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ON THE SYNGE'S THEOREM FOR COMPLEX FINSLER MANIFOLDS

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ABSTRACT. In [13], we developed a theory of complex Finsler manifolds to investigate the global geometry of complex Finsler manifolds. There we proved a version of Bonnet-Myers' theorem for complex Finsler manifolds with a certain condition on the Finsler metric which is a generalization of the Kähler condition for the Hermitian metric.

In this paper, we show that if the holomorphic sectional curvature of M is $\geq c^2 > 0$, then M is simply connected. This is a generalization of the Synge's theorem in the Riemannian geometry and the Tsukamoto's theorem for Kähler manifolds.

The main point of the proof lies in how we can circumvent the convex neighborhood theorem in the Riemannian geometry. A second variation formula of arc length for complex Finsler manifolds is also derived.

1. Introduction

One of the main themes in global differential geometry is probably that the curvature controls the topology of the underlying manifold. In this regard, we have Bonnet-Myers' theorem, Synge's theorem and Cartan-Hadamard's theorem to name a few. In particular, Synge's theorem in the Riemannian geometry gives simply connectedness for the manifold with curvature bounded below by some positive constant. Later in [12], Y. Tsukamoto proved a version for Kähler manifolds under a weaker assumption. He only assumed that the holomorphic sectional curvature is bounded below by some positive constant. Here, we generalize this theorem to the complex Finsler manifolds. The main difficulty

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is in that we can not use the convex neighborhood theorem in the Riemannian geometry. And so the proof of Proposition 3.3 on the existence of a certain closed geodesic is rather long but is of importance in itself. The proof of the main theorem also depends on our previous work [13].

In [6], S. Kobayashi paved a road to a modern approach to the study of holomorphic vector bundles with complex Finsler structures. His main observation lies in the following diagram:

$$\begin{array}{ccc} p^*E & \xrightarrow{\tilde{p}} & E \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}E & \xrightarrow[p]{} & M \end{array}$$

He considered a pull-back bundle of the holomorphic vector bundle $\pi : E \to M$ by the projection $p : \mathbb{P}E \to M$. Then the Finsler structure on E produces a Hermitian structure on the induced bundle $\tilde{\pi} : p^*E \to \mathbb{P}E$. And hence we can apply well-known techniques in the Hermitian geometry. In this paper, we essentially follow his idea for the case $E = TM$, the tangent bundle.

In §2, we set up the notations. And we recall some of definitions in Finsler geometry. More thorough treatments on the Finsler geometry can be found in the books [1], [2], [9]. In §3, we briefly go over a general theory of geodesics in metric spaces. A complex Finsler metric gives rise to an inner distance function on the underlying manifold. Generalized Hope-Rinow theorem(Lemma 3.1) applied to the complex Finsler manifold with this inner distance produces Proposition 3.2. The existence of minimizing geodesics joining any two points is the starting point of the proof of Proposition 3.3. For the proof of Proposition 3.4, we cook up a nice C^1-variation and we derive a second variation formula of arc length for Finsler metrics.

2. Preliminaries

Let M be an n-dimensional complex manifold with a local coordinate system (z^i), $i = 1, 2, \cdots, n$, where $z^i = x^i + \sqrt{-1}y^i$ so that (x^i, y^i), $i = 1, 2, \cdots, n$, is a local coordinate system of the underlying real manifold. We will use (z^i, ζ^i), $i = 1, 2, \cdots, n$ as a local coordinate system for the holomorphic tangent bundle $T^{1,0}M$. Let J be the natural complex structure of M. Hereafter we identify T_pM with $T_p^{1,0}M$ by the identification $\phi : T_pM \to T_p^{1,0}M$ defined by $\phi(v) = \frac{1}{2}(v - \sqrt{-1}Jv)$.

To a complex Finsler metric F on M, we can associate a real Finsler metric F^o on M via the identification map ϕ above. But the real Finsler metric F^o is not necessarily strongly convex, i.e., real Hessian of $(F^o)^2$ is not positive definite in general. As in [1], we assume that F^o is strongly convex. This Finsler structure is enough to define a length of a curve and in turn a notion of geodesics. A curve $c : [0, l] \to M$ in M is a geodesic for a complex Finsler metric F if it is a critical point of L_F, where

$$L_F(c) = \int_0^l F(c(t), \dot{c}(t))\, dt. \tag{2.1}$$

Since F^o is strongly convex, well-known theorems on the geodesics in real Finsler geometry are readily applicable. On this regard, see [1] and [13].

On the Hermitian vector bundle $\tilde{\pi} : p^*T^{1,0}M \to \mathbb{P}T^{1,0}M$, there exists a unique connection D of type $(1,0)$ which is compatible with the Hermitian structure. For the existence of such a connection, which is called the Chern connection, see [4, p. 73]. Then with respect to a local frame $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$, the curvature form $\Omega_i^{\ j}$ of the Chern connection D is defined by

$$\Omega_i^{\ j} = R_{i\ k\bar{l}}^{\ j} dz^k \wedge d\bar{z}^l + \text{ mixed terms in } dz^k \text{ and } d\zeta^l,$$

where

$$R_{i\ k\bar{l}}^{\ j} = -G^{j\bar{h}} \frac{\partial^2 G_{i\bar{h}}}{\partial z^k \partial \bar{z}^l} + G^{j\bar{h}} G^{a\bar{b}} \frac{\partial G_{a\bar{h}}}{\partial \bar{z}^l} \frac{\partial G_{i\bar{b}}}{\partial z^k}.$$

By lowering index, we have

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 G_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{a,b=1}^{n} G^{a\bar{b}} \frac{\partial G_{a\bar{j}}}{\partial \bar{z}^l} \frac{\partial G_{i\bar{b}}}{\partial z^k}.$$

Then for a nonzero tangent vector ζ at $z \in M$, the holomorphic sectional curvature H of ζ at $z \in M$ of a complex Finsler manifold (M, F) is defined by

$$H(z, \zeta) = \frac{1}{F^4(z, \zeta)} \sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}} \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^l.$$

As a generalization of the Kähler condition in the complex differential geometry, a complex Finsler metric F is called pseudo-Kähler if $\frac{\partial G_{i\bar{j}}}{\partial z^k}(z, \zeta) = \frac{\partial G_{k\bar{j}}}{\partial z^i}(z, \zeta)$ for all $(z, \zeta) \in \mathbb{P}T^{1,0}M$.

The main goal of this paper is to prove the following:

THEOREM 2.1. *Let (M, F) be a complete pseudo-Kähler Finsler manifold with strongly convex F^o. If the holomorphic sectional curvature of M is $\geq c^2 > 0$, then M is simply connected.*

The Riemannian version of this theorem is due to Synge. See, eg., [3, pp.98–99]. Its Kähler version is proved by Tsukamoto [12].

3. Proof of the main theorem

We begin with a general theory of metric spaces with inner metrics. Let (X, d) be a metric space. The distance function d on X defines a length function $L(c)$ of a curve $c : [0, l] \to X$ in X by

$$L(c) = \sup \sum_{i=1}^{k} d(c(t_{i-1}), c(t_i)), \tag{3.1}$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_k = l$ of the interval $[0, l]$. To this length function L, we associate a new distance function $d^i : X \times X \to \mathbb{R}$ by

$$d^i(p, q) = \inf\{L(c) \mid c : \text{ curve joining } p \text{ and } q\}, \text{ for } p, q \in X. \tag{3.2}$$

If the metric d satisfies $d^i = d$, we call this metric d an inner metric and the metric space (X, d) a length space.

Now go back to the Finsler manifold (M, F). The length function $L_F(c)$ of (2.1) gives rise to a distance function $d_F : M \times M \to \mathbb{R}$ defined by

$$d_F(p, q) = \inf\{L_F(c) \mid c : \text{ curve joining } p \text{ and } q\}, \text{ for } p, q \in M.$$

Indeed, this d_F is a metric and hence (M, d_F) is a metric space. Now this distance function d_F defines another length function $\tilde{L}(c)$ of a curve c in M as in (3.1). Then to this length function $\tilde{L}$, we associate a new distance function ${d_F}^i : M \times M \to \mathbb{R}$ as in (3.2). A distance function defined via (2.1) is automatically an inner metric, i.e., ${d_F}^i = d_F$. And the metric space (M, d_F) is a length space.

For a length space, we have a following generalization of the theorem of Hopf-Rinow.

LEMMA 3.1. *Let (M, d) be a locally compact complete length space. Then any two points can be joined by a minimizing geodesic.*

For its proof, we refer the reader to the books [5], [7] and [10]. This lemma applied to a complex Finsler manifold (M, F) is tantamount to the following.

PROPOSITION 3.2. *Let (M, F) be a complex Finsler manifold with strongly convex F^o. Then any two points can be joined by a minimizing geodesic.*

With the existence of a minimizing geodesic joining any two points, we are ready to establish the following propositions which are essential to the proof of Theorem 2.1.

PROPOSITION 3.3. *Let (M, F) be a compact complex Finsler manifold. Every nontrivial free homotopy class of closed curves of M contains a closed geodesic which is shortest in that class.*

Proof. Let $\widetilde{M}$ be the universal covering space of M and $\pi : \widetilde{M} \to M$ be the projection. And let $\mathcal{D}$ be the set of all covering transformations. The complex Finsler metric on M can be lifted to a complex Finsler metric on $\widetilde{M}$ by π. Let Γ be a nontrivial free homotopy class of closed curves in M. To Γ associate a conjugacy class in $\mathcal{D}$ in the following way. Let $c : [0,1] \to M$ be a closed curve in Γ, let $\tilde{c} : [0,1] \to \widetilde{M}$ be its lifting, starting at some $q \in \pi^{-1}(c(0))$. Then there exists a unique $\delta \in \mathcal{D}$ such that

$$\tilde{c}(1) = \delta(q) = \delta(\tilde{c}(0)) \,. \tag{3.3}$$

Note that $\delta \neq 0$ because c is not homotopic to a constant.

Now consider the conjugacy class $\{\phi \circ \delta \circ \phi^{-1} \mid \phi \in \mathcal{D}\}$ of $\delta \in \mathcal{D}$. This class does not depend on the choices of $q \in \pi^{-1}(c(0))$ and $c \in \Gamma$. In fact, for $q' \in \pi^{-1}(c(0))$, there exists $\phi \in \mathcal{D}$ such that $q' = \phi(q)$ and the lifting $\tilde{\tilde{c}}$ of c starting $q' = \phi(q)$ is $\phi \circ \tilde{c}$. So

$$\tilde{\tilde{c}}(1) = \phi(\tilde{c}(1)) = \phi(\delta(q)) = \phi \circ \delta \circ \phi^{-1}(\phi(q)) = \phi \circ \delta \circ \phi^{-1}(\tilde{\tilde{c}}(0)).$$

Let $c_1 : [0,1] \to M$ be another closed curve in Γ. Then there exists a homotopy $H : [0,1] \times [0,1] \to M$ satisfying $H(t,0) = c(t)$, $H(t,1) = c_1(t)$ and $H(0,s) = H(1,s)$.

Let $\widetilde{H} : [0,1] \times [0,1] \to \widetilde{M}$ be a lifting of H. Then $\widetilde{H}(1,s) = \delta(s)(\widetilde{H}(0,s))$ for some $\delta(s)$ and all $\delta(s)$ must be the same δ by continuity of $s \mapsto \delta(s)$. Thus both c and c_1 determine the same δ and hence the same conjugacy class.

Next we define $h_\delta : \widetilde{M} \to \mathbb{R}$ by $h_\delta(q) = \inf\{d(q, \phi \circ \delta \circ \phi^{-1}(q)) : \phi \in \mathcal{D}\}$. Since M is compact, we have a compact set $K \subseteq \widetilde{M}$ with $\pi(K) = M$ and consequently $\mathcal{D}(K) = \widetilde{M}$. Thus from the fact that h_δ is invariant under the action of $\mathcal{D}$, the minimum of h_δ on K is the minimum on all of $\widetilde{M}$. Say that h_δ takes its minimum at $q_o \in \widetilde{M}$ and that $h_\delta(q_o) =$

$d(q_o, \phi_o \circ \delta \circ \phi_o^{-1}(q_o))$. Note that such ϕ_o exists since $\mathcal{D}$ acts discretely. Let γ be a minimizing geodesic in $\widetilde{M}$ from q_o to $\phi_o \circ \delta \circ \phi_o^{-1}(q_o)$. The existence of such a minimizing geodesic follows from the completeness of M and $\widetilde{M}$ by Proposition 3.2.

We will show that the closed curve $\pi \circ \gamma$ is in Γ. Consider the lifting $\tilde{\tilde{c}}$ of c starting at $\phi_o(q)$, i.e., $\tilde{\tilde{c}}(t) = \phi_o \circ \tilde{c}(t)$, where $\tilde{c}$ is the lift of c with $\tilde{c}(0) = q$. Then by (3.3),

$$\tilde{\tilde{c}}(1) = \phi_o \circ \tilde{c}(1) = \phi_o \circ \delta \circ \tilde{c}(0) = \phi_o \circ \delta \circ \phi_o^{-1}(\tilde{\tilde{c}}(0)).$$

Let $\alpha : [0,1] \to \widetilde{M}$ be a curve from $\gamma(0)$ to $\tilde{\tilde{c}}(0)$. Then $\phi_o \circ \delta \circ \phi_o^{-1} \circ \alpha$ is a curve from $\gamma(1)$ to $\tilde{\tilde{c}}(1)$. So we can define a continuous map $H : \partial([0,1] \times [0,1]) \to \widetilde{M}$ such that

$$H(t,0) = \gamma(t), \quad H(t,1) = \tilde{\tilde{c}}(t),$$
$$H(0,s) = \alpha(s), \quad H(1,s) = \phi_o \circ \delta \circ \phi_o^{-1}(\alpha(s)).$$

Since $\widetilde{M}$ is simply connected, we can extend this to a map $H : [0,1] \times [0,1] \to \widetilde{M}$. Then $\pi \circ H : [0,1] \times [0,1] \to M$ is a homotopy of $\pi \circ \gamma$ and c. So $\pi \circ \gamma$ is in Γ.

We know that $\pi \circ \gamma$ is smooth possibly except for $\pi \circ \gamma(0) = \pi \circ \gamma(1) = \pi(q_o)$ since γ is a geodesic. To show that $\pi \circ \gamma$ is smooth at $\pi(q_o)$, let $\beta : [0,1] \to \widetilde{M}$ be such that

$$\beta(t) = \begin{cases} \gamma(t + 1/2), & \text{if } t \leq 1/2; \\ \phi_o \circ \delta \circ \phi_o^{-1}(\gamma(t - 1/2)), & \text{if } t \geq 1/2. \end{cases}$$

Then

$$\begin{aligned} L_{\tilde{F}}(\beta) &= L_{\tilde{F}}(\beta|_{[0,1/2]}) + L_{\tilde{F}}(\beta|_{[1/2,1]}) \\ &= L_{\tilde{F}}(\gamma|_{[1/2,1]}) + L_{\tilde{F}}(\phi_o \circ \delta \circ \phi_o^{-1} \circ \gamma|_{[0,1/2]}) \\ &= L_{\tilde{F}}(\gamma|_{[1/2,1]}) + L_{\tilde{F}}(\gamma|_{[0,1/2]}) \\ &= L_{\tilde{F}}(\gamma), \end{aligned}$$

since the covering transformation $\phi_o \circ \delta \circ \phi_o^{-1}$ preserves the length of a curve.

On the other hand,

$$\begin{aligned} h_\delta(\beta(0)) \geq h_\delta(q_o) = L_{\tilde{F}}(\gamma) &= L_{\tilde{F}}(\beta) \\ &\geq d(\beta(0), \beta(1)) = d(\beta(0), \phi_o \circ \delta \circ \phi_o^{-1}(\beta(0))) \\ &\geq h_\delta(\beta(0)). \end{aligned}$$

Thus $L_{\tilde{F}}(\beta) = d(\beta(0), \beta(1))$ and hence β is a minimizing geodesic. In particular, β is smooth at $t = 1/2$ and so is $\pi \circ \beta$ and so $\pi \circ \gamma$ is smooth at $\pi(q_o)$.

Finally, we will prove that $\pi \circ \gamma$ is of minimum length in Γ. Let c_1 be any curve in Γ and $\tilde{c}_1$ be any lifting, starting at some point q_1, then $\tilde{c}_1(1) = \psi \circ \delta \circ \psi^{-1}(q_1)$ for some $\psi \in \mathcal{D}$. Consequently,

$$\begin{aligned} L_F(c_1) = L_{\tilde{F}}(\tilde{c}_1) &\geq d(q_1, \psi \circ \delta \circ \psi^{-1}(q_1)) \\ &\geq h_\delta(q_1) \\ &\geq h_\delta(q_o) = L_{\tilde{F}}(\gamma) = L_F(\pi \circ \gamma). \end{aligned}$$

Since c_1 is arbitrary in Γ, $\pi \circ \gamma$ is of minimum length in Γ. □

Riemannian version of this theorem is due to É. Cartan. See, e.g., [11, pp.355–358]. Actually, the above proof is similar to the one in Riemannian geometry except for the smoothness of $\pi \circ \gamma$. Since we do not have convex neighborhood theorem in our case, we needed other trick.

PROPOSITION 3.4. *If a pseudo-Kähler Finsler manifold (M, F) has positive holomorphic sectional curvature, then there does not exist a closed geodesic of minimum type.*

Proof. Let $c_o : [0, l] \to M$ be a closed geodesic in M. Then as in Proposition 3.1 of [13], we have a C^1-variation $c : [0, l] \times (-\epsilon, \epsilon) \to M$ of c_o such that

(1) $\{\partial c/\partial t, \partial c/\partial s\}$ are linearly independent for all $t \in [0, l]$, $s \in (-\epsilon, \epsilon)$,
(2) $\partial c/\partial s(t, s)|_{s=0} = JT(t)$.

Note that the restriction of c to $[0, l'] \times (-\epsilon, \epsilon)$ is one-to-one if $l' < l$ and that $\partial c/\partial s(0, s) = \partial c/\partial s(l, s)$. So we can choose a continuous nowhere vanishing vector field X on some open set U containing $\{c(t, s) : t \in [0, l], s \in (-\epsilon, \epsilon)\}$ such that X agrees with $\partial c/\partial t(t, s)$ i.e.,

$$X(z) = \partial c/\partial t(t, s) \quad \text{if} \quad z = c(t, s).$$

On U, we have a Kähler metric defined by $g_{i\bar{j}}(z) = G_{i\bar{j}}(z, X(z))$.

We will use the induced Kähler metric on U to find the second variation formula of arc length. Let $\langle\ ,\ \rangle$ denote the real part of the induced Kähler metric and ∇ its Levi-Civita connection. We observe that, by the definition of X,

$$\begin{aligned} L_F(c_s) &= \int_0^l F(c_s(t), \dot{c}_s(t))\, dt \\ &= \int_0^l F(c_s(t), X(c_s(t)))\, dt \\ &= \int_0^l \langle T, T \rangle\, dt \\ &= L_U(c_s) \end{aligned}$$

and by the second variation formula of length in Riemannian metric $\langle\ ,\ \rangle$ of U,

$$\begin{aligned} &\frac{1}{2}\frac{d^2}{ds^2}\Big|_{s=0} L_F(c_s) \\ &= \langle T, \nabla_S S\rangle\Big|_{t=0}^{l} \\ &\quad - \int_0^l \langle \nabla_T S, \nabla_T S\rangle + \langle S, \mathcal{R}(T,S)T\rangle + (T\langle T, S\rangle)^2\Big|_{s=0} dt\,. \end{aligned}$$

Then the boundary term is zero since each c_s is closed.

And $\nabla_T S = \nabla_T JT = J(\nabla_T T) = 0$ at $s = 0$ because c_o is a geodesic in U with respect to $\langle\ ,\ \rangle$ and g is a Kähler metric on U. Furthermore, $\langle T, S\rangle = \langle T, JT\rangle = 0$ at $s = 0$.

Hence

$$\begin{aligned} \frac{1}{2}\frac{d^2}{ds^2}\Big|_{s=0} L_F(c_s) &= -\int_0^l \langle S, \mathcal{R}(T,S)T\rangle\Big|_{s=0} dt \\ &= -\int_0^l \langle JT, \mathcal{R}(T,JT)T\rangle\, dt \\ &< 0. \end{aligned}$$

Therefore, c_o cannot be of minimum type. □

Proof of Theorem 2.1. First note that M is compact by Theorem 2.1 in [13].

Suppose that M is not simply connected. Then there exists a nontrivial free homotopy class Γ of closed curves of M . By Proposition 3.3, Γ contains a closed geodesic which is shortest in that class and hence is of minimum type.

On the other hand, by Proposition 3.4, M does not contain a closed geodesic of minimum type. This is a contradiction.

Therefore M is simply connected. □

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