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# $FSI$-IDEALS AND $FSC$-IDEALS OF BCI-ALGEBRAS

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ABSTRACT. The notions of $FSI$-ideals and $FSC$-ideals in BCI-algebras are introduced. The characterization properties of $FSI$-ideals and $FSC$-ideals are obtained. We investigate the relations between $FSI$-ideals (resp. $FSC$-ideals) and other fuzzy ideals, between $FSI$-ideals (resp. $FSC$-ideals) and BCI-algebras, and show that a fuzzy subset of a BCI-algebra is an $FSI$-ideal if and only if it is both an $FSC$-ideal and a fuzzy BCI-positive implicative ideal.

## 1. Introduction

BCK-algebras and BCI-algebras are two classes of logical algebras, which were initiated by K. Iseki [3, 4]. The notion of fuzzy sets, invented by L. A. Zadeh [20], has been applied to many field. In 1991, O. G. Xi [19] applied it to BCK-algebras. Since then fuzzy BCI/BCK-algebras have been extensively investigated by several researchers. For BCK-algebras, Y. B. Jun et al. [6, 9] introduced the notions of fuzzy positive implicative ideals and fuzzy commutative ideals, J. Meng et al. [14] introduced the notion of fuzzy implicative ideals. For BCI-algebras, Y. B. Jun et al. [5, 7, 8] introduced the notions of fuzzy $q$-ideals (i.e., fuzzy quasi-associative ideals), fuzzy $p$-ideals and fuzzy BCI-commutative ideals, the first author et al. [11, 12] introduced the notions of fuzzy BCI-positive implicative ideals, fuzzy BCI-implicative ideals and fuzzy $a$-ideals. The aim of this paper is to introduce the notions of $FSI$-ideals and $FSC$-ideals and discuss their properties. The characterization properties of $FSI$-ideals and $FSC$-ideals are obtained. We investigate the relations between $FSI$-ideals (resp. $FSC$-ideals) and other fuzzy ideals, between $FSI$-ideals (resp. $FSC$-ideals) and BCI-algebras, and show that a fuzzy subset of a BCI-algebra is an $FSI$-ideal

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if and only if it is both an $FSC$-ideal and a fuzzy BCI-positive implicative ideal.

## 2. Preliminaries

By a BCI-algebra we mean a nonempty set $X$ with a binary operation $*$ and a constant 0 satisfying the following conditions:

(1) $((x*y)*(x*z))*(z*y)=0$,
(2) $(x*(x*y))*y=0$,
(3) $x*x=0$,
(4) $x*y=0$ and $y*x=0$ imply $x=y$,

for all $x,y,z\in X$.

In a BCI-algebra $X$, the partial ordering $\leq$ is defined by $x\leq y$ if and only if $x*y=0$. In a BCI-algebra $X$, the following hold:

(5) $(x*y)*z=(x*z)*y$,
(6) $x*(x*(x*y))=x*y$,
(7) $(x*z)*(y*z)\leq x*y$,
(8) $0*(x*y)=(0*x)*(0*y)$,
(9) $x*0=x$,
(10) $x\leq y$ implies $x*z\leq y*z$ and $z*y\leq z*x$.

We refer the reader to K. Iseki [3] for details of BCI-algebras. Throughout this paper $X$ always means a BCI-algebra without any specification. A nonempty subset $I$ of $X$ is called an ideal of $X$ if $(I_1)$: $0\in I$, $(I_2)$: $x*y\in I$ and $y\in I$ imply $x\in I$. A nonempty subset $I$ of $X$ is called a positive implicative ideal (i.e., weakly positive implicative ideal) of $X$ if it satisfies $(I_1)$ and $(I_3)$: $((x*z)*z)*(y*z)\in I$ and $y\in I$ imply $x*z\in I$ [13]. A nonempty subset $I$ of $X$ is called a sub-implicative ideal of $X$ if it satisfies $(I_1)$ and $(I_4)$: $((x*(x*y))*(y*x))*z\in I$ and $z\in I$ imply $y*(y*x)\in I$ [10]. A nonempty subset $I$ of $X$ is called a sub-commutative ideal of $X$ if it satisfies $(I_1)$ and $(I_5)$: $(y*(y*(x*(x*y))))*z\in I$ and $z\in I$ imply $x*(x*y)\in I$ [10]. Let $S$ be a set. A fuzzy subset of $S$ is a function $\mu$: $S\to[0,1]$. Let $\mu$ be a fuzzy subset of $S$. For $t\in[0,1]$, the set $\mu_t=\{s\in S\mid \mu(s)\geq t\}$ is called a level subset of $\mu$ [2].

DEFINITION 2.1 (Xi [19]). A fuzzy subset $\mu$ of $X$ is said to be a fuzzy ideal of $X$ if it satisfies

$(F_1)$ $\mu(0)\geq\mu(x)$ for all $x\in X$,
$(F_2)$ $\mu(x)\geq\min\{\mu(x*y),\mu(y)\}$ for all $x,y\in X$.

DEFINITION 2.2 (Liu and Meng [11]). A fuzzy subset $\mu$ of $X$ is called a fuzzy BCI-positive implicative ideal of $X$ if it satisfies $(F_1)$ and

($F_3$) $\mu(x * z) \geq \min\{\mu(((x * z) * z) * (y * z)), \mu(y)\}$ for all $x, y, z \in X$.

DEFINITION 2.3 (Jun and Meng [7]). A fuzzy subset $\mu$ of $X$ is called a fuzzy $p$-ideal of $X$ if it satisfies ($F_1$) and

($F_4$) $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$ for all $x, y, z \in X$.

THEOREM 2.4 (Jun et al. [6]). *Every fuzzy ideal $\mu$ of $X$ is order reversing.*

THEOREM 2.5 (Jun and Meng [8]). *Let $\mu$ be a fuzzy ideal of $X$. Then $x * y \leq z$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$ for all $x, y, z \in X$.*

## 3. $FSI$-ideals of BCI-algebras

DEFINITION 3.1. A fuzzy subset $\mu$ of $X$ is called a fuzzy sub-implicative ideals (briefly, $FSI$-ideals) of $X$ if it satisfies ($F_1$) and

($F_5$) $\mu(y * (y * x)) \geq \min\{\mu(((x * (x * y)) * (y * x)) * z), \mu(z)\}$ for all $x, y, z \in X$.

EXAMPLE 3.2. Let $X = \{0, 1, 2\}$ be a BCI-algebra with Cayley table as follows:

| * | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 2 | 0 |

Define $\mu : X \to [0, 1]$ by $\mu(0) = \mu(1) = t_0$ and $\mu(2) = t_1$, where $t_0, t_1 \in [0, 1]$ and $t_0 > t_1$. By routine calculations give that $\mu$ is an $FSI$-ideal of $X$.

Now we give some characterizations of $FSI$-ideals of $X$.

THEOREM 3.3. *Let $\mu$ be a fuzzy ideal of $X$. Then the following are equivalent:*

(i) *$\mu$ is an $FSI$-ideal of $X$,*

(ii) *$\mu(y * (y * x)) \geq \mu((x * (x * y)) * (y * x))$ for all $x, y \in X$,*

(iii) *$\mu(y * (y * x)) = \mu((x * (x * y)) * (y * x))$ for all $x, y \in X$.*

*Proof.* (i)$\Rightarrow$(ii) Suppose that $\mu$ is an $FSI$-ideal of $X$. By ($F_5$) and ($F_1$) we have $(y * (y * x)) \geq \min\{\mu(((x * (x * y)) * (y * x)) * 0), \mu(0)\} = \mu((x * (x * y)) * (y * x))$.

(ii)$\Rightarrow$(iii) Since $(x * (x * y)) * (y * x) \leq y * (y * x)$, we have $\mu((x * (x * y)) * (y * x)) \geq \mu(y * (y * x))$ as Theorem 2.4. Combining (ii) we have $\mu(y * (y * x)) = \mu((x * (x * y)) * (y * x))$.

(iii)⇒(i) Since $((x*(x*y))*(y*x))*(((x*(x*y))*(y*x))*z) \leq z$, by Theorem 2.5 we obtain $\mu((x*(x*y))*(y*x)) \geq \min\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$. From (iii), $\mu(y*(y*x)) \geq \min\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\}$. Hence $\mu$ is an $FSI$-ideal of $X$. The proof is complete. □

THEOREM 3.4. *Let $\mu$ be a fuzzy subset of $X$. Then $\mu$ is an $FSI$-ideal of $X$ if and only if for all $t \in [0,1]$, $\mu_t$ is either empty or a sub-implicative ideal of $X$.*

*Proof.* Let $\mu$ be a $FSI$-ideal of $X$ and $\mu_t \neq \emptyset$ for some $t \in [0,1]$. Since $\mu(0) \geq \mu(x) \geq t$ for some $x$, we have $0 \in \mu_t$. If $((x*(x*y))*(y*x))*z \in \mu_t$ and $z \in \mu_t$, then $\mu(((x*(x*y))*(y*x))*z) \geq t$ and $\mu(z) \geq t$. It follows from $(F_5)$ that $\mu(y*(y*x)) \geq \min\{\mu(((x*(x*y))*(y*x))*z), \mu(z)\} \geq t$, and so $y*(y*x) \in \mu_t$. Hence $\mu_t$ is a sub-implicative ideal of $X$ by $(I_4)$.

Conversely, suppose that for each $t \in [0,1]$, $\mu_t$ is either empty or a sub-implicative ideal of $X$. For any $x \in X$, putting $\mu(x) = t$, then $x \in \mu_t$. Since $\mu_t (\neq \emptyset)$ is a sub-implicative ideal of $X$, we have $0 \in \mu_t$ and hence $\mu(0) \geq t = \mu(x)$. Thus $\mu(0) \geq \mu(x)$ for all $x \in X$. Now we prove that $\mu$ satisfies $(F_5)$. If not, then there exist $x_0, y_0, z_0 \in X$ such that $\mu(y_0*(y_0*x_0)) < \min\{\mu(((x_0*(x_0*y_0))*(y_0*x_0))*z_0), \mu(z_0)\}$. Taking $t_0$ satisfying $\mu(y_0*(y_0*x_0)) < t_0 < \min\{\mu(((x_0*(x_0*y_0))*(y_0*x_0))*z_0), \mu(z_0)\}$, we have $((x_0*(x_0*y_0))*(y_0*x_0))*z_0 \in \mu_{t_0}$ and $z_0 \in \mu_{t_0}$ , but $y_0*(x_0*y_0) \notin \mu_{t_0}$. Thus $\mu_{t_0}$ is not a sub-implicative ideal of $X$. This is a contradiction with hypothesis. This completes the proof. □

Next we investigate the relations between $FSI$-ideals and other fuzzy ideals of $X$.

THEOREM 3.5. *Any $FSI$-ideal is a fuzzy ideal, but the converse does not hold.*

*Proof.* Assume that $\mu$ is an $FSI$-ideal of $X$ and let $y = x$ in $(F_5)$. We obtain $\mu(x) \geq \min\{\mu(x*z), \mu(z)\}$ for all $x, z \in X$. This means that $\mu$ is a fuzzy ideal of $X$. The last part is shown by the following example:

EXAMPLE 3.6. Let $X = \{0,1,2,3\}$ be a BCI-algebra with Cayley table as follows:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 3 |
| 1 | 1 | 0 | 0 | 3 |
| 2 | 2 | 2 | 0 | 3 |
| 3 | 3 | 3 | 3 | 0 |

Define $\mu : X \to [0,1]$ by $\mu(0) = 1$ and $\mu(1) = \mu(2) = \mu(3) = 1/2$. Then $\mu$ is a fuzzy ideal of $X$, but not an $FSI$-ideal of $X$ since $\mu((2*(2*1))*(1*2)) = \mu(0) = 1 > 1/2 = \mu(1) = \mu(1*(1*2))$. The proof is complete.

□

LEMMA 3.7 (Liu and Meng [11]). *A fuzzy ideal $\mu$ of $X$ is a fuzzy BCI-positive implicative ideal of $X$ if and only if $\mu(x*y) \geq \mu(((x*y)*y)*(0*y))$ for all $x, y \in X$.*

THEOREM 3.8. *Any FSI-ideal is a fuzzy BCI-positive implicative ideal, but the converse is not true.*

*Proof.* Suppose that $\mu$ is an $FSI$-ideal of $X$. From Theorem 3.5, $\mu$ is a fuzzy ideal. Since

$$\begin{aligned}&((y*x)*((y*x)*y))*(y*(y*x))\\&=((y*(y*(y*x)))*x)*((y*x)*y)\\&=((y*x)*x)*(0*x),\end{aligned}$$

we have $\mu[((y*x)*((y*x)*y))*(y*(y*x))] = \mu[((y*x)*x)*(0*x)]$. By Theorem 3.3 (iii), $\mu(y*(y*(y*x))) = \mu[((y*x)*x)*(0*x)]$, i.e., $\mu(y*x) = \mu[((y*x)*x)*(0*x)]$. Hence $\mu$ is a fuzzy BCI-positive implicative ideal of $X$ as Lemma 3.7.

The last half part is shown by Example 3.6. We have known that $\mu$ is not an $FSI$-ideal of $X$. But it is easy to check that $\mu$ is a fuzzy BCI-positive implicative ideal of $X$, completing the proof. □

LEMMA 3.9 (Jun and Meng [7]). *A fuzzy ideal $\mu$ of $X$ is a fuzzy p-ideal of $X$ if and only if $\mu(x) \geq \mu(0*(0*x))$ for all $x \in X$.*

THEOREM 3.10. *Any fuzzy p-ideal is an FSI-ideal, but the converse is not true.*

*Proof.* Let $\mu$ be a fuzzy $p$-ideal of $X$. Then $\mu$ is a fuzzy ideal [7]. In order to prove that $\mu$ is an $FSI$-ideal, from Theorem 3.3 (ii) it suffices to show that $\mu(y*(y*x)) \geq \mu((x*(x*y))*(y*x))$. Since

$$\begin{aligned}&[0*(0*(y*(y*x)))]*[(x*(x*y))*(y*x)]\\&=[0*((x*(x*y))*(y*x))]*[0*(y*(y*x))]\\&=[((0*x)*(0*(x*y)))*(0*(y*x))]*[(0*y)*(0*(y*x))]\\&\leq((0*x)*(0*(x*y)))*(0*y)\\&=((0*x)*(0*y))*(0*(x*y))=0,\end{aligned}$$

we have $0 * (0 * (y * (y * x))) \leq (x * (x * y)) * (y * x)$. Hence $\mu(0 * (0 * (y * (y * x)))) \geq \mu((x * (x * y)) * (y * x))$. By Lemma 3.9, $\mu(y * (y * x)) \geq \mu((x * (x * y)) * (y * x))$.

The last half part is shown by Example 3.2. Define $\nu : X \to [0, 1]$ by $\nu(0) = 1$ and $\nu(1) = \nu(2) = 0$. It is easy to verify that $\nu$ is an $FSI$-ideal of $X$, but not a fuzzy $p$-ideal of $X$ because $\nu(0 * (0 * 1)) = \nu(0) = 1 > 0 = \nu(1)$. The proof is complete. □

DEFINITION 3.11 (Liu and Zhang [12]). A fuzzy set $\mu$ of $X$ is called a fuzzy $a$-ideal of $X$ if it satisfies $(F_1)$ and
$(F_6)$ $\mu(y * x) \geq \min\{\mu((x * z) * (0 * y)), \mu(z)\}$ for any $x, y, z \in X$.

DEFINITION 3.12 (Jun [5]). A fuzzy set $\mu$ of $X$ is called a fuzzy $q$-ideal of $X$ if it satisfies $(F_1)$ and
$(F_7)$ $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$ for any $x, y, z \in X$.

LEMMA 3.13 (Liu and Zhang [12]). *A fuzzy subset $\mu$ of X is a fuzzy a-ideal if and only if it is both a fuzzy q-ideal and a fuzzy p-ideal.*

COROLLARY 3.14. *Any fuzzy a-ideal is an FSI-ideal, but the converse is not true.*

From Theorem 4.3 and 4.7 of [12], we have: (i) $X$ is an associative BCI-algebra if and only if every fuzzy ideal of $X$ is a fuzzy $a$-ideal ; (ii) $X$ is a $p$-semisimple BCI-algebra if and only if every fuzzy ideal of $X$ is a fuzzy $p$-ideal. Combining Theorem 3.10 and Corollary 3.14 we obtain the following

COROLLARY 3.15. *Any fuzzy ideal in an associative BCI-algebra (resp. a p-semisimple BCI-algebra) is an FSI-ideal.*

Next we investigate the relations between $FSI$-ideals and BCI-algebras.

DEFINITION 3.16 (Meng and Xin [15]). A BCI-algebra is said to be implicative if it satisfies $(x * (x * y)) * (y * x) = y * (y * x)$.

THEOREM 3.17. *If X is an implicative BCI-algebra, then every fuzzy ideal of X is an FSI-ideal.*

*Proof.* It is an immediate consequence of Definition 3.16 and Theorem 3.3 (iii). □

If $\mu$ is a fuzzy ideal of $X$, we let $\mu_* = \mu_{\mu(0)} = \{x \in X \mid \mu(x) = \mu(0)\}$ and $B(X) = \{x \in X \mid 0 \leq x\}$.

THEOREM 3.18. *Let $\mu$ be a fuzzy ideal of X. If $X/\mu$ is an implicative BCI-algebra, then $\mu$ is an FSI-ideal of X. Conversely, if $\mu$ is an FSI-ideal with $\mu_* \supseteq B(X)$, then $X/\mu$ is an implicative BCI-algebra.*

*Proof.* If $X/\mu$ is an implicative BCI-algebra, then for any $x, y \in X$, we have $(\mu_x * (\mu_x * \mu_y)) * (\mu_y * \mu_x) = \mu_y * (\mu_y * \mu_x)$. Namely $\mu_{(x*(x*y))*(y*x)} = \mu_{y*(y*x)}$. Hence $\mu[(y * (y * x)) * ((x * (x * y)) * (y * x))] = \mu(0)$. Thus $\mu(y * (y * x)) \geq \min\{\mu((y * (y * x)) * ((x * (x * y)) * (y * x))), \mu((x * (x * y)) * (y * x))\} = \mu((x * (x * y)) * (y * x))$. Therefore $\mu$ is an $FSI$-ideal of $X$.

Conversely, assume that $\mu$ is an $FSI$-ideal with $\mu_* \supseteq B(X)$. For any $x, y \in X$, since $(y*(y*x))*((x*(x*y))*(y*x)) \geq (y*(y*x))*(y*(y*x)) = 0$, we have $(y * (y * x)) * ((x * (x * y)) * (y * x)) \in B(X) \subseteq \mu_*$, and so $\mu[(y * (y * x)) * ((x * (x * y)) * (y * x))] = \mu(0)$. On the other hand, $((x * (x * y)) * (y * x)) * (y * (y * x)) \leq (y * (y * x)) * (y * (y * x)) = 0$, so $\mu[((x * (x * y)) * (y * x)) * (y * (y * x))] = \mu(0)$. Thus we obtain $\mu_{y*(y*x)} = \mu_{(x*(x*y))*(y*x)}$. Namely $\mu_y*(\mu_y*\mu_x) = (\mu_x*(\mu_x*\mu_y))*(\mu_y*\mu_x)$. It means that $X/\mu$ is an implicative BCI-algebra. The proof is complete. □

COROLLARY 3.19. *For any BCI-algebra X, the characteristic function $\chi_{B(X)}$ is always an FSI-ideal of X.*

## 4. $FSC$-ideals of BCI-algebras

DEFINITION 4.1. A fuzzy subset $\mu$ of $X$ is called a fuzzy sub-commutative ideals (briefly, $FSC$-ideals) of $X$ if it satisfies $(F_1)$ and

$(F_8)$ $\mu(x * (x * y)) \geq \min\{\mu((y * (y * (x * (x * y)))) * z), \mu(z)\}$ for all $x, y, z \in X$.

EXAMPLE 4.2. Let $X = \{0, 1, 2, 3\}$ be a BCI-algebra with Cayley table as follows:

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

Define $\mu : X \to [0, 1]$ by $\mu(0) = \mu(3) = 0.8$ and $\mu(1) = \mu(2) = 0.2$. It is easy to check that $\mu$ is an $FSC$-ideal of $X$.

Now we give some characterizations of $FSC$-ideals of $X$.

THEOREM 4.3. *Let $\mu$ be a fuzzy ideal of X. Then the following are equivalent:*

(i) *$\mu$ is an FSC-ideal of X,*

(ii) $\mu(x*(x*y)) \geq \mu(y*(y*(x*(x*y))))$ *for all* $x, y \in X$,

(iii) $\mu(x*(x*y)) = \mu(y*(y*(x*(x*y))))$ *for all* $x, y \in X$,

(iv) *if* $x \leq y$, *then* $\mu(x) = \mu(y*(y*x))$ *for all* $x, y \in X$,

(v) *if* $x \leq y$, *then* $\mu(x) \geq \mu(y*(y*x))$ *for all* $x, y \in X$.

*Proof.* (i)⇒(ii) Suppose that $\mu$ is an $FSC$-ideal of $X$. By $(F_8)$ and $(F_1)$ we have $\mu(x*(x*y)) \geq \min\{\mu((y*(y*(x*(x*y))))*0), \mu(0)\} = \mu(y*(y*(x*(x*y))))$.

(ii)⇒(iii) Since $y*(y*(x*(x*y))) \leq x*(x*y)$, we have $\mu(y*(y*(x*(x*y)))) \geq \mu(x*(x*y))$. Combining (ii) we obtain $\mu(x*(x*y)) = \mu(y*(y*(x*(x*y))))$.

(iii)⇒(iv) If $x \leq y$, then $x*y = 0$. By (iii), we have $\mu(x) = \mu(y*(y*x))$.

(iv)⇒(v) Trivial.

(v)⇒(i) Since $x*(x*y) \leq y$, by (v) we have $\mu(x*(x*y)) \geq \mu(y*(y*(x*(x*y)))) \geq \min\{\mu((y*(y*(x*(x*y))))*z), \mu(z)\}$. Hence $\mu$ is an $FSC$-ideal of $X$, completing the proof. □

THEOREM 4.4. *Let $\mu$ be a fuzzy subset of X. Then $\mu$ is an FSC-ideal of X if and only if for all $t \in [0,1]$, $\mu_t$ is either empty or a sub-commutative ideal of X.*

*Proof.* It is similar to the proof of Theorem 3.4 and is omitted. □

Next we investigate the relations between $FSC$-ideals and other fuzzy ideals in $X$.

THEOREM 4.5. *Any FSC-ideal is a fuzzy ideal, but the converse does not hold.*

*Proof.* Suppose that $\mu$ is an $FSC$-ideal of $X$ and let $y = x$ in $(F_8)$. We have $\mu(x) \geq \min\{\mu(x*z), \mu(z)\}$ for all $x, z \in X$. Hence $\mu$ is a fuzzy ideal of $X$. The last half part is shown by Example 3.6. We have known that $\mu$ is a fuzzy ideal, but it is not an $FSC$-ideal of $X$ because $\mu(2*(2*(1*(1*2)))) = \mu(0) = 1 > 1/2 = \mu(1) = \mu(1*(1*2))$. The proof is complete. □

THEOREM 4.6. *Any fuzzy p-ideal is an FSC-ideal, but the converse is not true.*

*Proof.* Let $\mu$ be a fuzzy $p$-ideal of $X$. Then $\mu$ is a fuzzy ideal. Because

$$\begin{aligned}&[0*(0*(x*(x*y)))]*[y*(y*(x*(x*y)))]\\&=[0*(y*(y*(x*(x*y))))]*[0*(x*(x*y))]\\&=[(0*y)*((0*y)*(0*(x*(x*y))))]*[0*(x*(x*y))]\\&\leq[0*(x*(x*y))]*[0*(x*(x*y))]=0,\end{aligned}$$

we have $0*(0*(x*(x*y)))\leq y*(y*(x*(x*y)))$, and so $\mu(0*(0*(x*(x*y))))\geq\mu(y*(y*(x*(x*y))))$. By Lemma 3.9, $\mu(x*(x*y))\geq\mu(y*(y*(x*(x*y))))$. Hence $\mu$ is an $FSC$-ideal of $X$ as Theorem 4.3 (ii).

To show the last half part , we see Example 4.2. It has known that $\mu$ is an $FSC$-ideal of $X$. But it is not a fuzzy $p$-ideal of $X$ since $\mu(0*(0*2))=\mu(0)=0.8>0.2=\mu(2)$. This completes the proof. □

THEOREM 4.7. *Any FSI-ideal is an FSC-ideal, but the converse is not true.*

*Proof.* Assume that $\mu$ is an $FSI$-ideal of $X$. Then $\mu$ is a fuzzy ideal as Theorem 4.5. Because

$$\begin{aligned}&[(y*(y*x))*(x*y)]*[y*(y*(x*(x*y)))]\\&=[(y*(y*(y*(x*(x*y)))))*(y*x)]*(x*y)\\&=[(y*(x*(x*y)))*(y*x)]*(x*y)\\&=[(y*(y*x))*(x*(x*y))]*(x*y)\\&\leq(x*(x*(x*y)))*(x*y)\\&=(x*y)*(x*y)=0,\end{aligned}$$

we have $(y*(y*x))*(x*y)\leq y*(y*(x*(x*y)))$, and so $\mu((y*(y*x))*(x*y))\geq\mu(y*(y*(x*(x*y))))$. By Theorem 3.3 (iii) we have $\mu(x*(x*y))\geq\mu(y*(y*(x*(x*y))))$. Hence $\mu$ is an $FSC$-ideal of $X$.

To show the last half part, we see Example 4.2. It has known that $\mu$ is an $FSC$-ideal of $X$. But it is not an $FSI$-ideal of $X$ since $\mu((1*(1*2))*(2*1))=\mu(0)=0.8>0.2=\mu(1)=\mu(2*(2*1))$. The proof is complete. □

Now we give a characterization of fuzzy BCI-positive implicative ideals of $X$, which is needed in the sequel.

THEOREM 4.8. *A fuzzy ideal $\mu$ of $X$ is a fuzzy BCI-positive implicative ideal if and only if for all $x,y\in X$,*

$$(*)\qquad \mu(x*(x*(y*(y*x))))\geq\mu((x*(x*y))*(y*x)).$$

*Proof.* Let $\mu$ be a fuzzy ideal satisfying $(*)$. Since

$$\begin{aligned} &((x*y)*((x*y)*x))*(x*(x*y)) \\ =\ &((x*(x*(x*y)))*y)*((x*y)*x) \\ =\ &((x*y)*y)*(0*y), \end{aligned}$$

we have $\mu[((x*y)*((x*y)*x))*(x*(x*y))] = \mu(((x*y)*y)*(0*y))$. Substituting $x*y$ for $x$ and $x$ for $y$ in $(*)$, we have $\mu[(x*y)*((x*y)*(x*(x*(x*y))))] \geq \mu(((x*y)*y)*(0*y))$. Since

$$\begin{aligned} &(x*y)*((x*y)*(x*(x*(x*y)))) \\ =\ &(x*y)*((x*y)*(x*y)) \\ =\ &x*y, \end{aligned}$$

we have $\mu(x*y) \geq \mu(((x*y)*y)*(0*y))$. By Lemma 3.7, $\mu$ is a fuzzy BCI-positive implicative ideal of $X$.

Conversely, let $\mu$ be a fuzzy BCI-positive implicative ideal of $X$. Since

$$\begin{aligned} &[((y*(y*x))*(y*x))*(x*y)]*[(x*(x*y))*(y*x)] \\ =\ &[((y*(y*x))*(x*y))*(y*x)]*[(x*(x*y))*(y*x)] \\ \leq\ &[(y*(y*x))*(x*y)]*(x*(x*y)) \\ \leq\ &(y*(y*x))*x = 0, \end{aligned}$$

we have $\mu[((y*(y*x))*(y*x))*(x*y)] \geq \mu[(x*(x*y))*(y*x)]$. Let $s = y*x$ in $((y*(y*x))*(y*x))*(x*y)$. Then

$$(a) \qquad \mu[((y*s)*s)*(x*y)] \geq \mu((x*(x*y))*(y*x)).$$

Let $t = x*(y*(y*x)) = x*(y*s)$. Because

$$\begin{aligned} &[(((y*t)*s)*s)*(0*s)]*[((y*s)*s)*(x*y)] \\ =\ &[(((y*s)*s)*(0*s))*(((y*s)*s)*(x*y))]*t \\ \leq\ &((x*y)*(0*s))*t \\ =\ &((x*t)*y)*(0*s) \\ =\ &((x*(x*(y*s)))*y)*(0*s) \\ \leq\ &((y*s)*y)*(0*s) \\ =\ &(0*s)*(0*s) = 0, \end{aligned}$$

we have $\mu[(((y*t)*s)*s)*(0*s)] \geq \mu[((y*s)*s)*(x*y)]$. By Lemma 3.7, we have

$$(b) \qquad \mu((y*t)*s) \geq \mu[((y*s)*s)*(x*y)].$$

Since

$$\begin{aligned}&[((x*t)*t)*(0*t)]*((y*t)*s)\\ &=[((x*t)*((y*s)*t)]*(0*t)\\ &\leq ((x*t)*(y*s))*(0*t)\\ &=[(x*(x*(y*s)))*(y*s)]*(0*t)\\ &\leq ((y*s)*(y*s))*(0*t)\\ &=0*(0*t),\end{aligned}$$

and

$$\begin{aligned}&0*t\\ &=0*(x*(y*(y*x)))\\ &\leq 0*(x*x)\\ &=0,\end{aligned}$$

we have $0*(0*t)=0$, and so $\mu[((x*t)*t)*(0*t)]\geq\mu((y*t)*s)$. By Lemma 3.7 again, we have

$$(c)\qquad \mu(x*t)\geq\mu((y*t)*s).$$

Combining $(a)$, $(b)$ and $(c)$, we obtain $\mu(x*t)\geq\mu((x*(x*y))*(y*x))$, i.e., $\mu(x*(x*(y*(y*x))))\geq\mu((x*(x*y))*(y*x))$. The proof is complete. □

The following theorem shows that the close relations among $FSI$-ideals, $FSC$-ideals and fuzzy BCI-positive implicative ideals.

THEOREM 4.9. *Let $\mu$ be a fuzzy subset of $X$. Then $\mu$ is an FSI-ideal if and only if it is both an FSC-ideal and a fuzzy BCI-positive implicative ideal.*

*Proof.* If $\mu$ is an $FSI$-ideal, by Theorem 3.8 and 4.7, $\mu$ is both an $FSC$-ideal and a fuzzy BCI-positive implicative ideal. Conversely, if $\mu$ is both an $FSC$-ideal and a fuzzy BCI-positive implicative ideal, by Theorem 4.5, $\mu$ is a fuzzy ideal. For any $x,y\in X$, by Theorem 4.3 (ii) and Theorem 4.8, we have $\mu(y*(y*x))\geq\mu(x*(x*(y*(y*x))))\geq\mu((x*(x*y))*(y*x))$. Hence $\mu$ is an $FSI$-ideal of $X$ as Theorem 3.3 (ii). The proof is complete. □

Next we investigate the relation between $FSC$-ideals and BCI-algebras.

DEFINITION 4.10 (Meng and Xin [16]). A BCI-algebra is commutative if and only if $x*(x*y)=(y*(y*(x*(x*y))))$.

THEOREM 4.11. *If $X$ is a commutative BCI-algebra, then every fuzzy ideal of $X$ is an FSC-ideal.*

*Proof.* It is an immediate consequence of Definition 4.10 and Theorem 4.3 (iii). □

THEOREM 4.12. *Let $\mu$ be a fuzzy ideal of $X$. If $X/\mu$ is commutative, then $\mu$ is an FSC-ideal. Conversely, if $\mu$ is an FSC-ideal with $B(X) \subseteq \mu_*$, then $X/\mu$ is a commutative BCI-algebra.*

*Proof.* It is similar to the proof of Theorem 3.16 and omitted. □

COROLLARY 4.13. *For any BCI-algebra $X$, the characteristic function $\chi_{B(X)}$ is always an FSC-ideal of $X$.*

## 5. Conclusion

BCK-algebras and BCI-algebras are two important classes of logical algebras. Many logical algebras can be represented in BCK-algebras or BCI-algebras. For example, Boolean algebras are equivalent to the bounded implicative BCK-algebras [4], MV-algebras are equivalent to the bounded commutative BCK-algebras [18], Hilbert algebras are equivalent to the positive implicative BCK-algebra [1]. In this paper we proposed the concepts of $FSI$-ideals and $FSC$-ideals in BCI-algebras, established the relations between $FSI$-ideals (resp. $FSC$-ideals) and some other fuzzy ideals, between $FSI$-ideals (resp. $FSC$-ideals) and BCI-algebras. But further properties of $FSI$-ideals and $FSC$-ideals remain to be revealed. For example, do the converses of Theorem 3.17 and Theorem 4.11 hold? In [11], the notion of fuzzy BCI-implicative ideals was introduced. What relations between $FSI$-ideals and fuzzy BCI-implicative ideals are?

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