CONDITIONAL FOURIER-FEYNMAN TRANSFORM AND CONVOLUTION PRODUCT OVER WIENER PATHS IN ABSTRACT WIENER SPACE: AN $L_p$ THEORY

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Abstract. In this paper, using a simple formula, we evaluate the conditional Fourier-Feynman transforms and the conditional convolution products of cylinder type functions, and show that the conditional Fourier-Feynman transform of the conditional convolution product is expressed as a product of the conditional Fourier-Feynman transforms. Also, we evaluate the conditional Fourier-Feynman transforms of the functions of the forms

$$\exp\left\{\int_0^T \theta(s, x(s))ds\right\}, \quad \exp\left\{\int_0^T \theta(s, x(s))ds\phi(x(T))\right\},$$

$$\exp\left\{\int_0^T \theta(s, x(s))d\zeta(s)\right\}, \quad \exp\left\{\int_0^T \theta(s, x(s))d\zeta(s)\phi(x(T))\right\},$$

which are of interest in Feynman integration theories and quantum mechanics.

1. Introduction and preliminaries

Let $C_0[0, T]$ denote the classical Wiener space, that is, the space of real-valued continuous functions $x(t)$ on $[0, T]$ with $x(0) = 0$. The concept of conditional Wiener integral on Wiener space was introduced by Yeh in [18, 19]. By a conditional Wiener integral we mean the conditional expectation $E[F|X]$ of a real or complex-valued Wiener integrable function $F$ conditioned by a Wiener measurable function $X$ on $C_0[0, T]$,
which is given as a function on the value space of $X$. We shall be concerned exclusively with $X$ given by $X(x) = (x(t_1), \ldots, x(t_k))$, where $0 < t_1 < \cdots < t_k = T$ for any fixed positive integer $k$. In [14], Park and Skoug derived a simple formula for the conditional Wiener integral with the conditioning function $X$. They then used this formula to express conditional Wiener integrals directly in terms of ordinary Wiener integrals. In [15], Park and Skoug introduced the concept of a conditional Fourier-Feynman transform and the concept of a conditional convolution product and obtained several formulas relating these concepts; in particular see equations (3.13) and (4.11) in [15]. In [5], using ideas and formulas developed in [15], Chang and Skoug examined the effects that drift had on conditional Fourier-Feynman transforms and conditional convolution products. In section 4 of [5] and in section 4 of [15] the authors established various formulas involving conditional transforms and conditional convolution products for functions in the Banach algebra $S$ which was introduced by Cameron and Storvick in [1].

In [12], Kuelbs and Lepage introduced $C_0(\mathbb{B})$, the space of continuous functions on $[0, T]$ into $\mathbb{B}$ which vanish at 0. In [16], Ryu established various properties involving $C_0(\mathbb{B})$. In [20], Yoo introduced the Banach algebra $S''_E$ which corresponds to Cameron and Storvick’s space $S''$ in [1]. Chang and his coworkers introduced the class $A^{(p)}_{n,s}(1 \leq p \leq \infty)$ of cylinder type functions defined on $C_0(\mathbb{B})$, and then established relationships between the Fourier-Feynman transform and the convolution product of functions in $A^{(p)}_{n,s}(\mathbb{B})$.

In [2], Chang and his coworkers defined the $L_1$ conditional Fourier-Feynman transform and the conditional convolution product on the space $C_0(\mathbb{B})$, and they investigate relationships between them.

In this paper, we define the $L_p$ conditional Fourier-Feynman transform on $C_0(\mathbb{B})$ for $1 < p \leq \infty$, and we evaluate the $L_p$ conditional Fourier-Feynman transforms and the conditional convolution products of functions in $A^{(p)}_{n,s}$, and then, we show that the conditional Fourier-Feynman transform of the conditional convolution product can be expressed as a product of the conditional Fourier-Feynman transforms for these functions. Also, for $1 \leq p \leq \infty$, we evaluate the $L_p$ conditional Fourier-Feynman transforms of the functions of the forms

$$\exp\left\{ \int_0^T \theta(s, x(s)) ds \right\}, \quad \exp\left\{ \int_0^T \theta(s, x(s)) ds \right\} \phi(x(T)),$$

$$\exp\left\{ \int_0^T \theta(s, x(s)) d\zeta(s) \right\}, \quad \exp\left\{ \int_0^T \theta(s, x(s)) d\zeta(s) \right\} \phi(x(T)),$$
which are of interest in Feynman integration theories and quantum mechanics.

Let \((\Omega, \mathcal{A}, P)\) be a probability space, let \(B\) be a real normed linear space with norm \(\| \cdot \|\) and let \(\mathcal{B}(B)\) be the Borel \(\sigma\)-field on \(B\). Let \(X : (\Omega, \mathcal{A}, P) \to (B, \mathcal{B}(B))\) be a random variable and let \(F : \Omega \to \mathbb{C}\) be an integrable function. Let \(P_X\) be the probability distribution of \(X\) on \((B, \mathcal{B}(B))\) and let \(\mathcal{D}\) be the \(\sigma\)-field \(\{X^{-1}(A) : A \in \mathcal{B}(B)\}\). Let \(P_\mathcal{D}\) be the probability measure induced by \(P\), that is, \(P_\mathcal{D}(E) = P(E)\) for \(E \in \mathcal{D}\). By the definition of conditional expectation there exists a \(\mathcal{D}\)-measurable function \(E[F|X](\text{the conditional expectation of } F \text{ given } X)\) defined on \(\Omega\) such that the relation

\[
\int_E E[F|X](\omega) \, dP_\mathcal{D}(\omega) = \int_E F(\omega) \, dP(\omega)
\]

holds for every \(E \in \mathcal{D}\). But there exists a \(P_X\)-integrable function \(\psi\) defined on \(B\) which is unique up to \(P_X\)-a.e. such that \(E[F|X](\omega) = (\psi \circ X)(\omega)\) for \(P_\mathcal{D}\)-a.e. \(\omega\) in \(\Omega\). \(\psi\) is also called the conditional expectation of \(F\) given \(X\) and without loss of generality, it is denoted by \(E[F|X](\xi)\) for \(\xi \in B\). Throughout this paper, we will consider the function \(\psi\) as the conditional expectation of \(F\) given \(X\).

2. Wiener paths in abstract Wiener space

Let \((\mathcal{H}, \mathbb{B}, m)\) be an abstract Wiener space ([13]). Let \(\{e_j : j \geq 1\}\) be a complete orthonormal set in the real separable Hilbert space \(\mathcal{H}\) such that \(e_j\)'s are in \(\mathbb{B}^*\), the dual space of real separable Banach space \(\mathbb{B}\). For each \(h \in \mathcal{H}\) and \(y \in \mathbb{B}\), define the stochastic inner product \((h, y)\sim\) by

\[
(h, y)\sim = \lim_{n \to \infty} \sum_{j=1}^{n} (h, e_j)(y, e_j),
\]

if the limit exists;

\[
0,
\]

otherwise,

where \((\cdot, \cdot)\sim\) denotes the dual pairing between \(\mathbb{B}\) and \(\mathbb{B}^*([11])\). Note that for each \(h(\neq 0)\) in \(\mathcal{H}\), \((h, \cdot)\sim\) is a Gaussian random variable on \(\mathbb{B}\) with mean zero, variance \(|h|^2\); also \((h, y)\sim\) is essentially independent of the choice of the complete orthonormal set used in its definition and further, \((h, \lambda y)\sim = (\lambda h, y)\sim = \lambda (h, y)\sim\) for all \(\lambda \in \mathbb{R}\). It is well-known that if \(\{h_1, h_2, \ldots, h_n\}\) is an orthogonal set in \(\mathcal{H}\), then the random variables \((h_j, \cdot)\sim\)'s are independent. Moreover, if both \(h\) and \(y\) are in \(\mathcal{H}\), then \((h, y)\sim = (h, y)\).

Let \(C_0(\mathbb{B})\) denote the set of all continuous functions on \([0, T]\) into \(\mathbb{B}\) which vanish at 0. Then \(C_0(\mathbb{B})\) is a real separable Banach space with the norm \(\|x\|_{C_0(\mathbb{B})} \equiv \sup_{0 \leq t \leq T} \|x(t)\|_{\mathbb{B}}\). The minimal \(\sigma\)-field making the
mapping $x \to x(t)$ measurable is $B(C_0(\mathbb{B}))$, the Borel $\sigma$-field on $C_0(\mathbb{B})$.

Further, Brownian motion in $\mathbb{B}$ induces a probability measure $m_\mathbb{B}$ on $(C_0(\mathbb{B}), B(C_0(\mathbb{B})))$ which is mean-zero Gaussian. We will find a concrete form of $m_\mathbb{B}$. Let $\vec{t} = (t_1, t_2, \ldots, t_k)$ be given with $0 = t_0 < t_1 < t_2 < \cdots < t_k \leq T$. Let $T_\vec{t} : \mathbb{B}^k \to \mathbb{B}^k$ be given by

$$T_\vec{t}(x_1, x_2, \ldots, x_k) = \left( (t_1 - t_0)^{\frac{1}{2}} x_1, (t_1 - t_0)^{\frac{1}{2}} x_1 + (t_2 - t_1)^{\frac{1}{2}} x_2, \ldots, \sum_{j=1}^{k} (t_j - t_{j-1})^{\frac{1}{2}} x_j \right)$$

We define a set function $\nu_\vec{t}$ on $B(\mathbb{B}^k)$ by

$$\nu_\vec{t}(B) = \left( \prod_{1}^{k} m \right) \left( T_\vec{t}^{-1}(B) \right)$$

for $B \in B(\mathbb{B}^k)$. Then $\nu_\vec{t}$ is a Borel measure. Let $f_\vec{t} : C_0(\mathbb{B}) \to \mathbb{B}^k$ be the function defined by

$$f_\vec{t}(x) = (x(t_1), x(t_2), \ldots, x(t_k)).$$

For Borel subsets $B_1, B_2, \ldots, B_k$ of $\mathbb{B}$, $f_\vec{t}^{-1}(\prod_{j=1}^{k} B_j)$ is called the $I$-set with respect to $B_1, B_2, \ldots, B_k$. Then the collection $\mathcal{I}$ of all $I$-sets is a semi-algebra. We define a set function $m_\mathbb{B}$ on $\mathcal{I}$ by

$$m_\mathbb{B} \left( f_\vec{t}^{-1} \left( \prod_{j=1}^{k} B_j \right) \right) = \nu_\vec{t} \left( \prod_{j=1}^{k} B_j \right)$$

Then $m_\mathbb{B}$ is well-defined and countably additive on $\mathcal{I}$. Using Carathéodory extension process, we have a Borel measure $m_\mathbb{B}$ on $B(C_0(\mathbb{B}))$.

Now, we introduce Wiener integration theorem without proof. For the proof see [16].

**Theorem 1** (Wiener integration theorem). Let $\vec{t} = (t_1, t_2, \ldots, t_k)$ be given with $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T$ and let $f : \mathbb{B}^k \to \mathbb{C}$ be a Borel measurable function. Then

$$\int_{C_0(\mathbb{B})} f(x(t_1), x(t_2), \ldots, x(t_k)) \, dm_\mathbb{B}(x) \overset{\text{def}}{=} \int_{\mathbb{B}^k} (f \circ T_\vec{t})(x_1, x_2, \ldots, x_k) \, d \left( \prod_{j=1}^{k} m \right) (x_1, x_2, \ldots, x_k),$$
where by \( * \) we mean that if either side exists, then both sides exist and they are equal.

Let \( \tau : 0 = t_0 < t_1 < \cdots < t_k = T \) be a partition of \([0, T]\) and let \( x \) be in \( C_0(\mathbb{B}) \). Define the polygonal function \([x]\) of \( x \) on \([0, T]\) by

\[
[x](t) = \sum_{j=1}^{k} \chi(t_{j-1}, t_j)(t) \left[ x(t_{j-1}) + \frac{t-t_{j-1}}{t_j-t_{j-1}} (x(t_j) - x(t_{j-1})) \right],
\]

where \( t \in [0, T] \). For each \( \bar{\xi} = (\xi_1, \ldots, \xi_k) \in \mathbb{B}^k \), let \( \bar{[\xi]} \) be the polygonal function of \( \bar{\xi} \) on \([0, T]\) given by (1) with replacing \( x(t_j) \) by \( \xi_j \) for \( j = 0, 1, \ldots, k \) (\( \xi_0 = 0 \)).

The following lemmas are useful to the next sections. For the detailed proof, see [3].

**Lemma 2.** If \( \{x(t) : 0 \leq t \leq T\} \) is the Wiener process on \( C_0(\mathbb{B}) \times [0, T] \), then \( \{x(t) - [x](t) : t_{j-1} \leq t \leq t_j\} \), where \( j = 1, \ldots, k \), are stochastically independent.

**Lemma 3.** Let \( F \) be defined and integrable on \( C_0(\mathbb{B}) \). Let \( X_\tau : C_0(\mathbb{B}) \to \mathbb{B}^k \) be a random variable given by \( X_\tau(x) = (x(t_1), \ldots, x(t_k)) \). Then for every Borel measurable subset \( B \) of \( \mathbb{B}^k \),

\[
\mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x) \, dm_\mathbb{B}(x) = \int_B E[F(x - [x] + [\bar{\xi}])] \, dP_{X_\tau}(\bar{\xi}),
\]

where \( P_{X_\tau} \) is the probability distribution of \( X_\tau \) on \((\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))\).

By the definition of conditional expectation and Lemma 3, we have

\[
E[F|X_\tau](\bar{\xi}) = E[F(x - [x] + [\bar{\xi}])] \quad \text{for } P_{X_\tau}\text{-a.e. } \bar{\xi}.
\]

The function \( E[F|X_\tau] \) is called the conditional Wiener integral of \( F \) given \( X_\tau \). Note that, for the definition of the conditional Wiener integral, \( X_\tau \) may be any random variable defined on \( C_0(\mathbb{B}) \). The equation (2) is called a simple formula for conditional Wiener integral on the space \( C_0(\mathbb{B}) \).

For \( \lambda > 0 \) and \( \bar{\xi} \in \mathbb{B}^k \), suppose \( E[F(\lambda^{-\frac{1}{2}} \cdot)|X_\tau(\lambda^{-\frac{1}{2}} \cdot)](\bar{\xi}) \) exists. From (2) we have

\[
E[F(\lambda^{-\frac{1}{2}} \cdot)|X_\tau(\lambda^{-\frac{1}{2}} \cdot)](\bar{\xi}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\bar{\xi}])]
\]

for a.e. \( \bar{\xi} \in \mathbb{B}^k \). If, for \( \bar{\xi} \in \mathbb{B}^k \), \( E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\bar{\xi}])] \) has the analytic extension \( J_\lambda(\bar{\xi}) \) on \( \mathbb{C}_+ \equiv \{ \lambda \in \mathbb{C} : \text{Re } \lambda > 0 \} \), then we write

\[
J_\lambda(\bar{\xi}) = E^{anw, \lambda}[F|X_\tau](\bar{\xi})
\]
for $\lambda \in \mathbb{C}_+$. In this case, we call $J_\lambda(\tilde{\xi})$ a version of conditional analytic Wiener integral of $F$ given $X_\tau$.

For non-zero real $q$ and $\tilde{\xi} \in \mathbb{B}^k$, if the limit
\[
\lim_{\lambda \to -iq} E^{\text{anw}_\lambda}[F|X_\tau](\tilde{\xi})
\]
exists, where $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$, then we write
\[
\lim_{\lambda \to -iq} E^{\text{anw}_\lambda}[F|X_\tau](\tilde{\xi}) = E^{\text{anf}_q}[F|X_\tau](\tilde{\xi}).
\]
In this case, we call $E^{\text{anf}_q}[F|X_\tau](\tilde{\xi})$ a version of conditional analytic Feynman integral of $F$ given $X_\tau$.

3. Conditional Fourier-Feynman transform of cylinder type functions

In this section, we define $L_p(1 < p \leq \infty)$ conditional Fourier-Feynman transform over Wiener paths in abstract Wiener space and evaluate them for cylinder type functions. We also evaluate conditional convolution products for these type of functions.

A subset $E$ of $C_0(\mathbb{B})$ is called a scale-invariant null set if $m_{\mathbb{B}}(\lambda E) = 0$ for any $\lambda > 0$ and a property is said to hold s-a.e. if it holds except for a scale-invariant null set.

For a given real number $p$ with $1 < p \leq \infty$, suppose $p$ and $p'$ are related by $\frac{1}{p} + \frac{1}{p'} = 1$ (possibly $p' = 1$ if $p = \infty$). Let $G_n$ and $G$ be measurable functions such that, for each $\gamma > 0$,
\[
\lim_{n \to \infty} \int_{C_0(\mathbb{B})} |G_n(\gamma x) - G(\gamma x)|^{p'} dm_{\mathbb{B}}(x) = 0.
\]
Then we write
\[
l.i.m.(w_s^{p'})(G_n) \approx G
\]
and call $G$ the scale-invariant limit in the mean of order $p'$. A similar definition is understood when $n$ is replaced by a continuously varying parameter.

Let $\mathcal{H}$ be a real separable infinite dimensional Hilbert space, let $n$ be a positive integer and let $\{h_1, \ldots, h_n\}$ be an orthonormal set in $\mathcal{H}$. For $1 \leq p < \infty$ let $\mathcal{A}^{(p)}_{n,s}$ be the space of all cylinder type functions $F$ defined on $C_0(\mathbb{B})$ of the form
\[
F(x) = f((h_1, x(s))^{\sim}, \ldots, (h_n, x(s))^{\sim})
\]
for $s$-a.e. $x$ in $C_0(\mathbb{B})$ where $f : \mathbb{R}^n \to \mathbb{R}$ is in $L_p(\mathbb{R}^n)$ and $s \in (0,T]$. Let $\mathcal{A}^{(0)}_{n,s}$ be the space of all functions of the form (3) with $f \in C_0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ which vanish at infinity and let $\mathcal{A}^{(\infty)}_{n,s}$ be the space of all functions of the form (3) with $f \in L_\infty(\mathbb{R}^n)$, the space of essentially bounded functions on $\mathbb{R}^n$. Note that, without loss of generality, we can take $f$ to be Borel measurable.

For convenience, we let

$$\Gamma = \frac{t_{p^*} - t_{p^* - 1}}{(t_{p^*} - s)(s - t_{p^* - 1})}$$

for $t_{p^* - 1} < s < t_{p^*}$ and

$$\phi_\lambda(\tilde{u}) = \left(\frac{\lambda}{2\pi}\right)^\frac{n}{2} \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^{n} u_j^2\right\} \quad (\lambda^{\frac{1}{2}} \in \mathbb{C}_+$$

where $\tilde{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}_+ \equiv \{\lambda \in \mathbb{C} : \text{Re } \lambda \geq 0\} - \{0\}$. Also, we let for $\tilde{\xi} \in \mathbb{B}^k$

$$\tilde{w}_{\tilde{\xi}} = (w_{\xi_1}, \ldots, w_{\xi_n}) = ((h_1, |\xi|(s))\sim, \ldots, (h_n, |\xi|(s))\sim)$$

and

$$\tilde{w}_{\tilde{y}} = (w_{y_1}, \ldots, w_{y_n}) = ((h_1, y(s))\sim, \ldots, (h_n, y(s))\sim)$$

for $y$ in $C_0(\mathbb{B})$.

**DEFINITION 4.** Let $F$ be defined on $C_0(\mathbb{B})$ and let $X_\tau$ be given as in Lemma 3. For $\lambda \in \mathbb{C}_+$ and for $s$-a.e. $\tilde{\xi} \in \mathbb{B}^k$ let

$$T_{\lambda}[F|X_\tau](y, \tilde{\xi}) = E^{\text{anw},\lambda}[F(y + \cdot)|X_\tau](\tilde{\xi})$$

for $s$-a.e. $y \in C_0(\mathbb{B})$ if it exists. For non-zero real $q$ and for $s$-a.e. $\tilde{\xi} \in \mathbb{B}^k$, we define the $L_1$ conditional Fourier-Feynman transform $T_q^{(1)}[F|X_\tau]$ of $F$ given $X_\tau$ by the formula

$$T_q^{(1)}[F|X_\tau](y, \tilde{\xi}) = \lim_{\lambda \to -iq} T_{\lambda}[F|X_\tau](y, \tilde{\xi})$$

if it exists for $s$-a.e. $y \in C_0(\mathbb{B})$ and for $1 < p \leq \infty$ we define the $L_p$ conditional Fourier-Feynman transform $T_q^{(p)}[F|X_\tau]$ of $F$ given $X_\tau$ by the formula

$$T_q^{(p)}[F|X_\tau](\cdot, \tilde{\xi}) \approx \lim_{\lambda \to -iq} \lambda \text{-i.m.}(w_p^\lambda)(T_{\lambda}[F|X_\tau](\cdot, \tilde{\xi}))$$

where $\lambda$ approaches to $-iq$ through $\mathbb{C}_+$. 
THEOREM 5. Let \( F \in \mathcal{A}_{n,s}^{(p)}(1 \leq p \leq \infty) \) be given by (3). Let \( X_r \) be given as in Lemma 3 and let \( t_{p^*} - 1 < s \leq t_{p^*} \) for some \( p^* \in \{1, \ldots, k\} \). Then, for any \( \lambda \in \mathbb{C}_+ \) and for s-a.e. \( \xi \in \mathbb{B}^k \), \( T_\lambda[F|X_r](y, \xi) \) exists for s-a.e. \( y \in C_0(\mathbb{B}) \) and \( T_\lambda[F|X_r](\cdot, \xi) \in \mathcal{A}_{n,s}^{(p)} \). Moreover, when \( t_{p^*} - 1 < s < t_{p^*} \), we have

\[
T_\lambda[F|X_r](y, \xi) = (\phi_{\lambda \Gamma} \ast f)(\tilde{w}_y + \tilde{w}_\xi)
\]

and, when \( s = t_{p^*} \), we have

\[
T_\lambda[F|X_r](y, \xi) = F(y + [\xi]) = f(\tilde{w}_y + \tilde{w}_\xi),
\]

where \( \Gamma, \phi_{\lambda \Gamma}, \tilde{w}_\xi \) and \( \tilde{w}_y \) are given by (4), (5), (6) and (7), respectively.

Proof. Let \( t_{p^*} - 1 < s < t_{p^*} \) and let \( \lambda > 0 \). Using the same method in the proof of Theorem 4.2 in [2], we have

\[
E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\xi])] = (\phi_{\lambda \Gamma} \ast f)(\tilde{w}_y + \tilde{w}_\xi)
\]

for s-a.e. \( \xi \in \mathbb{B}^k \) and for s-a.e. \( y \in C_0(\mathbb{B}) \). By Morera’s theorem, we have (8). The fact \( T_\lambda[F|X_r](\cdot, \xi) \in \mathcal{A}_{n,s}^{(p)} \) follows from Young’s inequality in [7, p.232].

When \( s = t_{p^*} \), the equation (9) follows easily since \( E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\xi])] = F(y + [\xi]) \) for \( \lambda > 0 \). \( \square \)

THEOREM 6. For \( 1 \leq p \leq 2 \) let \( F \in \mathcal{A}_{n,s}^{(p)} \) be given by (3) and let \( \frac{1}{p} + \frac{1}{p'} = 1(p' = \infty \text{ if } p = 1) \). Let \( X_r \) be given as in Lemma 3 and let \( t_{p^*} - 1 < s \leq t_{p^*} \) for some \( p^* \in \{1, \ldots, k\} \). Then, for any non-zero real \( q \), and for s-a.e. \( \xi \in \mathbb{B}^k \), \( T_q^{(p)}[F|X_r](y, \xi) \) exists for s-a.e. \( y \in C_0(\mathbb{B}) \). Moreover, when \( t_{p^*} - 1 < s < t_{p^*} \), we have

\[
T_q^{(p)}[F|X_r](y, \xi) = (\phi_{-iq \Gamma} \ast f)(\tilde{w}_y + \tilde{w}_\xi)
\]

with \( T_q^{(p)}[F|X_r](\cdot, \xi) \in \mathcal{A}_{n,s}^{(q)} \) (in fact, \( T_q^{(1)}[F|X_r](\cdot, \xi) \in \mathcal{A}_{n,s}^{(0)} \)) and, when \( s = t_{p^*} \), we have

\[
T_q^{(p)}[F|X_r](y, \xi) = F(y + [\xi]) = f(\tilde{w}_y + \tilde{w}_\xi)
\]

with \( T_q^{(p)}[F|X_r](\cdot, \xi) \in \mathcal{A}_{n,s}^{(p)} \), where \( \Gamma, \phi_{-iq \Gamma}, \tilde{w}_\xi \) and \( \tilde{w}_y \) are given by (4), (5), (6) and (7), respectively.

Proof. When \( p = 1 \), the result follows from Theorem 4.2 in [2].
Let \( 1 < p \leq 2, \ t_{p^* - 1} < s < t_{p^*} \). Then, for \( \gamma > 0 \), for s-a.e. \( \tilde{\xi} \in \mathbb{B}^k \) and by Theorems 1, 5, we have
\[
\lim_{\lambda \to -iq} \int_{C_0(\mathbb{B})} |(\phi_{\lambda \Gamma} * f)(\gamma \tilde{w}_y + \tilde{w}_{\tilde{\xi}}) - (\phi_{-iq \Gamma} * f)(\gamma \tilde{w}_y + \tilde{w}_{\tilde{\xi}})|^{p'} \ dm_{\mathbb{B}}(y)
\]
\[
= \lim_{\lambda \to -iq} \int_{\mathbb{B}} |(\phi_{\lambda \Gamma} * f)(\gamma \sqrt{s}((h_1, x_1)^{\sim}, \ldots, (h_n, x_1)^{\sim}) + \tilde{w}_{\tilde{\xi}}) - (\phi_{-iq \Gamma} * f)(\gamma \sqrt{s}((h_1, x_1)^{\sim}, \ldots, (h_n, x_1)^{\sim}) + \tilde{w}_{\tilde{\xi}})|^{p'} \ dm(x_1)
\]
\[
\leq \lim_{\lambda \to -iq} \left( \frac{1}{2 \pi s^{\gamma^2}} \right)^{\frac{n}{2}} \| (\phi_{\lambda \Gamma} * f)(\cdot + \tilde{w}_{\tilde{\xi}}) - (\phi_{-iq \Gamma} * f)(\cdot + \tilde{w}_{\tilde{\xi}}) \|^{p'}_{p'}
\]
which converges to 0 as \( \lambda \) approaches to \( -iq \) through \( \mathbb{C}_+ \), by Lemmas 1.1, 1.2 in [10] and since \((h_j, \cdot)^{\sim}\) is normally distributed with mean 0, variance 1.

When \( s = t_{p^*} \), the result follows trivially by Theorem 5. \( \square \)

**Remark 1.** \((10)\) holds for \( 1 \leq p \leq \infty \).

Next we establish an inverse conditional Fourier-Feynman transform theorem for functions in \( A^{(p)}_{n,s} \).

**Theorem 7.** For \( 1 \leq p < \infty \) let \( F \in A^{(p)}_{n,s} \) be given by \((3)\) and let \( X_\tau \) be given as in Lemma 3. Let \( q \) be a non-zero real number and let \( \tilde{w}_y \) be given by \((7)\). For \((\tilde{\xi}_1, \tilde{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k \) and for \( y \in C_0(\mathbb{B}) \), let \( \tilde{T}_{\tilde{\xi}_1,\tilde{\xi}_2}(y) = f(\tilde{w}_y + \tilde{w}_1 + \tilde{w}_2) \), where \( \tilde{w}_1 = (w_{11}, \ldots, w_{1n}) = ((h_1, [\tilde{\xi}_1](s))^{\sim}, \ldots, (h_n, [\tilde{\xi}_1](s))^{\sim}) \), \( \tilde{w}_2 = (w_{21}, \ldots, w_{2n}) = ((h_1, [\tilde{\xi}_2](s))^{\sim}, \ldots, (h_n, [\tilde{\xi}_2](s))^{\sim}) \). Then we have

(i) for s-a.e. \((\tilde{\xi}_1, \tilde{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k \) and for \( \gamma > 0 \), we have
\[
\lim_{\lambda \to -iq} \int_{C_0(\mathbb{B})} |\tilde{T}_{\tilde{\xi}}[\tilde{T}_\lambda[F(X_\tau)(\cdot, \tilde{\xi}))(X_\tau)](\gamma y, \tilde{\xi}_2) - \tilde{T}_{\tilde{\xi}_1,\tilde{\xi}_2}(\gamma y)|^{p'} \ dm_{\mathbb{B}}(y) = 0
\]
and

(ii) for s-a.e. \((\tilde{\xi}_1, \tilde{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k \) and for s-a.e. \( y \in C_0(\mathbb{B}) \), we have
\[
\tilde{T}_{\tilde{\xi}}[\tilde{T}_\lambda[F(X_\tau)(\cdot, \tilde{\xi}))X_\tau](y, \tilde{\xi}_2) \to \tilde{T}_{\tilde{\xi}_1,\tilde{\xi}_2}(y),
\]
where \( \lambda \) approaches to \( -iq \) through \( \mathbb{C}_+ \).

**Proof.** Let \( t_{p^* - 1} < s < t_{p^*} \) for some \( p^* \in \{1, \ldots, k\} \) and let \( \tilde{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n \). Then for \( \lambda \in \mathbb{C}_+ \), for s-a.e. \((\tilde{\xi}_1, \tilde{\xi}_2) \in \mathbb{B}^k \times \mathbb{B}^k \) and
for s.a.e. \( y \in C_0(\mathbb{B}) \), using the same method in the proof of Theorem 4.4 in [2], we have

\[
T_{\mathbb{X}}[T_{\lambda}[F|X_{\tau}](\cdot, \xi_1)|X_{\tau}](y, \xi_2) = \left( \frac{|\lambda|^2 \Gamma}{4\pi \text{Re}\lambda} \right)^{\frac{n}{2}} \frac{1}{\Gamma_n} f(\bar{u}) \exp \left\{ -\frac{|\lambda|^2 \Gamma}{4\pi \text{Re}\lambda} \sum_{j=1}^{n} (u_j - w_{yj} - w_{1j} - w_{2j})^2 \right\} d\bar{u}
\]

where \( \Gamma \) is given by (4). Let \( \gamma > 0 \) and for \( \bar{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \) let

\[
k_{\xi_1, \xi_2}^\bar{v}(\lambda, \bar{v}) = \left( \frac{|\lambda|^2 \Gamma}{4\pi \text{Re}\lambda} \right)^{\frac{n}{2}} \frac{1}{\Gamma_n} f(\bar{u}) \exp \left\{ -\frac{|\lambda|^2 \Gamma}{4\pi \text{Re}\lambda} \sum_{j=1}^{n} (u_j - v_j - w_{1j} - w_{2j})^2 \right\} d\bar{u}.
\]

Then, by Wiener integration theorem (Theorem 1) and the change of variable theorem, we have

\[
\int_{\mathbb{X}} |T_{\mathbb{X}}[T_{\lambda}[F|X_{\tau}](\cdot, \xi_1)|X_{\tau}](\gamma y, \xi_2) - F_{\xi_1, \xi_2}^\gamma(\gamma y)|^p \, dm_\mathbb{B}(y)
\]

\[
= \int_{\mathbb{X}} |k_{\xi_1, \xi_2}^\gamma(\lambda, \gamma \sqrt{s}((h_1, x_1)^*, \ldots, (h_n, x_1)^*)) - f(\gamma \sqrt{s}((h_1, x_1)^* + w_1, \bar{w} + \bar{w}_2))|^p \, dm(x_1)
\]

\[
= \left( \frac{1}{2\pi s \gamma^2} \right)^{\frac{n}{2}} \frac{1}{\Gamma_n} f(\bar{w} + \bar{w}_1 + \bar{w}_2)^p \exp \left\{ -\frac{1}{2\pi s \gamma^2} \sum_{j=1}^{n} z_j^2 \right\} d\bar{z}
\]

where \( \bar{z} = (z_1, \ldots, z_n) \), since \( (h_j, \cdot)^* \) is normally distributed with mean 0 and variance 1. Let \( \epsilon = (\frac{2\text{Re}\lambda}{|\lambda|^2 \Gamma})^{\frac{1}{2}} \) and let \( \varphi_\epsilon(\bar{u}) = \frac{1}{\epsilon^n} \phi_1(\frac{\bar{u}}{\epsilon}) \), where \( \phi_1 \) is given by (5). Then, \( \int_{\mathbb{X}} \rho_1(\bar{u}) d\bar{u} = 1 \) and hence in view of Theorem 1.18 in [17], we have

\[
\lim_{\lambda \to -iq} \int_{\mathbb{X}} |T_{\mathbb{X}}[T_{\lambda}[F|X_{\tau}](\cdot, \xi_1)|X_{\tau}](\gamma y, \xi_2) - F_{\xi_1, \xi_2}^\gamma(\gamma y)|^p \, dm_\mathbb{B}(y)
\]

\[
\leq \left( \frac{1}{2\pi s \gamma^2} \right)^{\frac{n}{2}} \left\| (f * \varphi_\epsilon)(\cdot + \bar{w}_1 + \bar{w}_2) - f(\cdot + \bar{w}_1 + \bar{w}_2) \right\|^p
\]

\[
= 0
\]

where \( \lambda \) approaches to \(-iq\) through \( \mathbb{C}_+ \). This proves (i). Also, (ii) follows from Theorem 1.25 in [17].
Suppose that \( s = t_{p^*} \) for some \( p^* \in \{1, \ldots, k\} \) and let \( \gamma > 0 \). Then, for \( \lambda \in \mathbb{C}_+ \), for s.a.e. \((\xi_1, \xi_2) \in \mathbb{B}^k \times \mathbb{B}^k \) and for s.a.e. \( y \in C_0(\mathbb{B}) \), we have
\[
T_{\lambda}[T_{\lambda}[F|X_\tau]|X_\tau](\gamma y, \xi_2) = F_{\xi_1,\xi_2}(\gamma y)
\]
by Theorem 5. Then (i) follows trivially and (ii) follows when \( \gamma = 1 \). \( \square \)

**Remark 2.** (ii) of Theorem 7 holds for \( p = \infty \).

4. **Conditional convolution product and conditional transformation of conditional convolution product**

Now we define conditional convolution product and investigate relationships between conditional convolution product and conditional Fourier Feynman transform.

**Definition 8.** Let \( X_\tau \) be given as in Lemma 3 and let \( F, G \) be defined on \( C_0(\mathbb{B}) \). We define the conditional convolution product \([ (F * G)_\lambda |X_\tau] \) of \( F, G \) given \( X_\tau \) by the formula, for s.a.e. \( \xi \in \mathbb{B}^k \)
\[
[(F * G)_\lambda |X_\tau](y, \xi) = \begin{cases} 
E^{\text{anw}_\lambda} \left[ F \left( \frac{y + \cdot}{2^{\lambda^2}} \right) G \left( \frac{y - \cdot}{2^{\lambda^2}} \right) \right] |X_\tau| (\xi), & \lambda \in \mathbb{C}_+; \\
E^{\text{anf}_q} \left[ F \left( \frac{y + \cdot}{2^{q^2}} \right) G \left( \frac{y - \cdot}{2^{q^2}} \right) \right] |X_\tau| (\xi), & \lambda = -iq, \quad q \in \mathbb{R} - \{0\}
\end{cases}
\]
if they exist for s.a.e. \( y \in C_0(\mathbb{B}) \).

Using similar method in the proof Theorem 4.5 in [2], we have the following theorem.

**Theorem 9.** Let \( F_1 \in \mathcal{A}^{(p_1)}_{n_1, s} \) and \( F_2 \in \mathcal{A}^{(p_2)}_{n_2, s} \) be given by (3) with replacing \( f \) by \( f_1 \) and \( f_2 \), respectively, and let \( \frac{1}{p_1} + \frac{1}{p_2} = 1(1 \leq p_1, p_2 \leq \infty) \). Let \( X_\tau \) be given as in Lemma 3 and let \( t_{p^*-1} < s \leq t_{p^*} \) for some \( p^* \in \{1, \ldots, k\} \). Then, for \( \lambda \) in \( \mathbb{C}_+ \) and for s.a.e. \( \xi \in \mathbb{B}^k \), \([(F_1 * F_2)_\lambda |X_\tau]|(y, \xi) \) exists for s.a.e. \( y \in C_0(\mathbb{B}) \). Moreover, when \( t_{p^*-1} < s \leq t_{p^*} \), we have
\[
[(F_1 * F_2)_\lambda |X_\tau](y, \xi) = \int_{\mathbb{R}^n} f_1 \left( \frac{\bar{w}_y + \bar{w}_\xi + \bar{l}}{2^{\lambda^2}} \right) f_2 \left( \frac{\bar{w}_y - \bar{w}_\xi - \bar{l}}{2^{q^2}} \right) \phi_{\lambda^2}(\bar{l}) d\bar{l}
\]
with \([ [(F_1 \ast F_2)_{\lambda}|X_{\tau}] (\cdot, \xi) \in \mathcal{A}^{(1)}_{n,s} \text{ if } p_2 \leq p'_1 \) and \([ [(F_1 \ast F_2)_{\lambda}|X_{\tau}] (\cdot, \xi) \in \mathcal{A}^{(p_2)}_{n,s} \text{ if } p_2 \geq p'_1 \), where \( \Gamma, \phi_{\lambda \Gamma}, \bar{w}_{\xi} \) and \( \bar{w}_{y} \) are given by (4), (5), (6) and (7), respectively, and when \( s = t_{p^*} \), we have

\[
[[F_1 \ast F_2]_{\lambda} | X_{\tau}] (y, \xi) = \left[ F_1 \left( \frac{1}{2^{\frac{1}{2}}} (y + [\xi]) \right) \right] \left[ F_2 \left( \frac{1}{2^{\frac{1}{2}}} (y - [\xi]) \right) \right].
\]

The following corollary follows from Theorem 9 immediately.

**Corollary 10.** Let \( F_1, F_2 \in \bigcup_{1 \leq p \leq \infty} \mathcal{A}^{(p)}_{n,s} \). Let \( X_{\tau} \) be given as in Lemma 3 and let \( s = t_{p^*} \) for some \( p^* \in \{1, \ldots, k\} \). Then, for \( \lambda \in \mathbb{C}_{+} \) and for s.a.e. \( \xi \in \mathbb{B}^k \), \( [[F_1 \ast F_2]_{\lambda} | X_{\tau}] (y, \xi) \) exists for s.a.e. \( y \in C_0 (\mathbb{B}) \) and it is given by

\[
[[F_1 \ast F_2]_{\lambda} | X_{\tau}] (y, \xi) = \left[ F_1 \left( \frac{1}{2^{\frac{1}{2}}} (y + [\xi]) \right) \right] \left[ F_2 \left( \frac{1}{2^{\frac{1}{2}}} (y - [\xi]) \right) \right].
\]

**Theorem 11.** Let \( X_{\tau} \) be given as in Lemma 3 and let \( t_{p^*} - 1 < s < t_{p^*} \) for some \( p^* \in \{1, \ldots, k\} \). For s.a.e. \( \xi \in \mathbb{B}^k \) and for \( \lambda \in \mathbb{C}_{+} \), we have the following:

1. if \( F_1, F_2 \in \mathcal{A}^{(1)}_{n,s} \), then \( [[F_1 \ast F_2]_{\lambda} | X_{\tau}] (\cdot, \xi) \in \mathcal{A}^{(1)}_{n,s} \),
2. if \( F_1, F_2 \in \mathcal{A}^{(2)}_{n,s} \), then \( [[F_1 \ast F_2]_{\lambda} | X_{\tau}] (\cdot, \xi) \in \mathcal{A}^{(0)}_{n,s} \),
3. if \( F_1 \in \mathcal{A}^{(1)}_{n,s} \) and \( F_2 \in \mathcal{A}^{(2)}_{n,s} \), then \( [[F_1 \ast F_2]_{\lambda} | X_{\tau}] (\cdot, \xi) \in \mathcal{A}^{(2)}_{n,s} \),
4. if \( F_1 \in \mathcal{A}^{(1)}_{n,s} \) and \( F_2 \in \mathcal{A}^{(1)}_{n,s} \), then \( [[F_1 \ast F_2]_{\lambda} | X_{\tau}] (\cdot, \xi) \in \mathcal{A}^{(1)}_{n,s} \cap \mathcal{A}^{(2)}_{n,s} \),
5. if \( F_1 \in \mathcal{A}^{(1)}_{n,s} \) and \( F_2 \in \mathcal{A}^{(0)}_{n,s} \), then \( [[F_1 \ast F_2]_{\lambda} | X_{\tau}] (\cdot, \xi) \in \mathcal{A}^{(0)}_{n,s} \).

**Proof.** Let \( F_1, F_2 \) be given by (3) with replacing \( f \) by \( f_1, f_2 \), respectively. For \( \lambda \in \mathbb{C}_{+} \) let

\[
\begin{align*}
\lambda, \bar{u}, \bar{w}_{\xi} & = \int_{\mathbb{R}^n} f_1 \left( \frac{\bar{u} + \bar{w}_{\xi} + \bar{I}}{2^{\frac{1}{2}}} \right) f_2 \left( \frac{\bar{u} - \bar{w}_{\xi} - \bar{I}}{2^{\frac{1}{2}}} \right) \phi_{\lambda \Gamma} (\bar{I}) d\bar{I}
\end{align*}
\]

where \( \bar{u} \in \mathbb{R}^n \) and \( \Gamma, \phi_{\lambda \Gamma}, \bar{w}_{\xi} \) are given by (4), (5), (6), respectively.

1. By Theorem 9, it suffices to show that \( h(\lambda, \cdot, \bar{w}_{\xi}) \) is in \( L_1 (\mathbb{R}^n) \) for \( \lambda \in \mathbb{C}_{+} \). But this fact follows from the following:

\[
\int_{\mathbb{R}^n} | h(\lambda, \bar{u}, \bar{w}_{\xi}) | d\bar{u} \leq \left( \frac{|\lambda \Gamma|}{2\pi} \right)^{\frac{3}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| f_1 \left( \frac{\bar{u} + \bar{w}_{\xi} + \bar{I}}{2^{\frac{1}{2}}} \right) f_2 \left( \frac{\bar{u} - \bar{w}_{\xi} - \bar{I}}{2^{\frac{1}{2}}} \right) \right| d\bar{I} d\bar{u}
\]
\[
\begin{align*}
= & \left( \frac{|\lambda| |\Gamma|}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_1 \left( \bar{u}_1 + \frac{\bar{w}_\xi}{2^{\frac{1}{2}}} \right) f_2 \left( \bar{v}_2 - \frac{\bar{w}_\xi}{2^{\frac{1}{2}}} \right) d\bar{u}_1 d\bar{v}_2 \\
= & \left( \frac{|\lambda| |\Gamma|}{2\pi} \right)^{\frac{n}{2}} \|f_1\|_1 \|f_2\|_1,
\end{align*}
\]
where \( \bar{u} + \bar{v} = \bar{v}_1 \) and \( \bar{u} - \bar{v} = \bar{v}_2 \), by the change of variable theorem.

2. Note that, for \( \lambda \in C_0^\infty \), \( h(\lambda, \cdot, \bar{w}_\xi) \) is \( L_\infty(\mathbb{R}^n) \) since
\[
|h(\lambda, \bar{u}, \bar{w}_\xi)| \leq \left( \frac{|\lambda| |\Gamma|}{2\pi} \right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \left| f_1 \left( \frac{\bar{u} + \bar{w}_\xi + \bar{l}}{2^{\frac{1}{2}}} \right) f_2 \left( \frac{\bar{u} - \bar{w}_\xi - \bar{l}}{2^{\frac{1}{2}}} \right) \right| d\bar{l}
\leq \left( \frac{|\lambda| |\Gamma|}{\pi} \right)^{\frac{n}{2}} \|f_1\|_2 \|f_2\|_2
\]
for \( \bar{u} \in \mathbb{R}^n \) by Hölder inequality. Thus we have \( h(\lambda, \cdot, \bar{w}_\xi) \in C_0(\mathbb{R}^n) \) from a standard argument.

3. By Theorem 9, it suffices to show that \( h(\lambda, \cdot, \bar{w}_\xi) \) is in \( L_2(\mathbb{R}^n) \) for \( \lambda \in C_0^\infty \). But this fact follows from the following;
\[
\begin{align*}
\int_{\mathbb{R}^n} |h(\lambda, \bar{u}, \bar{w}_\xi)|^2 d\bar{u} \\
\leq \left( \frac{|\lambda| |\Gamma|}{2\pi} \right)^n \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} f_1 \left( \frac{\bar{u} + \bar{w}_\xi + \bar{l}}{2^{\frac{1}{2}}} \right) f_2 \left( \frac{\bar{u} - \bar{w}_\xi - \bar{l}}{2^{\frac{1}{2}}} \right) d\bar{l} \right]^2 d\bar{u}
= \left( \frac{|\lambda| |\Gamma|}{2\pi} \right)^n \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} f_1 \left( \frac{\bar{u} + \bar{w}_\xi + \bar{l}}{2^{\frac{1}{2}}} \right) f_2 \left( \frac{\bar{u} - \bar{w}_\xi - \bar{l}}{2^{\frac{1}{2}}} \right) d\bar{l} \right]^2
\times \left[ \int_{\mathbb{R}^n} f_1 \left( \frac{\bar{u} + \bar{w}_\xi + \bar{l}}{2^{\frac{1}{2}}} \right) f_2 \left( \frac{\bar{u} - \bar{w}_\xi - \bar{l}}{2^{\frac{1}{2}}} \right) d\bar{l} \right]
\times \left[ \int_{\mathbb{R}^n} f_1 \left( \frac{\bar{u} + \bar{w}_\xi + \bar{l}}{2^{\frac{1}{2}}} \right) f_2 \left( \frac{\bar{u} - \bar{w}_\xi - \bar{l}}{2^{\frac{1}{2}}} \right) d\bar{l} \right] d\bar{l}.
\end{align*}
\]
Let \( \bar{r} = \frac{\bar{u} + \bar{l}}{2^{\frac{1}{2}}} \) and \( \bar{s} = \frac{\bar{u} - \bar{l}}{2^{\frac{1}{2}}} \). Then we have
\[
\begin{align*}
\int_{\mathbb{R}^n} |h(\lambda, \bar{u}, \bar{w}_\xi)|^2 d\bar{u} \\
\leq \left( \frac{|\lambda| |\Gamma|}{2\pi} \right)^n 2^n \int_{\mathbb{R}^n} |f_1(\bar{r} + 2^{-\frac{1}{2}} \bar{w}_\xi)| \int_{\mathbb{R}^n} |f_1(\bar{s} + 2^{-\frac{1}{2}} \bar{w}_\xi)|
\times \int_{\mathbb{R}^n} |f_2(\sqrt{2} \bar{u} - \bar{r} - 2^{-\frac{1}{2}} \bar{w}_\xi)||f_2(\sqrt{2} \bar{u} - \bar{s} - 2^{-\frac{1}{2}} \bar{w}_\xi)| d\bar{u} d\bar{s} d\bar{r}
\leq \left( \frac{|\lambda| |\Gamma|}{2\pi} \right)^n 2^n \|f_1\|_1^2 \|f_2\|_2^2
\end{align*}
\]
by Hölder inequality.
4. It follows from 1 and 3.
5. It follows immediately and it is trivial.

By Theorems 5 and 9, using similar method in the proof of Theorem 4.8 in [2], we have the following theorem.

**THEOREM 12.** Let $F_1, F_2 \in \bigcup_{1 \leq p \leq \infty} \mathcal{A}^{(p)}_{n,s}$ be given by (3) with replacing $f$ by $f_1, f_2$, respectively, and let $X_\tau$ be given as in Lemma 3. Then, for $\lambda \in \mathbb{C}_+$ and for s-a.e. $(\xi_1, \xi_2) \in \mathbb{B}^k \times \mathbb{B}^k$, we have

$$T_\lambda \left[ ((F_1 * F_2)_\lambda | X_\tau)(\cdot, \xi_1) | X_\tau \right](y, \xi_2) = \left[ T_\lambda \left[ F_1 | X_\tau \right] \left( \frac{y}{2^{\frac{1}{2}}} , \frac{\xi_2 + \xi_1}{2^{\frac{1}{2}}} \right) \right] \left[ T_\lambda \left[ F_2 | X_\tau \right] \left( \frac{y}{2^{\frac{1}{2}}} , \frac{\xi_2 - \xi_1}{2^{\frac{1}{2}}} \right) \right],$$

for s-a.e. $y \in C_0(\mathbb{B})$.

The following theorem shows that the conditional Fourier-Feynman transform of conditional convolution product of some cylinder functions is a product of conditional transforms. For convenience, if $\lambda = -iq (q \in \mathbb{R} \setminus \{0\})$, then we denote $[(F_1 * F_2)_\lambda | X_\tau](\cdot, \xi_1)$ by $[(F_1 * F_2)_q | X_\tau](\cdot, \xi_1)$ in the following theorem.

**THEOREM 13.** Let $X_\tau$ be given as in Lemma 3 and let $q$ be a non-zero real number.

1. Let $F_1, F_2 \in \mathcal{A}^{(1)}_{n,s}$. Then, for s-a.e. $(\xi_1, \xi_2) \in \mathbb{B}^k \times \mathbb{B}^k$, we have

$$T_q^{(1)} \left[ ((F_1 * F_2)_q | X_\tau)(\cdot, \xi_1) | X_\tau \right](y, \xi_2) = \left[ T_q^{(1)} \left[ F_1 | X_\tau \right] \left( \frac{y}{2^{\frac{1}{2}}} , \frac{\xi_2 + \xi_1}{2^{\frac{1}{2}}} \right) \right] \left[ T_q^{(1)} \left[ F_2 | X_\tau \right] \left( \frac{y}{2^{\frac{1}{2}}} , \frac{\xi_2 - \xi_1}{2^{\frac{1}{2}}} \right) \right],$$

for s-a.e. $y \in C_0(\mathbb{B})$.

2. Let $F_1 \in \mathcal{A}^{(1)}_{n,s}$ and $F_2 \in \mathcal{A}^{(2)}_{n,s}$. Then, for s-a.e. $(\xi_1, \xi_2) \in \mathbb{B}^k \times \mathbb{B}^k$, we have

$$T_q^{(2)} \left[ ((F_1 * F_2)_q | X_\tau)(\cdot, \xi_1) | X_\tau \right](y, \xi_2) = \left[ T_q^{(1)} \left[ F_1 | X_\tau \right] \left( \frac{y}{2^{\frac{1}{2}}} , \frac{\xi_2 + \xi_1}{2^{\frac{1}{2}}} \right) \right] \left[ T_q^{(2)} \left[ F_2 | X_\tau \right] \left( \frac{y}{2^{\frac{1}{2}}} , \frac{\xi_2 - \xi_1}{2^{\frac{1}{2}}} \right) \right],$$

for s-a.e. $y \in C_0(\mathbb{B})$. 
3. Let \( F_1 \in \mathcal{A}_{n,s}^{(1)} \) and \( F_2 \in \mathcal{A}_{n,s}^{(1)} \cap \mathcal{A}_{n,s}^{(2)} \). Then, for s.a.e. \((\xi_1, \xi_2) \in \mathbb{B}^k \times \mathbb{B}^k\), we have
\[
T_q^{(1)}[[F_1 \ast F_2]_q|X_{\tau}](y, \xi_1, \xi_2) = \left[ T_q^{(1)}[F_1|X_{\tau}](y, \frac{\xi_2 + \xi_1}{2}) \right] \left[ T_q^{(2)}[F_2|X_{\tau}](y, \frac{\xi_2 - \xi_1}{2}) \right]
\]
and
\[
T_q^{(2)}[[F_1 \ast F_2]_q|X_{\tau}](y, \xi_1, \xi_2) = \left[ T_q^{(1)}[F_1|X_{\tau}](y, \frac{\xi_2 + \xi_1}{2}) \right] \left[ T_q^{(2)}[F_2|X_{\tau}](y, \frac{\xi_2 - \xi_1}{2}) \right]
\]
for s.a.e. \( y \in C_0(\mathbb{B}) \).

**Proof.** The results follow from Corollary 10, Theorems 6, 11 and 12. \(\square\)

5. **Stability theories**

Let \( \mathcal{H} \) be an infinite dimensional separable real Hilbert space and let \( \Delta_n = \{(s_1, s_2, \ldots, s_n) \in [0, T]^n : 0 = s_0 < s_1 < s_2 < \cdots < s_n \leq T \} \) for any fixed \( n \in \mathbb{N} \).

Let \( \mathcal{M}''_n = \mathcal{M}''_n(\Delta_n \times \mathcal{H}^n) \) be the class of all complex Borel measures on \( \Delta_n \times \mathcal{H}^n \) and let \( \|\mu\| = \text{var} \mu \), the total variation of \( \mu \) in \( \mathcal{M}''_n \). Let \( \mathcal{S}''_{n,\mathbb{B}} = \mathcal{S}''_{n,\mathbb{B}}(\Delta_n \times \mathcal{H}^n) \) be the space of functions of the form
\[
(11) \quad F_n(x) = \int_{\Delta_n \times \mathcal{H}^n} \exp \left\{ i \sum_{j=1}^{n} (h_j, x(s_j)) \right\} \, d\mu_{F_n}(s, \bar{h})
\]
for s.a.e. \( x \in C_0(\mathbb{B}) \), where \( s = (s_1, \ldots, s_n) \), \( \bar{h} = (h_1, \ldots, h_n) \) and \( \mu_{F_n} \in \mathcal{M}''_n \). Here we take \( \|F_n\|_n = \inf\{\|\mu_{F_n}\|\} \), where the infimum is taken over all \( \mu_{F_n} \)'s so that \( F_n \) and \( \mu_{F_n} \) are related by (11).

Let \( \mathcal{M}'' = \mathcal{M}''(\sum \Delta_n \times \mathcal{H}^n) \) be the class of all sequences \( \{\mu_n\} \) of measures such that each \( \mu_n \in \mathcal{M}''_n \) and \( \sum_{n=1}^{\infty} \|\mu_n\| < \infty \). Let \( \mathcal{S}''_{\mathbb{B}} = \mathcal{S}''_{\mathbb{B}}(\sum \Delta_n \times \mathcal{H}^n) \) be the space of functions of the form
\[
(12) \quad F(x) = \sum_{n=1}^{\infty} F_n(x),
\]
s.a.e. $x \in C_0(\mathbb{B})$, where each $F_n \in S''_{n,\mathbb{B}}$ and $\sum_{n=1}^{\infty} \|F_n\|'' < \infty$. The norm of $F$ is defined by $\|F\|'' = \inf \{\sum_{n=1}^{\infty} \|F_n\|''\}$, where the infimum is taken over all representations of $F$ given by (12).

Note that if $n$ and $l$ are positive integers then $S''_{n,\mathbb{B}} \subset S''_{n+l,\mathbb{B}}$ and $S''_{n,\mathbb{B}} \subset S''_{n',\mathbb{B}}$ for all $n \in \mathbb{N}$. Moreover we can show that $S''_{n,\mathbb{B}}$ is a Banach space and $S''_{\mathbb{B}}$ is a Banach algebra.

**Theorem 14.** Let $F_n \in S''_{n,\mathbb{B}}$ be given by (11) and let $X_T$ be given as in Lemma 3. Let $q \in \mathbb{R} \setminus \{0\}$ and $1 \leq p \leq \infty$. Then, for s.a.e. $\vec{\xi}$ in $\mathbb{B}^k$, $T_q^{(p)}[F_n|X_T](y, \vec{\xi})$ exists for s.a.e. $y \in C_0(\mathbb{B})$ and it is given by

$$T_q^{(p)}[F_n|X_T](y, \vec{\xi}) = \sum_{j_1 + \cdots + j_k = n} \int_{\Delta'_{n,j_1,\ldots,j_k} \times H'} H_n(y, \vec{\xi}, \vec{s}, \vec{h})G_n(-iq, \bar{\tau}, \vec{s}, \vec{h})d\mu_F(s, h)$$

where for $j_1 + \cdots + j_k = n$

$$\vec{s} = (s_{1,1}, \ldots, s_{1,j_1}, s_{2,1}, \ldots, s_{2,j_2}, \ldots, s_{k,1}, \ldots, s_{k,j_k});$$

$$\Delta'_{n,j_1,\ldots,j_k} = \{\vec{s}: 0 = s_{1,0} < s_{1,1} < \cdots < s_{1,j_1} \leq t_1 < s_{2,1} < \cdots < s_{2,j_2} \leq t_2 < \cdots < s_{k,j_k} \leq t_k = T\},$$

$$\vec{h} = (h_{1,1}, \ldots, h_{1,j_1}, h_{2,1}, \ldots, h_{2,j_2}, \ldots, h_{k,1}, \ldots, h_{k,j_k});$$

$$H_n(y, \vec{\xi}, \vec{s}, \vec{h}) = \exp \left\{ i \sum_{\alpha=1}^{k} \sum_{\beta=1}^{j_\alpha} (h_{\alpha,\beta}, y(s_{\alpha,\beta}) + [\xi](s_{\alpha,\beta})) \right\},$$

$$t_0 = s_{1,0} = 0, \quad t_\alpha = s_{\alpha+1,0} = s_{\alpha,j_\alpha+1} (\alpha = 1, \ldots, k-1),$$

$$s_{k,j_k+1} = t_k = T;$$

$$l_{\alpha,v} = s_{\alpha,v} - s_{\alpha,v-1}, \text{ for } \alpha = 1, \ldots, k; \text{ for } v = 1, \ldots, j_\alpha+1,$$

$$\bar{\tau} = (t_1, \ldots, t_k);$$
(21) \[ G_n(\lambda, \vec{\tau}, \vec{s}, \vec{h}) \]
\[ = \exp \left\{ -\frac{1}{2\lambda} \sum_{\alpha=1}^{k} \sum_{v=1}^{j_\alpha+1} l_{\alpha,v} \left( \sum_{\beta=1}^{\beta=v} \frac{t_{\alpha-1} - s_{\alpha,\beta}}{t_{\alpha} - t_{\alpha-1}} h_{\alpha,\beta} \right) \right\} \]
\[ + \sum_{\beta=v}^{j_\alpha} \frac{t_{\alpha} - s_{\alpha,\beta}}{t_{\alpha} - t_{\alpha-1}} h_{\alpha,\beta} \right\}^2 \right\} \]
with \( \lambda \in \mathbb{C}_+ \).

Proof. For \( \lambda > 0 \), for s-a.e. \( \vec{\xi} \) in \( \mathbb{B}^k \) and for s-a.e. \( y \) in \( C_0(\mathbb{B}) \), we have
\[
E[F_n(y + \lambda^{-1/2}(x - [x]) + [\vec{\xi}])]
\]
\[
= \int_{\Delta_n \times \mathbb{H}^n} \exp \left\{ i \sum_{j=1}^{n} (h_j, y(s_j) + [\vec{\xi}](s_j)) \right\} \int_{C_0(\mathbb{B})} \exp \left\{ i\lambda^{-1/2} \sum_{j=1}^{n} (h_j, x(s_j) - [x](s_j)) \right\} dm_{\mathbb{B}}(x) d\mu_{F_n}(\vec{s}, \vec{h})
\]
by Fubini theorem where \( \vec{s} = (s_1, \ldots, s_n) \) and \( \vec{h} = (h_1, \ldots, h_n) \). Let \( \vec{s}, \Delta'_{n,j_1,\ldots,j_k}, \vec{h} \) and \( H_n \) be given by (14), (15), (16) and (17), respectively. Then, by Lemma 2, we have
\[
E[F_n(y + \lambda^{-1/2}(x - [x]) + [\vec{\xi}])]
\]
\[
= \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n,j_1,\ldots,j_k} \times \mathbb{H}^n} H_n(y, \vec{\xi}, \vec{s}, \vec{h}) \prod_{\alpha=1}^{k} \left[ \int_{\mathbb{B}^{j_\alpha+1}} \exp \left\{ i\lambda^{-1/2} \sum_{\beta=1}^{j_\alpha} \right\} \left( h_{\alpha,\beta}, \sum_{v=1}^{\beta} \sqrt{l_{\alpha,v} x_{\alpha,v}} - \frac{s_{\alpha,\beta} - t_{\alpha-1}}{t_{\alpha} - t_{\alpha-1}} \sum_{v=1}^{j_\alpha+1} \sqrt{l_{\alpha,v} x_{\alpha,v}} \right) \right] \right\} \right) \]
\[ \left\{ dm^{j_\alpha+1}(\vec{x}_\alpha) \right\} d\mu_{F_n}(\vec{s}, \vec{h}) \]
by Wiener integration theorem (Theorem 1) where \( \vec{x}_\alpha = (x_{\alpha,1}, \ldots, x_{\alpha,j_\alpha+1}) \) and \( l_{\alpha,v} \) is given by (19) with (18). Let \( \vec{\tau} \) and \( G_n \) be given by (20) and (21), respectively. Then we have
\[
E[F_n(y + \lambda^{-1/2}(x - [x]) + [\vec{\xi}])]
\]
\[
= \sum_{j_1+\cdots+j_k=n} \int_{\Delta'_{n,j_1,\ldots,j_k} \times \mathbb{H}^n} H_n(y, \vec{\xi}, \vec{s}, \vec{h}) \prod_{\alpha=1}^{k} \left[ \int_{\mathbb{B}^{j_\alpha+1}} \exp \left\{ i\lambda^{-1/2} \sum_{v=1}^{j_\alpha+1} \right\} \left( x_{\alpha,v} \right) \right] \right\} \right) \right) \]
\[ \left\{ dm^{j_\alpha+1}(\vec{x}_\alpha) \right\} d\mu_{F_n}(\vec{s}, \vec{h}) \]
\[
\left( \sqrt{\frac{1}{\alpha_v} \left( \sum_{\beta=1}^{\nu-1} \frac{t_{\alpha-1} - s_{\alpha,\beta}}{t_{\alpha} - t_{\alpha-1}} h_{\alpha,\beta} + \sum_{\beta=\nu}^{j_\alpha} \frac{t_{\alpha} - s_{\alpha,\beta}}{t_{\alpha} - t_{\alpha-1}} h_{\alpha,\beta} \right), \bar{x}_{\alpha,v} \right) \sim \right) \nonumber
\]

\[
dm^{j_\alpha+1}(\bar{x}_\alpha) \right] d\mu_{F_{\alpha}}(\bar{s}, \bar{h}) \nonumber
\]

\[
= \sum_{j_1 + \ldots + j_k = n} \int_{A_{n;1,\ldots,n}^k \times \mathcal{H}^n} H_n(y, \xi, \bar{s}, \bar{h}) G_n(\lambda, \tau, \bar{s}, \bar{h}) d\mu_{F_{\alpha}}(\bar{s}, \bar{h}) \nonumber
\]

since \((h, \cdot)^\sim \) is normally distributed with mean 0 and variance \(|h|^2 (h \neq 0)\). By Morera’s theorem, we have

\[
T_\lambda[F_n|X_\tau](y, \xi) \nonumber
\]

\[
= \sum_{j_1 + \ldots + j_k = n} \int_{A_{n;1,\ldots,n}^k \times \mathcal{H}^n} H_n(y, \xi, \bar{s}, \bar{h}) G_n(\lambda, \tau, \bar{s}, \bar{h}) d\mu_{F_{\alpha}}(\bar{s}, \bar{h}) \nonumber
\]

for \(\lambda \in \mathbb{C}_+\). For \(1 \leq p \leq \infty\) let \(T_q^{(p)}[F_n|X_\tau](y, \xi)\) be given by (13). When \(p = 1\) we have

\[
|T_\lambda[F_n|X_\tau](y, \xi) - T_q^{(1)}[F_n|X_\tau](y, \xi)| \nonumber
\]

\[
\leq \sum_{j_1 + \ldots + j_k = n} \int_{A_{n;1,\ldots,n}^k \times \mathcal{H}^n} |G_n(\lambda, \tau, \bar{s}, \bar{h}) - G_n(-iq, \tau, \bar{s}, \bar{h})| \nonumber
\]

\[
d||\mu_{F_{\alpha}}||(|\bar{s}, \bar{h}) \nonumber
\]

and when \(1 < p \leq \infty\), for \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(\gamma > 0\), we have

\[
\int_{C_0(\mathbb{B})} |T_\lambda[F_n|X_\tau](\gamma y, \xi) - T_q^{(p)}[F_n|X_\tau](\gamma y, \xi)|^{p'} dm_B(y) \nonumber
\]

\[
\leq \left[ \sum_{j_1 + \ldots + j_k = n} \int_{A_{n;1,\ldots,n}^k \times \mathcal{H}^n} |G_n(\lambda, \tau, \bar{s}, \bar{h}) - G_n(-iq, \tau, \bar{s}, \bar{h})| \nonumber
\]

\[
d||\mu_{F_{\alpha}}||(|\bar{s}, \bar{h}) \right]^{p'} \nonumber
\]

Letting \(\lambda \to -iq\) through \(\mathbb{C}_+\), we have the results by the dominated convergence theorem. \(\square\)

**Theorem 15.** Let \(F \in S'_\mathbb{B}\) be given by (12) and let \(X_\tau\) be given as in Lemma 3. Let \(q \in \mathbb{R} - \{0\}\) and \(1 \leq p \leq \infty\). Then, for \(s\)-a.e. \(\xi\) in \(\mathbb{B}^k\), \(T_q^{(p)}[F|X_\tau](y, \xi)\) exists for \(s\)-a.e. \(y \in C_0(\mathbb{B})\) and it is given by

\[
T_q^{(p)}[F|X_\tau](y, \xi) = \sum_{n=1}^{\infty} T_q^{(p)}[F_n|X_\tau](y, \xi) \nonumber
\]
where $T_q^{(p)}[F_n|X_\tau]$ is given by (13) in Theorem 14.

Proof. Without loss of generality, we can assume $\sum_{n=1}^{\infty} \|\mu F_n\| < \infty$ where $F_n$ and $\mu F_n$ are related by (11). For $\lambda > 0$, for $s$-a.e. $\xi \in \mathbb{B}^k$ and for $s$-a.e. $y \in \mathcal{C}_0(\mathbb{B})$, we have

$$T_\lambda[F|X_\tau](y, \xi) = \sum_{n=1}^{\infty} T_\lambda[F_n|X_\tau](y, \xi)$$

by Theorem 14 and the dominated convergence theorem. Since $\sum_{n=1}^{\infty} T_\lambda[F_n|X_\tau](y, \xi)$ converges uniformly on $\mathcal{C}_+$ by Theorem 14, we have (23) for all $\lambda \in \mathcal{C}_+$. For $1 \leq p \leq \infty$ let $T_q^{(p)}[F|X_\tau](y, \xi)$ be given by (22). When $p = 1$ we have

$$|T_\lambda[F|X_\tau](y, \xi) - T_q^{(1)}[F|X_\tau](y, \xi)|$$

$$\leq \sum_{n=1}^{\infty} \sum_{j_1 + \ldots + j_k = n} \int_{H^0} |G_n(\lambda, \tau, \xi) - G_n(-iq, \rho, \xi, h)|$$

$$d\|\mu F_n\|(\xi, h)$$

and when $1 < p \leq \infty$, for $\frac{1}{p} + \frac{1}{p'} = 1$ and $\gamma > 0$, we have

$$\int_{\mathcal{C}_0(\mathbb{B})} |T_\lambda[F|X_\tau](\gamma y, \xi) - T_q^{(p)}[F|X_\tau](\gamma y, \xi)|^{p'} dm_{\mathbb{B}}(y)$$

$$\leq \left[ \sum_{n=1}^{\infty} \sum_{j_1 + \ldots + j_k = n} \int_{H^0} |G_n(\lambda, \tau, \xi, h) - G_n(-iq, \rho, \xi, h)|$$

$$d\|\mu F_n\|(\xi, h) \right]^{p'}.$$

Letting $\lambda \to -iq$ through $\mathcal{C}_+$, we have the results by the dominated convergence theorem.

Let $\mathcal{M}(\mathcal{H})$ be the class of all complex Borel measures on $\mathcal{H}$ and let $\mathcal{G}$ be the set of all $\mathbb{C}$-valued functions $\theta$ on $[0, T] \times \mathbb{B}$ which have the following form

$$(24) \quad \theta(s, y) = \int_{\mathcal{H}} \exp\{i(h, y)^\sim\} d\sigma_s(h)$$

where $\{\sigma_s : s \in [0, T]\}$ is the family from $\mathcal{M}(\mathcal{H})$ satisfying the following conditions;

1. for each Borel subset $E$ of $\mathcal{H}$, $\sigma_s(E)$ is a Borel measurable function of $s$ on $[0, T]$,
2. \(\|\sigma_s\| \in L_1([0, T])\).

Let \(\theta \in \mathcal{G}\) be given by (24) and for s-a.e. \(x\) in \(C_0(\mathbb{B})\) let

\[
F_n(x) = \left[ \int_0^T \theta(s, x(s))ds \right]^n
\]  

and

\[
F(x) = \exp \left\{ \int_0^T \theta(s, x(s))ds \right\}
\]

where \(n\) is any fixed natural number. For any Borel measurable subset \(E\) of \((0, T) \times \mathcal{H}\), let

\[
\mu(E) = \int_0^T \sigma_s(E(s))ds,
\]

where \(E(s) = \{h \in \mathcal{H} : (s, h) \in E\}\). By unsymmetric Fubini theorem ([8]), we have, for s-a.e. \(x\) in \(C_0(\mathbb{B})\),

\[
F_1(x) = \int_0^T \theta(s, x(s))ds = \int_0^T \int_{\mathcal{H}} \exp\{i(h, x(s))\} d\sigma_s(h) ds
\]

\[
= \int_{(0,T) \times \mathcal{H}} \exp\{i(h, x(s))\} d\mu(s, h),
\]

and \(\|\mu\| \leq \int_0^T \|\sigma_s\| ds\), so that \(F_1 \in S''_{1, \mathbb{B}} \subset S''_{\mathbb{B}}\). Moreover, we can show that \(F_n \in S''_{n, \mathbb{B}}\) for each \(n \in \mathbb{N}\). Thus we have the following theorems by Theorems 14 and 15.

**Theorem 16.** Let \(F_n\) be given by (25) and let \(X_\tau\) be given as in Lemma 3. Let \(q \in \mathbb{R} - \{0\}\) and \(1 \leq p \leq \infty\). For any Borel subset \(E\) of \(\Delta_n \times \mathcal{H}^n\) let

\[
\mu_{F_n} (E) = n! \int_{\Delta_n} \int_{\mathcal{H}^n} \chi_E(\vec{s}, \vec{\tilde{h}}) d\left( \prod_{j=1}^n \sigma_{s_j}(\vec{\tilde{h}}) \right) d\vec{s}
\]

where \(\vec{s} = (s_1, \ldots, s_n)\). Then, for s-a.e. \(\vec{\xi} \in \mathbb{B}^k\), \(T^{(p)}_q[F_n|X_\tau](y, \vec{\xi})\) exists for s-a.e. \(y \in C_0(\mathbb{B})\) and it is given by (13) with replacing \(\mu_{F_n}\) by \(\mu'_{F_n}\).

**Theorem 17.** Let \(F\) be given by (26) and let \(X_\tau\) be given as in Lemma 3. Let \(q \in \mathbb{R} - \{0\}\) and \(1 \leq p \leq \infty\). Then, for s-a.e. \(\vec{\xi} \in \mathbb{B}^k\), \(T^{(p)}_q[F|X_\tau](y, \vec{\xi})\) exists for s-a.e. \(y \in C_0(\mathbb{B})\) and it is given by

\[
T^{(p)}_q[F|X_\tau](y, \vec{\xi}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} T^{(p)}_q[F_n|X_\tau](y, \vec{\xi})
\]

where \(T^{(p)}_q[F_n|X_\tau]\) is given as in Theorem 16.
Let $\mathcal{F}(\mathbb{B})$ be the class of all functions of the form

$$
(27) \quad \phi(x_1) = \int_{\mathcal{H}} \exp\{i(h, x_1)\} d\nu(h)
$$

for s.a.e. $x_1$ in $\mathbb{B}$ where $\nu \in \mathcal{M}(\mathcal{H})$. For s.a.e. $x$ in $C_0(\mathbb{B})$ let

$$
(28) \quad K_n(x) = F_n(x) \phi(x(T)) \quad \text{and} \quad K(x) = F(x) \phi(x(T))
$$

where $F_n$ and $F$ are given by (25) and (26), respectively. Then for $\lambda > 0$, $x, y \in C_0(\mathbb{B})$ and $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{B}^k$, we have

$$
(29) \quad |\phi(y(T) + \lambda^{-\frac{1}{2}} (x(T) - [x](T)) + [\xi](T))| = |\phi(y(T) + \xi_k)| \leq \|\nu\|.
$$

Thus we have the following theorem.

**Theorem 18.** Let $K_n, K$ be given by (28) and let $X_\tau$ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. Then, for s.a.e. $\xi$ in $\mathbb{B}^k$, both $T_q^{(p)}[K_n|X_\tau](y, \xi)$ and $T_q^{(p)}[K|X_\tau](y, \xi)$ exist for s.a.e. $y \in C_0(\mathbb{B})$, and they are given by

$$
T_q^{(p)}[K_n|X_\tau](y, \xi) = T_q^{(p)}[F_n|X_\tau](y, \xi) \phi(y(T) + \xi_k)
$$

and

$$
T_q^{(p)}[K|X_\tau](y, \xi) = T_q^{(p)}[F|X_\tau](y, \xi) \phi(y(T) + \xi_k) + \sum_{n=1}^{\infty} \frac{1}{n!} T_q^{(p)}[K_n|X_\tau](y, \xi),
$$

where $T_q^{(p)}[F_n|X_\tau]$ and $T_q^{(p)}[F|X_\tau]$ are given as in Theorems 16 and 17, respectively.

Let $\zeta$ be a complex-valued Borel measure on $[0, T]$. Then $\zeta = \mu + \nu_d$ can be decomposed uniquely into the sum of a continuous measure $\mu$ and a discrete measure $\nu_d([6, p.142])$. Let $\delta_{\tau_j}$ denote the Dirac measure with total mass one concentrated at $\tau_j$.

Let $G^*$ be the set of all $C$-valued functions $\theta$ on $[0, T] \times \mathbb{B}$ which have the form (24) where $\{\sigma_s : s \in [0, T]\}$ is the family from $\mathcal{M}(\mathcal{H})$ satisfying the following conditions:

1. for each Borel subset $E$ of $\mathcal{H}$, $\sigma_s(E)$ is a Borel measurable function of $s$ on $[0, T]$,
2. $\|\sigma_s\| \in L_1([0, T], \mathcal{B}([0, T]), \|\zeta\|)$. 
In convenience, let

$$\zeta = \mu + \sum_{j=1}^{r} w_j \delta_{\tau_j}$$

where \(0 \leq \tau_1 < \cdots < \tau_r \leq T\) and the \(w_j\)'s are in \(\mathbb{C}\) for \(j = 1, \ldots, r(\in \mathbb{N})\), and let \(\theta \in \mathcal{G}^*\) be given by (24). For s-a.e. \(x\) in \(C_0(\mathbb{B})\) let

$$F_n(x) = \left[ \int_0^T \theta(s, x(s)) d\zeta(s) \right]^n$$

and

$$F(x) = \exp \left\{ \int_0^T \theta(s, x(s)) d\zeta(s) \right\}.$$

**Theorem 19.** Let \(F_n\) be given by (31) and let \(X_r\) be given as in Lemma 3. Let \(q \in \mathbb{R} - \{0\}\) and \(1 \leq p \leq \infty\). By reordering \(\tau_j\)'s, \(t_j\)'s and renaming \(\tau_j\)'s by \(\tau_{\alpha, u}\)'s \((\alpha = 1, \ldots, k; u = 1, \ldots, r_\alpha)\) let \(0 \leq \tau_{1, 1} < \tau_{1, 2} < \cdots < \tau_{1, r_1} \leq t_1 < \tau_{2, 1} < \cdots < \tau_{2, r_2} \leq t_2 < \cdots \leq t_{k-1} < \tau_{k, 1} < \cdots < \tau_{k, r_k} \leq t_k = T\), where \(r_1 + \cdots + r_k = r\). Then, for s-a.e. \(\tilde{x}\) in \(\mathbb{B}^k\), \(T_q^{(p)}[F_n|X_r](y, \tilde{x})\) exists for s-a.e. \(y \in C_0(\mathbb{B})\) and it is given by

$$T_q^{(p)}[F_n|X_r](y, \tilde{x})$$

$$= n! \sum_{q_1 + \cdots + q_k = n, \alpha = 1}^{k} \prod_{l_0 + l_1 + \cdots + l_{r_\alpha} = q_\alpha} L(q_\alpha) W(q_\alpha) \sum_{j_1 + \cdots + j_{r_\alpha} + 1 = l_0} \int_{\Delta_{l_0, j_1, \ldots, j_{r_\alpha} + 1}} \int_{H_{q_\alpha}} H_{q_\alpha}(y, \tilde{s}, \tilde{h}, \tilde{h}', \tilde{\tau}, \tilde{\tau}) G_{q_\alpha}(-iq, \tilde{s}, \tilde{h}, \tilde{h}', \tilde{\tau}) d(\sigma_\tau \times \sigma_{\tau_\alpha})(\tilde{h}, \tilde{h}') d\mu_0^l(\tilde{s})$$

where for \(q_1 + \cdots + q_k = n\) and for \(\alpha = 1, \ldots, k\); for \(l_0 + l_1 + \cdots + l_{r_\alpha} = q_\alpha\)

$$L(q_\alpha) = \frac{1}{l_1! \cdots l_{r_\alpha}!}$$

and

$$w_{\alpha, u} = w_{r_1 + \cdots + r_{\alpha-1} + u} (u = 1, \ldots, r_\alpha);$$

$$W(q_\alpha) = \prod_{u=1}^{r_\alpha} w_{\alpha, u}^{l_\alpha}.$$
for \( j_1 + \cdots + j_{r_{a} + 1} = l_0 \)

\[ (37) \quad \vec{s} = (s_{0,1}, \ldots, s_{0,j_1}, s_{1,1}, \ldots, s_{1,j_2}, \ldots, s_{r_{a},1}, \ldots, s_{r_{a},j_{r_{a} + 1}}); \]

\[ (38) \quad \Delta_{0:j_1, \ldots, j_{r_{a} + 1}} = \{ \vec{s} : 0 \leq t_{\alpha-1} \leq s_{0,1} \leq \cdots \leq s_{0,j_1} \leq \tau_{\alpha,1} < s_{1,1} < \cdots < s_{1,j_2} < \tau_{\alpha,2} < \cdots < \tau_{\alpha,\alpha} \leq s_{r_{a},1} \leq \cdots \leq s_{r_{a},j_{r_{a} + 1}} \leq t_{\alpha} \leq T \}; \]

\[ (39) \quad \vec{h} = (h_{0,1}, \ldots, h_{0,j_1}, h_{1,1}, \ldots, h_{1,j_2}, \ldots, h_{r_{a},1}, \ldots, h_{r_{a},j_{r_{a} + 1}}) \]

and

\[ (40) \quad \vec{\tau}_{\alpha} = (\tau_{\alpha,1}, \ldots, \tau_{\alpha,\alpha}); \]

\[ (41) \quad \sigma_{\vec{s}} = \prod_{u=0}^{r_{a}} \prod_{v=1}^{j_{u}+1} \sigma_{s_{u,v}}, \quad \sigma_{\vec{\tau}_{\alpha}} = \prod_{u=1}^{r_{a}} \prod_{l_u=1}^{l_u} \sigma_{\tau_{\alpha,u}} \]

and

\[ (42) \quad \vec{h}' = (h_{0,1}'_1, \ldots, h_{0,j_1}'_1, h_{1,1}'_2, \ldots, h_{1,j_2}'_2, \ldots, h_{r_{a},1}'_{r_{a}}, \ldots, h_{r_{a},j_{r_{a} + 1}}'_{r_{a}}); \]

for \( u = 0, \ldots, r_{a} - 1 \)

\[ (43) \quad \begin{align*} h_{u,j_{u}+1+1} &= \sum_{\beta=1}^{l_{u}+1} h_{u+1,1}\beta, \quad s_{u,j_{u}-1+1} = \tau_{\alpha,u+1} = s_{u+1,0}, \\
    h_{r_{a},j_{r_{a} + 1} + 1} &= 0 \text{ and } s_{r_{a},j_{r_{a} + 1} + 1} = t_{\alpha}; \end{align*} \]

\[ (44) \quad \begin{align*} H_{\alpha} (y, \vec{s}, \vec{h}, \vec{h}', \vec{\xi}, \vec{\tau}_{\alpha}) &= \exp \left\{ i \sum_{u=0}^{r_{a}} \sum_{v=1}^{j_{u}+1} (h_{u,v}, y(s_{u,v}) + [\vec{\xi}(s_{u,v})]) \right\} \end{align*} \]

and for \( a = 0, 1, \ldots, r_{a} \); for \( b = 1, \ldots, j_{a+1} + 1 \)

\[ (45) \quad \beta_{a,b} = s_{a,b} - s_{a,b-1} \text{ with } s_{0,0} = t_{a-1}; \]
for $\lambda \in \mathbb{C}_-^+$

\begin{align}
G_{q_\alpha}(\lambda, \bar{s}, \bar{h}, \bar{h}', \bar{r}_\alpha) &= \exp\left\{ -\frac{1}{2\lambda} \sum_{u=0}^{r_\alpha} \sum_{v=1}^{j_{u+1}+1} \beta_{u,v} \left[ \sum_{a=0}^{u-1} \sum_{b=1}^{r_{a+1}+1} \frac{t_{a-1} - s_{a,b}}{t_{a} - t_{a-1}} h_{a,b} + \sum_{b=1}^{v-1} \frac{t_{a-1} - s_{u,b}}{t_{a} - t_{a-1}} h_{u,b} + \sum_{b=v}^{j_{u+1}+1} \frac{t_{a} - s_{a,b}}{t_{a} - t_{a-1}} h_{a,b} \right]^2 \right\}.
\end{align}

Proof. For $q_1 + \cdots + q_k = n$ let

\[ Q(n) = q_1! \cdots q_k! \]

and let $w_{a,u}$ be given by (35). For $\lambda > 0$, for s-a.e. $\xi \in \mathbb{B}^k$ and for s-a.e. $y \in C_0(\mathbb{B})$, we have

\[ E[F_n(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\xi])] = \int_{C_0(\mathbb{B})} \left[ \sum_{a=1}^{k} \left[ \int_{t_{a-1}}^{t_a} \theta(s, y(s) + \lambda^{-\frac{1}{2}}(x(s) - [x](s)) + [\xi](s)) d\mu(s) \right. \right. \\
+ \left. \left. \sum_{u=1}^{r_a} w_{a,u} \theta(\tau_{a,u}, y(\tau_{a,u}) + \lambda^{-\frac{1}{2}}(x(\tau_{a,u}) - [x](\tau_{a,u})) + [\xi](\tau_{a,u})) \right] \right]^{n} dm_{\mathbb{B}}(x) \]

by binomial expansion and Lemma 2 where $t_0 = 0$. Let $L(q_\alpha)$ and $W(q_\alpha)$ be given by (34) and (36), respectively. For $q_1 + \cdots + q_k = n$ and for $\alpha = 1, \ldots, k$; for $l_0 + l_1 + \cdots + l_r = q_\alpha$, let $\bar{s}_0 = (s_1, s_2, \ldots, s_{l_0})$ and
\[ \Delta_{t_0} = \{ \tilde{s}_0 : t_{\alpha-1} < s_1 < \cdots < s_{l_0} \leq t_\alpha \}. \text{ Then, we have} \]

\[ E[F_n(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\xi])] \]

\[ = n! \sum_{q_1 + \cdots + q_k = n} \prod_{\alpha = 1}^{k} \left[ \sum_{l_0 + l_1 + \cdots + l_\alpha = q_\alpha} L(q_\alpha) W(q_\alpha) \int_{C_0(B)} \prod_{j_1 = 1}^{l_0} \theta(s_{j_1}, y(s_{j_1})) \right] \]

\[ \left[ \prod_{u=1}^{r_\alpha} \left[ \theta(\tau_{\alpha,u}, y(\tau_{\alpha,u})) + \lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u}) - [x](\tau_{\alpha,u})) \right] d\mu_{l_0}^{(\tilde{s}_0)} \right] \int_{\Delta_{t_0,j_1,\ldots,j_{r_\alpha+1}}} \prod_{u=0}^{\alpha} \prod_{v=1}^{j_u+1} \theta(s_{u,v}, y(s_{u,v})) + \lambda^{-\frac{1}{2}}(x(s_{u,v}) - [x](s_{u,v})) + \left[ \xi(s_{u,v}) \right] d\mu^{\tilde{s}}(s) \int_{u=1}^{r_\alpha} \left[ \theta(\tau_{\alpha,u}, y(\tau_{\alpha,u})) + \lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u}) - [x](\tau_{\alpha,u})) + \left[ \xi(\tau_{\alpha,u}) \right] \right] d\mu^{(\tilde{s}_0)}(x) \]

where \( \tilde{s} \) and \( \Delta_{t_0,j_1,\ldots,j_{r_\alpha+1}} \) are given by \((37)\) and \((38)\), respectively. Let \( \tilde{h} \) and \( \tilde{h}_{\alpha} \) be given by \((39)\) and \((40)\), respectively. Then we have

\[ E[F_n(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\xi])] \]

\[ = n! \sum_{q_1 + \cdots + q_k = n} \prod_{\alpha = 1}^{k} \left[ \sum_{l_0 + l_1 + \cdots + l_\alpha = q_\alpha} L(q_\alpha) W(q_\alpha) \int_{C_0(B)} \prod_{j_1 = 1}^{l_0} \theta(s_{j_1}, y(s_{j_1})) \right] \]

\[ \left[ \prod_{u=1}^{r_\alpha} \left[ \theta(\tau_{\alpha,u}, y(\tau_{\alpha,u})) + \lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u}) - [x](\tau_{\alpha,u})) \right] d\mu_{l_0}^{(\tilde{s}_0)} \right] \int_{\Delta_{t_0,j_1,\ldots,j_{r_\alpha+1}}} \int_{H_{l_0}} \exp \left\{ \sum_{u=0}^{r_\alpha} \sum_{v=1}^{j_u+1} (h_{u,v}, y(s_{u,v})) + \lambda^{-\frac{1}{2}}(x(s_{u,v}) - [x](s_{u,v})) + \left[ \xi(s_{u,v}) \right] \right\} d\sigma_{\tilde{s}}(\tilde{h}) d\mu_{l_0}^{(\tilde{s}_0)}(\tilde{s}) \int_{H_{l_1+\cdots+l_{r_\alpha}}} \exp \left\{ \sum_{u=1}^{r_\alpha} \sum_{\beta=1}^{l_u} (h'_{u,\beta}, y(\tau_{\alpha,u})) + \lambda^{-\frac{1}{2}}(x(\tau_{\alpha,u}) - [x](\tau_{\alpha,u})) + \left[ \xi(\tau_{\alpha,u}) \right] \right\} d\sigma_{\tilde{\tau}_{\alpha}}(\tilde{h}') d\mu_{l_\alpha}^{(\tilde{\tau}_{\alpha})}(\tilde{h}') \]

\[ d\mu_\mathbb{B}(x) \]
where \( \sigma_{\tilde{\gamma}} \), \( \sigma_{\tilde{\tau}} \), are given by (41) and \( \tilde{h}' \) is given by (42). For \( u = 0, \ldots, r_{\alpha} - 1 \), let \( h_{u,j_{u+1}+1} \) and \( s_{u,j_{u+1}+1} \) be given by (43). Then we have

\[
E[F_n(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\tilde{\xi}])]
\]

\[
= n! \sum_{q_1 + \cdots + q_k = n} \prod_{\alpha=1}^{k} \left[ \sum_{l_0 + l_1 + \cdots + l_{\alpha} = q_{\alpha}} L(q_{\alpha}) W(q_{\alpha}) \int_{C_0(\mathbb{R})} \left[ \sum_{j_1 + \cdots + j_{\alpha+1} = l_0} \right. \right. \\
\left. \left. \int_{\Delta_{l_0,j_1,\ldots,j_{\alpha+1}}} \int_{H^{q_{\alpha}}} \exp \left\{ i \sum_{u=0}^{r_{\alpha}} \sum_{v=1}^{j_{u+1}+1} (h_{u,v}, y(s_{u,v}) + [\tilde{\xi}](s_{u,v})) \right\} \\
\times \exp \left\{ i \lambda^{-\frac{1}{2}} \sum_{u=0}^{r_{\alpha}} \sum_{v=1}^{j_{u+1}+1} (h_{u,v}, x(s_{u,v}) - [x]) \right\} \\
d(\sigma_{\tilde{\gamma}} \times \sigma_{\tilde{\tau}})(\tilde{h}, \tilde{h}') d\mu^{l_0}(\tilde{\gamma}) \right] d\mu_{q_{\alpha}}(x) \right]
\]

where \( h_{r_{\alpha},j_{r_{\alpha}+1}+1} = 0 \) and \( s_{r_{\alpha},j_{r_{\alpha}+1}+1} = t_{\alpha} \). For \( a = 0, 1, \ldots, r_{\alpha} \), for \( b = 1, \ldots, j_{a+1} + 1 \) let \( \beta_{a,b} \) be given by (45) and let \( H_{q_{\alpha}} \) be given by (44). Then we have

\[
E[F_n(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\tilde{\xi}])]
\]

\[
= n! \sum_{q_1 + \cdots + q_k = n} \prod_{\alpha=1}^{k} \left[ \sum_{l_0 + l_1 + \cdots + l_{\alpha} = q_{\alpha}} L(q_{\alpha}) W(q_{\alpha}) \int_{C_0(\mathbb{R})} \left[ \sum_{j_1 + \cdots + j_{\alpha+1} = l_0} \right. \right. \\
\left. \left. \int_{\Delta_{l_0,j_1,\ldots,j_{\alpha+1}}} \int_{H^{q_{\alpha}}} H_{q_{\alpha}}(y, \tilde{s}, \tilde{h}, \tilde{h}', \tilde{\xi}, \tilde{\tau}_{\alpha}) \int_{\mathbb{R}^{l_0+r_{\alpha+1}}} \exp \left\{ i \lambda^{-\frac{1}{2}} \sum_{u=0}^{r_{\alpha}} \right. \\
\sum_{u=0}^{j_{a+1}+1} \left( h_{u,v}, \sum_{a=0}^{u-1} \sum_{b=1}^{j_{a+1}+1} \sqrt{\beta_{a,b} y_{a,b}} + \sum_{b=1}^{u} \sqrt{\beta_{a,b} y_{a,b}} - \frac{s_{u,v} - t_{a-1}}{t_{\alpha} - t_{\alpha-1}} \sum_{a=0}^{r_{\alpha}} \right. \\
\left. \sum_{b=1}^{j_{a+1}+1} \sqrt{\beta_{a,b} y_{a,b}} \right) \right\} \right] \\
dm_{l_0+r_{\alpha}+1}(\tilde{\gamma})d(\sigma_{\tilde{\gamma}} \times \sigma_{\tilde{\tau}})(\tilde{h}, \tilde{h}') d\mu^{l_0}(\tilde{\gamma}) \right] d\mu_{q_{\alpha}}(x) \right]
\]

by Theorem 1(Wiener integration theorem) and Fubini theorem where \( \tilde{\gamma} = (y_0, 1, \ldots, y_0, j_1 + 1, y_1, 1, \ldots, y_1, j_2 + 1, \ldots, y_{r_{\alpha}-1}, 1, \ldots, y_{r_{\alpha}}, j_{r_{\alpha}+1} + 1) \). Let
$G_{qx}$ be given by (46). Then we have
\[
E[F_n(y + \lambda^{-1/2}(x - [x]) + [\xi])]
\]
\[
= n! \sum_{q_1 + \cdots + q_k = n} \prod_{\alpha=1}^{k} \sum_{l_0 + l_1 + \cdots + l_{r_\alpha} = q_\alpha} L(q_\alpha) W(q_\alpha) \left[ \sum_{j_1 + \cdots + j_{r_{\alpha} + 1} = l_0} \int_{\Delta_{l_0,j_1,\ldots,j_{r_{\alpha}+1}}} \int_{H_{q_\alpha}} H_{q_\alpha}(y, \vec{s}, \vec{h}, \vec{h'}, \vec{\xi}, \vec{\tau}) \exp \left\{ i \lambda^{-1/2} \sum_{u=0}^{r_\alpha} \sum_{j_u + 1} \beta_{u,v} \left( \sum_{a=0}^{u-1} \sum_{b=1}^{j_{u+1}} \frac{t_{a-1} - s_{a,b}}{t_{a} - t_{a-1}} h_{a,b} + \sum_{b=1}^{v-1} \frac{t_{a-1} - s_{u,b}}{t_{a} - t_{a-1}} h_{u,b} + \right) \right\} \right] \right. \left. \right. \left. dm_{l_0+r_{\alpha}+1}(\vec{y}) d(\sigma_\vec{s} \times \sigma_\vec{\tau}) (\vec{h}, \vec{h'}) d\mu_{l_0}(\vec{s}) \right]
\]
\[
= n! \sum_{q_1 + \cdots + q_k = n} \prod_{\alpha=1}^{k} \sum_{l_0 + l_1 + \cdots + l_{r_\alpha} = q_\alpha} L(q_\alpha) W(q_\alpha) \left[ \sum_{j_1 + \cdots + j_{r_{\alpha} + 1} = l_0} \int_{\Delta_{l_0,j_1,\ldots,j_{r_{\alpha}+1}}} \int_{H_{q_\alpha}} H_{q_\alpha}(y, \vec{s}, \vec{h}, \vec{h'}, \vec{\xi}, \vec{\tau}) G_{q_\alpha} (\lambda, \vec{s}, \vec{h}, \vec{h'}, \vec{\tau}) \right] \left. \right. \left. d(\sigma_\vec{s} \times \sigma_\vec{\tau}) (\vec{h}, \vec{h'}) d\mu_{l_0}(\vec{s}) \right]
\]
since $(h, \cdot)\sim$ is normally distributed with mean 0 and variance $|h|^2 (h \neq 0)$. By analytic extension, $T_\lambda[F_n|X_\tau](y, \vec{\xi})$ exists and it is given by the above result for $\lambda \in \mathbb{C}_+$. For $1 \leq p \leq \infty$ let $T_\lambda^{(p)}[F_n|X_\tau](y, \vec{\xi})$ be given by (33). When $p = 1$ we have
\[
|T_\lambda[F_n|X_\tau](y, \vec{\xi}) - T_\lambda^{(1)}[F_n|X_\tau](y, \vec{\xi})| \leq 2 \left[ \int_0^T \|\sigma_s\|d\|\xi\||(s) \right]^n
\]
and when $1 < p \leq \infty$, for $\frac{1}{p} + \frac{1}{p'} = 1$ and $\gamma > 0$, we have
\[
\int_{C_0(\mathbb{R})} |T_\lambda[F_n|X_\tau](\gamma y, \vec{\xi}) - T_\lambda^{(p)}[F_n|X_\tau](\gamma y, \vec{\xi})|^{p'} dm_{\mathbb{R}}(y)
\]
\[
\leq \left[ 2 \left[ \int_0^T \|\sigma_s\|d\|\xi\||(s) \right]^n \right]^{p'}.
\]
Letting $\lambda \to -iq$ through $\mathbb{C}_+$, we have the results by the dominated convergence theorem.

\begin{remark}
Let $\zeta_1 = \mu + \sum_{j=1}^{\infty} w_j \delta_{\tau_j}$, where the $\tau_j$'s are in $[0,T]$ and the $w_j$'s are in $\mathbb{C}$. Using the following version of the $\delta_0$-nomial formula([9, p.41])

$$\left(\sum_{j=0}^{\infty} b_j\right)^n = \sum_{j=0}^{\infty} \sum_{q_0+\ldots+q_j = n, \text{q}_j \neq 0} \frac{n!}{q_0! \ldots q_j!} b_{q_0} \ldots b_{q_j},$$

we can show that the results in Theorem 19 hold with replacing $\zeta$ in (30), (31) by $\zeta_1$.
\end{remark}

\begin{theorem}
Let $F$ be given by (32) and let $X_\tau$ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. Then, for s-a.e. $\tilde{\xi}$ in $\mathbb{B}^k$, $T_{q}^{(p)}[F|X_\tau](y, \tilde{\xi})$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and it is given by

$$T_{q}^{(p)}[F|X_\tau](y, \tilde{\xi}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} T_{q}^{(p)}[F_n|X_\tau](y, \tilde{\xi})$$

where $T_{q}^{(p)}[F_n|X_\tau]$ is given by (33) in Theorem 19.
\end{theorem}

**Proof.** For s-a.e. $x \in C_0(\mathbb{B})$, by Maclaurin series expansion, we have

$$F(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} F_n(x)$$

where $F_n$ is given by (31), and for $\lambda \in \mathbb{C}_+$ we also have

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} |T_\lambda[F_n|X_\tau](y, \tilde{\xi})| \leq \exp\left\{ \int_0^T \|\sigma_s\|d\|\xi\|(s) \right\} < \infty$$

for s-a.e. $\tilde{\xi} \in \mathbb{B}^k$ and for s-a.e. $y \in C_0(\mathbb{B})$. Hence we have

$$T_\lambda[F|X_\tau](y, \tilde{\xi}) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} T_\lambda[F_n|X_\tau](y, \tilde{\xi})$$

for $\lambda \in \mathbb{C}_+$. For $1 \leq p \leq \infty$ let $T_{q}^{(p)}[F|X_\tau](y, \tilde{\xi})$ be given by (47). When $p = 1$ we have

$$|T_\lambda[F|X_\tau](y, \tilde{\xi}) - T_{q}^{(1)}[F|X_\tau](y, \tilde{\xi})| \leq 2 \exp\left\{ \int_0^T \|\sigma_s\|d\|\xi\|(s) \right\}$$
and when $1 < p \leq \infty$, for $\frac{1}{p'} + \frac{1}{p'} = 1$ and $\gamma > 0$, we have

$$\int_{C_0(\mathbb{B})} \left| T_n[F_n|X_T]((\gamma y, \xi) - T_n^{(p)}[F_n|X_T](\gamma y, \xi))^{p'} d\mu_{\mathbb{B}}(y) \right| \leq \left[ \frac{1}{2} \exp \left\{ \int_0^T \| \sigma_s \| d\| \zeta \|(s) \right\} \right]^{p'}.$$ 

Letting $\lambda \to -iq$ through $\mathbb{C}_+$, we have the results by the dominated convergence theorem.

For s.a.e. $x$ in $C_0(\mathbb{B})$ let

$$K_n(x) = F_n(x)\phi(x(T))$$

where $\phi$, $F_n$ and $F$ are given by (27), (31) and (32), respectively. By (29), we have the following theorem.

**Theorem 21.** Let $K_n$, $K$ be given by (48) and let $X_T$ be given as in Lemma 3. Let $q \in \mathbb{R} - \{0\}$ and $1 \leq p \leq \infty$. Then, for s.a.e. $\xi$ in $\mathbb{B}_k$, both $T_q^{(p)}[K_n|X_T](y, \xi)$ and $T_q^{(p)}[F|X_T](y, \xi)$ exist for s.a.e. $y \in C_0(\mathbb{B})$ and they are given by

$$T_q^{(p)}[K_n|X_T](y, \xi) = T_q^{(p)}[F_n|X_T](y, \xi)\phi(y(T) + \xi)$$

and

$$T_q^{(p)}[K|X_T](y, \xi) = T_q^{(p)}[F|X_T](y, \xi)\phi(y(T) + \xi)$$

$$= \phi(y(T) + \xi) + \sum_{n=1}^{\infty} \frac{1}{n!} T_q^{(p)}[K_n|X_T](y, \xi),$$

where $T_q^{(p)}[F_n|X_T]$ and $T_q^{(p)}[F|X_T]$ are given by (33) in Theorem 19 and (47) in Theorem 20, respectively.

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**References**


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