CRITICAL POINTS AND
WARPED PRODUCT METRICS

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ABSTRACT. It has been conjectured that, on a compact orientable manifold \( M \), a critical point of the total scalar curvature functional restricted the space of unit volume metrics of constant scalar curvature is Einstein. In this paper we show that if a manifold is a 3-dimensional warped product, then \((M, g)\) cannot be a critical point unless it is isometric to the standard sphere.

1. Introduction

Let \( M \) be an \( n \)-dimensional compact orientable manifold and let \( \mathcal{M} \) denote the space of smooth Riemannian metrics on \( M \). Also let \( s'_g \) denote the linearization of the scalar curvature \( s_g \) on \((M, g)\) given by

\[
s'_g(h) = -\Delta_g tr h + \delta^* \delta_g h - g(h, ric_g)
\]

where \( \Delta_g \) is the Laplacian, \( \delta \) is the divergence operator, \( ric_g \) is the Ricci curvature tensor of \( g \), and \( \delta^* \) is the formal adjoint of \( \delta \). Then the \( L^2 \)-adjoint operator \( s'^*_g \) of \( s'_g \) is given by

\[
s'^*_g(f) = -g \Delta_g f + D_g df - fr_g.
\]

On the other hand, due to the resolution of the Yamabe problem, it is known that within each conformal class there exists a metric of constant scalar curvature. Thus we may consider the space \( \mathcal{C} \) of metrics of constant scalar curvature on \( M \). Let \( \mathcal{C}_1 \) denote corresponding space of unit volume metrics. Then the total scalar curvature functional \( S : \mathcal{C}_1 \rightarrow \mathbb{R} \)

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given by

\[ S(g) = \int_{M^n} s_g dv_g \]

has the following Euler-Lagrange equation

\[ z_g = s_g^*(f) \tag{1.2} \]

for a critical point \( g \), where \( z_g \) is the traceless Ricci tensor, \( f \) is a function on \( M^n \) with vanishing mean value, and \( \Delta_g f = -\frac{s_g}{n-1} f \). It has been conjectured that any smooth Riemannian \( n \)-dimensional manifold \((M, g)\) satisfying (1.2) for some smooth function \( f \) is Einstein, or \( f \in \text{Ker} \ s_g^* \) (Conjecture A). If the metric \( g \) is Einstein, it turns out that \((M, g)\) is isometric to a standard sphere \( S^n \) [7]. For the partial answers to this conjecture, see [3], [6]. One of them states that if \( g \) is conformally flat, that \((M, g)\) is isometric to a standard sphere [6].

The motivation of this paper is to find the validity of this conjecture. In order to do so, we prove a rigidity theorem. In [2], Fisher and Marsden conjectured that if \( f \in \text{Ker} \ s_g^* \) for some smooth function \( f \), then such a Riemannian manifold \((M, g)\) is isometric to the standard sphere (F-M conjecture, thereafter). F-M conjecture is closely related to our Conjecture A (see, for example, [4]). It turns out that there are counter-examples of F-M conjecture. However, all the known counter-examples of this conjecture are warped products. Therefore, it is a natural question whether a warped product can be a counter-example of our Conjecture A. In this paper we prove that no warped products can satisfy the equation (1.2) in dimension 3 unless it is isometric to a standard sphere. In a forthcoming paper, using the technique developed in this paper, we will show that no warped product metrics can be a solution of (1.2) in higher dimensions.

Now our main result can be stated as follows:

**Theorem 1.1.** Let \((M, g)\) be a 3-dimensional warped product given by \( B \times_{\psi^2} F \), \( C_1 \) the space of unit volume metrics of constant scalar curvature, and \( S_{\mid C_1} \) be the total scalar curvature functional restricted \( C_1 \). Then \((M, g)\) cannot be a critical point of \( S_{\mid C_1} \) unless it is isometric to the standard sphere.

**2. Proof of Theorem 1**

This section is devoted to the proof of Theorem 1. Let \((M, g)\) is a warped product given by \((B, \tilde{g}) \times_{\psi^2} (F, \hat{g})\) with \( g = \tilde{g} + \psi^2 \hat{g}, \; \psi \geq 0 \). In our further considerations in the present paper, we assume that \( g \) is
a solution to (1.2) on a 3-dimensional compact manifold $M$. We also assume that the scalar curvature $s_g$ is positive, otherwise the solution of $\Delta f = -\frac{s_g}{2} f$ is trivial, which implies that $g$ is Einstein by (1.2). We now consider the following two cases, and prove that our assumption leads to a desired conclusion in both cases.

**Case 1.** $\dim B = 1$ and $\dim F = 2$.

In this case, $g = dt^2 + \psi^2 \hat{g}$, where $\psi = \psi(t)$. In virtue of the formula for the warped product, we have

\begin{align*}
(2.1) \quad r(X, X) &= -\frac{2\psi''}{\psi} \\
(2.2) \quad r(X, U) &= 0 \\
(2.3) \quad r(U, V) &= \hat{r}(U, V) + \langle U, V \rangle \left(-\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2}\right)
\end{align*}

for any horizontal vector $X$ and any vertical vectors $U, V$.

Now we are going to prove that $(M, g)$ is isometric to a 3-sphere. It consists of the following three contentions.

**Contention 1.** $F$ is isometric to the standard sphere $S^2$.

**Proof.** In virtue of (2.1), (2.2) and (2.3), the intrinsic scalar curvature $\hat{s}$ of $F$ is given by

\begin{equation}
(2.4) \quad \hat{s} = s \psi^2 + 4 \psi'' \psi + 2 \psi'^2
\end{equation}

where we used the fact that $\sum_{i=1}^2 \hat{r}(\psi U_i, \psi U_i) = \hat{s}$ since $\hat{g}(U_i, U_i) = \frac{1}{\psi^2}$. Since $\hat{s}$ is independent of $t$, $\hat{s}$ has to be a constant in virtue of (2.4). It implies that $F$ is of constant curvature of dimension 2. Since $M$ is assumed to be orientable, $F$ should be isometric to a standard sphere of dimension 2. \hfill \Box

From the above proof, we examine a property of $f$ in Case 1. The equation (1.2) can be rewritten as

\begin{equation}
(2.5) \quad (1 + f)z_g = D_g df + \frac{s_g f}{6} g.
\end{equation}

Therefore, putting the following

$$
\langle D_X df, U \rangle = \langle D_X (df)^\top, U \rangle = \frac{\psi'}{\psi} \langle (df)^\top, U \rangle
$$

into the equation (1.2), we have, for any vertical $U$,

$$
0 = (1 + f)z(X, U) = \langle D_X df, U \rangle = \frac{\psi'}{\psi} \langle (df)^\top, U \rangle.
$$
Then it may be shown that either \( f = f(t) \), or \( \psi \) is constant. The proof of this fact will be shown in a forthcoming paper. The following lemma shows that \( f \) should be a function of \( t \) alone in Case 1. We remark that we do not use this fact to prove our Theorem.

**Lemma 2.1.** \( \psi \) cannot be constant.

*Proof.* Suppose that \( \psi \) is constant, say \( \psi = 1 \). Then \( g \) is a standard product metric. In virtue of (2.4), we have \( \dot{s} = s \), which implies that \( F \) is of positive constant curvature metric. Therefore \( F \) is isometric to \( S^2 \), since \( M \) is orientable by assumption. Thus \( (M, g) = (S^1 \times S^2, g) \), which is conformally flat. It follows from [6] that \( g \) should be Einstein if \( g \) is conformally flat, contradicting the fact that there is no Einstein metric on \( S^1 \times S^2 \). Therefore \( g \) cannot be a solution of (1.2). \( \square \)

**Contention 2.** \( B \) cannot be \( S^1 \).

*Proof.* Assume that \( B = S^1 \). Then \( \psi(t) \) cannot vanish at any \( t \in S^1 \). In virtue of Conjecture 1, we have \( M = S^1 \times \psi S^2 \) with the metric \( g \) given by \( g = dt^2 + \psi(t)^2 g_0 \). This metric is conformally flat, since \( g_0 \) is of constant curvature, cf. [5]. It follows from [6] and [7] that, if \( g \) is a solution of (1.2) and conformally flat, \( g \) should be isometric to a standard sphere \( S^3 \), which is a contradiction since there is no \( \psi \) satisfying \( S^3 = S^1 \times \psi S^2 \). \( \square \)

**Contention 3.** \( M \) is isometric to \( S^3 \).

*Proof.* In virtue of contention 1 and 2, we have \( M \) should be of form \([a, b] \times \psi S^2 \). In order for \( M \) to be a complete manifold, we have
\[
(2.6) \quad \psi(a) = \psi(b) = 0.
\]
Also, in order for \( M \) to be smooth, we have (cf. [1], p.269)
\[
(2.7) \quad \psi'(a) = -\psi'(b) = \frac{2}{\dot{s}}.
\]
Note that the equation (2.4) can be rewritten as the following
\[
(2.8) \quad \psi'' = \frac{1}{4} \left( \frac{\dot{s}}{\psi} - s \psi' - 2\frac{\psi'^2}{\psi} \right).
\]
First, for a solution \( \psi \) of (2.8), we observe that \( \psi''(a) = 0 \), since we have
\[
\psi \psi' = \frac{1}{4} (\dot{s} - s \psi'^2 - 2\psi'^2)
\]
\[
\psi' \psi'' + 2 \psi \psi''' = -\frac{s}{2} \psi \psi'.
\]
Note that \( \psi_0(t) = k_1 \cos k_2(2t - b - a) \) with \( k_1 = \sqrt{\frac{3\delta}{s}} \) and \( k_2 = \frac{\pi}{(b-a) \sqrt{24}} \) is a solution of (2.8) with the initial condition (2.6) and (2.7). Now we claim that the above \( \psi_0 \) is the unique solution of (2.8) with the condition (2.6) and (2.7). Let \( \psi \) be another solution of (2.8) satisfying the same initial conditions, and let \( F = \frac{\psi}{\psi_0} \). Our claim follows if we show that \( F \equiv 1 \), i.e., \( \psi \equiv \psi_0 \). It is easy to see that \( F \) is well-defined on \([a, b]\) and \( F(a) = 1 \). Since \( \psi = \psi_0 F \) is a solution of (2.8), we also have

\[
\frac{1}{4} \delta(F^2 - 1) + 3\psi_0 \psi' F F' + \psi_0^2 (F F'' + \frac{1}{2} F'^2) = 0,
\]

where we used the fact that \( \psi_0 \psi'' = \frac{1}{4} (\delta - s \psi_0^2 - 2 \psi_0'^2) \). Thus we have a differential equation

\[
(2.9) \quad F'' = -\frac{1}{\psi_0^2} \left( \frac{1}{4} \delta(F^2 - 1) + 3\psi_0 \psi' F F' + \frac{1}{2} \psi_0^2 F'^2 \right),
\]

with the initial conditions

\[
(2.10) \quad F(a) = 1, \quad F'(a) = 0,
\]

where the second condition follows from the fact that \( \psi'' = \psi_0'^2 F + 2 \psi_0' F' + \psi_0 F'' \) and \( \psi''(a) = 0 \) as mentioned above. Let

\[
\xi(t, y_1, y_2) = -\frac{1}{\psi_0^2 y_1} \left( \frac{1}{4} \delta(y_1^2 - 1) + 3\psi_0 \psi' y_1 y_2 + \frac{1}{2} \psi_0^2 y_2^2 \right),
\]

where \( \xi \) is obtained just by the substitution \( F \) and \( F' \) with \( y_1 \) and \( y_2 \) respectively in (2.9). In order to prove the uniqueness of solution (2.9) with initial conditions (2.10), from the ODE theory, it is enough to show that \( \xi(t, y_1, y_2) \) is continuous with respect to \( t \) and Lipschitz with respect to \( y_1 \) and \( y_2 \) in the rectangle \( R = [a, b] \times [r, R_1] \times [-R_2, R_2] \), for all \( 0 < r < 1 < R_1 \) and \( R_2 > 0 \). First, it is easy to show that \( \xi \) is continuous for \( t \) by letting \( \xi(0, y_1(0), y_2(0)) = 0 \). Secondly, since \( \xi \) is smooth with respect to \( y_1, y_2 \) in \( R, \xi \) is Lipschitz in the sense that for some \( M > 0 \),

\[
|\xi(t, y_1(t), y_2(t)) - \xi(t, \tilde{y}_1(t), \tilde{y}_2(t))| < M( |y_1(t) - \tilde{y}_1(t)| + |y_2(t) - \tilde{y}_2(t)| )
\]

for all \( (t, y_1, y_2) \) and \( (t, \tilde{y}_1, \tilde{y}_2) \) in \( R \). So \( F \equiv 1 \) is the unique solution of (2.9), proving our claim. Therefore, the given warped product metric \( g \) should be isometric to a standard sphere. As noting that \( f \) is a function of \( t \) alone, it is also easy to see that \( f(t) = k \sin \sqrt{\frac{\delta}{6}} t \).

\( \square \)

**Case 2.** \( \dim B = 2 \) and \( \dim F = 1 \).

In this case, we will show that there is no solution metrics of (1.2).
Let \( \{X, Y, U\} \) be an orthonormal vectors, where \( X, Y \) are horizontal vectors and \( U \) is a vertical vector. By the formula of warped product manifold, we have

\[
(2.11) \quad r(X,Y) = \bar{r}(\bar{X}, \bar{Y}) - \frac{1}{\psi} \bar{D}d\psi(\bar{X}, \bar{Y}) \\
(2.12) \quad r(X,U) = 0 \\
(2.13) \quad r(U,U) = -\frac{\bar{\Delta}\psi}{\psi}
\]

where \( \bar{X}, \bar{Y} \) are the projections of \( X, Y \) respectively. Also \( \bar{D} \) is the induced connection and \( \bar{\Delta} \) is the induced Laplacian on \( B \).

In the virtue of the property of \( f \), we have

\[
(2.14) \quad \Delta f = \langle D_X df, X \rangle + \langle D_Y df, Y \rangle + \langle D_U df, U \rangle = -\frac{s}{2} f.
\]

And by definition of \( \bar{\Delta} f \), we have

\[
(2.15) \quad \bar{\Delta} f = \Delta f - \langle D_U df, U \rangle = -\frac{s}{2} f - \langle D_U df, U \rangle.
\]

Thus by applying \( X, Y, U \) in the equation (2.5) and using (2.11), (2.13), we have

\[
(2.16) \quad (1 + f)(\bar{r}(\bar{X}_i, \bar{X}_i) - \frac{1}{\psi} \bar{D}d\psi(\bar{X}_i, \bar{X}_i) - \frac{s}{3}) = \langle D_X df, X_i \rangle + \frac{s}{6} f \\
(2.17) \quad (1 + f)(-\frac{\bar{\Delta}\psi}{\psi} - \frac{s}{3}) = \langle D_U df, U \rangle + \frac{s}{6} f,
\]

where \( X_1 = X \) and \( X_2 = Y \).

In virtue of the above observations, we get the following proposition, which completes the proof that there is no solution metrics of (1.2).

**Proposition 2.2.** \( B \times_{\psi^2} F \) cannot be a critical point of \( S_{\mid C_1} \), when \( \dim B = 2 \) and \( \dim F = 1 \).

**Proof.** Suppose that \( B \times_{\psi^2} F \) is a solution of (1.2). Then it satisfies all the above equations. Substituting (2.14) into (2.17), we have

\[
(2.18) \quad (1 + f)\left(\frac{\bar{\Delta}\psi}{\psi} + \frac{s}{3}\right) = \bar{\Delta} f + \frac{s}{3} f,
\]

or

\[
(2.19) \quad (1 + f)\bar{\Delta} \psi + \frac{s}{3} \psi = \psi \bar{\Delta} f.
\]
Now taking the integration over $B$ we have
\begin{equation}
(2.20) \quad \int_B f \Delta \psi + \frac{s}{3} \int_B \psi \Delta f = \int_B f \Delta \psi
\end{equation}
which implies that, being $s$ positive,
\begin{equation}
(2.21) \quad \int_B \psi = 0.
\end{equation}

Since $\psi$ is smooth and $\psi \geq 0$, (2.21) tells that $\psi \equiv 0$. This is a contradiction. \hfill \square

References


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