It has been conjectured that, on a compact orientable manifold M, a critical point of the total scalar curvature functional restricted the space of unit volume metrics of constant scalar curvature is Einstein. In this paper we show that if a manifold is a 3-dimensional warped product, then (M, g) cannot be a critical point unless it is isometric to the standard sphere.

1. Introduction

Let M be an n-dimensional compact orientable manifold and let $\mathcal{M}$ denote the space of smooth Riemannian metrics on M. Also let s'_g denote the linearization of the scalar curvature s_g on (M, g) given by

$$s'_g(h) = -\Delta_g trh + \delta^*_g \delta_g h - g(h, ric_g)$$

where Δ_g is the Laplacian, δ is the divergence operator, ric_g is the Ricci curvature tensor of g, and δ^* is the formal adjoint of δ. Then the L^2-adjoint operator s'^*_g of s'_g is given by

$$s'^*_g(f) = -g\Delta_g f + D_g df - fr_g. \tag{1.1}$$

On the other hand, due to the resolution of Yamabe problem, it is known that within each conformal class there exists a metric of constant scalar curvature. Thus we may consider the space $\mathcal{C}$ of metrics of constant scalar curvature on M. Let $\mathcal{C}_1$ denote corresponding space of unit volume metrics. Then the total scalar curvature functional $\mathcal{S} : \mathcal{C}_1 \to \mathbb{R}$

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given by

$$\mathcal{S}(g) = \int_{M^n} s_g dv_g$$

has the following Euler-Lagrange equation

$$z_g = s_g'^*(f) \tag{1.2}$$

for a critical point g, where z_g is the traceless Ricci tensor, f is a function on M^n with vanishing mean value, and $\Delta_g f = -\frac{s_g}{n-1}f$. It has been conjectured that any smooth Riemannian n-dimensional manifold (M, g) satisfying (1.2) for some smooth function f is Einstein, or $f \in Ker\ s_g'^*$ (Conjecture A). If the metric g is Einstein, it turns out that (M, g) is isometric to a standard sphere S^n [7]. For the partial answers to this conjecture, see [3], [6]. One of them states that if g is conformally flat, that (M, g) is isometric to a standard sphere [6].

The motivation of this paper is to find the validity of this conjecture. In order to do so, we prove a rigidity theorem. In [2], Fisher and Marsden conjectured that if $f \in Ker\ s_g'^*$ for some smooth function f, then such a Riemannian manifold (M, g) is isometric to the standard sphere (F-M conjecture, thereafter). F-M conjecture is closely related to our Conjecture A (see, for example, [4]). It turns out that there are counter-examples of F-M conjecture. However, all the known counter-examples of this conjecture are warped products. Therefore, it is a natural question whether a warped product can be a counter-example of our Conjecture A. In this paper we prove that no warped products can satisfy the equation (1.2) in dimension 3 unless it is isometric to a standard sphere. In a forthcoming paper, using the technique developed in this paper, we will show that no warped product metrics can be a solution of (1.2) in higher dimensions.

Now our main result can be stated as follows:

THEOREM 1.1. *Let (M, g) be a 3-dimensional warped product given by $B \times_{\psi^2} F$, $\mathcal{C}_1$ the space of unit volume metrics of constant scalar curvature, and $\mathcal{S}_{|\mathcal{C}_1}$ be the total scalar curvature functional restricted $\mathcal{C}_1$. Then (M, g) cannot be a critical point of $\mathcal{S}_{|\mathcal{C}_1}$ unless it is isometric to the standard sphere.*

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Let (M, g) is a warped product given by $(B, \check{g}) \times_{\psi^2} (F, \hat{g})$ with $g = \check{g} + \psi^2\hat{g}$, $\psi \geq 0$. In our further considerations in the present paper, we assume that g is a solution to (1.2) on a 3-dimensional compact manifold M. We also assume that the scalar curvature s_g is positive, otherwise the solution of $\Delta f = -\frac{s_g}{2} f$ is trivial, which implies that g is Einstein by (1.2). We now consider the following two cases, and prove that our assumption leads to a desired conclusion in both cases.

CASE 1. dim $B = 1$ and dim $F = 2$.

In this case, $g = dt^2 + \psi^2 \hat{g}$, where $\psi = \psi(t)$. In virtue of the formula for the warped product, we have

$$r(X,X) = -\frac{2\psi''}{\psi} \tag{2.1}$$

$$r(X,U) = 0 \tag{2.2}$$

$$r(U,V) = \hat{r}(U,V) + \langle U,V\rangle \left(-\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right) \tag{2.3}$$

for any horizontal vector X and any vertical vectors U, V.

Now we are going to prove that (M,g) is isometric to a 3-sphere. It consists of the following three contentions.

CONTENTION 1. F is isometric to the standard sphere S^2.

Proof. In virtue of (2.1), (2.2) and (2.3), the intrinsic scalar curvature $\hat{s}$ of F is given by

$$\hat{s} = s\psi^2 + 4\psi''\psi + 2\psi'^2 \tag{2.4}$$

where we used the fact that $\sum_{i=1}^{2} \hat{r}(\psi U_i, \psi U_i) = \hat{s}$ since $\hat{g}(U_i,U_i) = \frac{1}{\psi^2}$. Since $\hat{s}$ is independent of t, $\hat{s}$ has to be a constant in virtue of (2.4). It implies that F is of constant curvature of dimension 2. Since M is assumed to be orientable, F should be isometric to a standard sphere of dimension 2. □

From the above proof, we examine a property of f in Case 1. The equation (1.2) can be rewritten as

$$(1+f)z_g = D_g df + \frac{s_g f}{6} g. \tag{2.5}$$

Therefore, putting the following

$$\langle D_X df, U\rangle = \langle D_X (df)^\top, U\rangle = \frac{\psi'}{\psi}\langle (df)^\top, U\rangle$$

into the equation (1.2), we have, for any vertical U,

$$0 = (1+f)z(X,U) = \langle D_X df, U\rangle = \frac{\psi'}{\psi}\langle (df)^\top, U\rangle.$$

Then it may be shown that either $f = f(t)$, or ψ is constant. The proof of this fact will be shown in a forthcoming paper. The following lemma shows that f should be a function of t alone in Case 1. We remark that we do not use this fact to prove our Theorem.

LEMMA 2.1. *ψ cannot be constant.*

Proof. Suppose that ψ is constant, say $\psi = 1$. Then g is a standard product metric. In virtue of (2.4), we have $\hat{s} = s$, which implies that F is of positive constant curvature metric. Therefore F is isometric to S^2, since M is orientable by assumption. Thus $(M, g) = (S^1 \times S^2, g)$, which is conformally flat. It follows from [6] that g should be Einstein if g is conformally flat, contradicting the fact that there is no Einstein metric on $S^1 \times S^2$. Therefore g cannot be a solution of (1.2). □

CONTENTION 2. B cannot be S^1.

Proof. Assume that $B = S^1$. Then $\psi(t)$ cannot vanish at any $t \in S^1$. In virtue of Contention 1, we have $M = S^1 \times_\psi S^2$ with the metric g given by $g = dt^2 + \psi(t)^2 g_0$. This metric is conformally flat, since g_0 is of constant curvature, cf. [5]. It follows from [6] and [7] that, if g is a solution of (1.2) and conformally flat, g should be isometric to a standard sphere S^3, which is a contradiction since there is no ψ satisfying $S^3 = S^1 \times_\psi S^2$. □

CONTENTION 3. M is isometric to S^3.

Proof. In virtue of contention 1 and 2, we have M should be of form $[a, b] \times_\psi S^2$. In order for M to be a complete manifold, we have

$$\psi(a) = \psi(b) = 0. \tag{2.6}$$

Also, in order for M to be smooth, we have (cf. [1], p.269)

$$\psi'(a) = -\psi'(b) = \frac{2}{\hat{s}}. \tag{2.7}$$

Note that the equation (2.4) can be rewritten as the following

$$\psi'' = \frac{1}{4}\Big(\frac{\hat{s}}{\psi} - s\psi - 2\frac{\psi'^2}{\psi}\Big). \tag{2.8}$$

First, for a solution ψ of (2.8), we observe that $\psi''(a) = 0$, since we have

$$\begin{aligned} \psi\psi' &= \frac{1}{4}(\hat{s} - s\psi^2 - 2\psi'^2) \\ \psi'\psi'' + 2\psi\psi''' &= -\frac{s}{2}\psi\psi'. \end{aligned}$$

Note that $\psi_0(t) = k_1 \cos k_2(2t-b-a)$ with $k_1 = \sqrt{\frac{3\hat{s}}{s}}$ and $k_2 = \frac{\pi}{(b-a)}\sqrt{\frac{s}{24}}$ is a solution of (2.8) with the initial condition (2.6) and (2.7). Now we claim that the above ψ_0 is the unique solution of (2.8) with the condition (2.6) and (2.7). Let ψ be another solution of (2.8) satisfying the same initial conditions, and let $F = \frac{\psi}{\psi_0}$. Our claim follows if we show that $F \equiv 1$, i.e., $\psi \equiv \psi_0$. It is easy to see that F is well-defined on $[a, b]$ and $F(a) = 1$. Since $\psi = \psi_0 F$ is a solution of (2.8), we also have

$$\frac{1}{4}\hat{s}(F^2 - 1) + 3\psi_0\psi_0' FF' + \psi_0^2(FF'' + \frac{1}{2}F'^2) = 0,$$

where we used the fact that $\psi_0\psi_0'' = \frac{1}{4}(\hat{s} - s\psi_0^2 - 2\psi_0'^2)$. Thus we have a differential equation

$$F'' = -\frac{1}{\psi_0^2 F}\Big(\frac{1}{4}\hat{s}(F^2 - 1) + 3\psi_0\psi_0' FF' + \frac{1}{2}\psi_0^2 F'^2\Big), \tag{2.9}$$

with the initial conditions

$$F(a) = 1, \quad F'(a) = 0, \tag{2.10}$$

where the second condition follows from the fact that $\psi'' = \psi_0'' F + 2\psi_0' F' + \psi_0 F''$ and $\psi''(a) = 0$ as mentioned above. Let

$$\xi(t, y_1, y_2) = -\frac{1}{\psi_0^2 y_1}\Big(\frac{1}{4}\hat{s}({y_1}^2 - 1) + 3\psi_0\psi_0' y_1 y_2 + \frac{1}{2}\psi_0^2 {y_2}^2\Big),$$

where ξ is obtained just by the substitution F and F' with y_1 and y_2 respectively in (2.9). In order to prove the uniqueness of solution (2.9) with initial conditions (2.10), from the ODE theory, it is enough to show that $\xi(t, y_1, y_2)$ is continuous with respect to t and Lipschitz with respect to y_1 and y_2 in the rectangle $R = [a, b] \times [r, R_1] \times [-R_2, R_2]$, for all $0 < r < 1 < R_1$ and $R_2 > 0$. First, it is easy to show that ξ is continuous for t by letting $\xi(0, y_1(0), y_2(0)) = 0$. Secondly, since ξ is smooth with respect to y_1, y_2 in R, ξ is Lipschitz in the sense that for some $M > 0$, $|\xi(t, y_1(t), y_2(t)) - \xi(t, \bar{y}_1(t), \bar{y}_2(t))| < M(|y_1(t) - \bar{y}_1(t)| + |y_2(t) - \bar{y}_2(t)|)$. So $F \equiv 1$ is the unique solution of (2.9), proving our claim. Therefore, the given warped product metric g should be isometric to a standard sphere. As noting that f is a function of t alone, it is also easy to see that $f(t) = k \sin\sqrt{\frac{s}{6}}t$. □

CASE 2. dim $B = 2$ and dim $F = 1$.

In this case, we will show that there is no solution metrics of (1.2).

Let $\{X, Y, U\}$ be an orthonormal vectors, where X, Y are horizontal vectors and U is a vertical vector. By the formula of warped product manifold, we have

$$
\begin{aligned}
(2.11)\qquad r(X,Y) &= \breve{r}(\breve{X},\breve{Y}) - \frac{1}{\psi}\breve{D}d\psi(\breve{X},\breve{Y}) \\
(2.12)\qquad r(X,U) &= 0 \\
(2.13)\qquad r(U,U) &= -\frac{\breve{\Delta}\psi}{\psi}
\end{aligned}
$$

where $\breve{X}, \breve{Y}$ are the projections of X, Y respectively. Also $\breve{D}$ is the induced connection and $\breve{\Delta}$ is the induced Laplacian on B.

In the virtue of the property of f, we have

$$(2.14)\qquad \Delta f = \langle D_X df, X\rangle + \langle D_Y df, Y\rangle + \langle D_U df, U\rangle = -\frac{s}{2}f.$$

And by definition of $\breve{\Delta} f$, we have

$$(2.15)\qquad \breve{\Delta} f = \Delta f - \langle D_U df, U\rangle = -\frac{s}{2}f - \langle D_U df, U\rangle.$$

Thus by applying X, Y, U in the equation (2.5) and using (2.11), (2.13), we have

$$(2.16)\quad (1+f)(\breve{r}(\breve{X}_i,\breve{X}_i) - \frac{1}{\psi}\breve{D}d\psi(\breve{X}_i,\breve{X}_i) - \frac{s}{3}) = \langle D_{X_i}df, X_i\rangle + \frac{s}{6}f$$

$$(2.17)\qquad (1+f)(-\frac{\breve{\Delta}\psi}{\psi} - \frac{s}{3}) = \langle D_U df, U\rangle + \frac{s}{6}f,$$

where $X_1 = X$ and $X_2 = Y$.

In virtue of the above observations, we get the following proposition, which completes the proof that there is no solution metrics of (1.2).

PROPOSITION 2.2. *$B \times_{\psi^2} F$ cannot be a critical point of $\mathcal{S}_{|C_1}$, when* dim $B = 2$ *and* dim $F = 1$.

Proof. Suppose that $B \times_{\psi^2} F$ is a solution of (1.2). Then it satisfies all the above equations. Substituting (2.14) into (2.17), we have

$$(2.18)\qquad (1+f)(\frac{\breve{\Delta}\psi}{\psi} + \frac{s}{3}) = \breve{\Delta} f + \frac{s}{3}f,$$

or

$$(2.19)\qquad (1+f)\breve{\Delta}\psi + \frac{s}{3}\psi = \psi\breve{\Delta} f.$$

Now taking the integration over B we have

$$\int_B f\check{\Delta}\psi + \frac{s}{3}\int_B \psi = \int_B \psi\check{\Delta} f = \int_B f\check{\Delta}\psi \tag{2.20}$$

which implies that, being s positive,

$$\int_B \psi = 0. \tag{2.21}$$

Since ψ is smooth and $\psi \geq 0$, (2.21) tells that $\psi \equiv 0$. This is a contradiction. □

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