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EXISTENCE OF THE INTEGRAL SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we prove the existence of integral solutions, defined on a compact interval, for functional differential inclusion with nonlocal conditions and impulsive functional differential inclusions in a Banach space.

1. Introduction

In this paper, we prove the existence of the integral solutions, defined on a compact interval, for functional differential inclusion with nonlocal conditions and impulsive functional differential inclusions.

Let $(E, |\cdot|)$ be a Banach space. First we study the functional differential inclusion of the form

$$\begin{aligned} &y'(t) \in Ay(t) + F(t, y_t), \qquad \text{a.e. } t \in J = [0, b], \\ &y(t) + (\xi(y_{t_1}, \dots, y_{t_p}))(t) = \phi(t), \quad t \in J_0 = [-r, 0], \end{aligned} \tag{1.1}$$

where $A : D(A) \subseteq E \to E$ is a closed linear operator, $F : J \times C(J_0, E) \to 2^E$ is a bounded, closed, convex multivalued map, $\phi \in C(J_0, E)$, $0 < t_1 < t_2 < \cdots < t_p \leq b$, $p \in \mathbb{N}$, $\xi : [C(J_0, E)]^p \to C(J_0, E)$. For any continuous function y defined on the interval $J_1 = [-r, b]$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in J_0.$$

Next, we devoted to the study of the impulsive functional differential inclusion of the form

$$\begin{aligned} &y'(t) \in Ay(t) + F(t, y_t), \ t \in J = [0, b], \ t \neq t_k, \ k = 1, 2, \dots, m, \\ &\Delta y|_{t=t_k} = I_k(y(t_k^-)), \ k = 1, 2, \dots, m, \\ &y(t) = \phi(t), \ t \in J_0 = [-r, 0], \end{aligned} \tag{1.2}$$

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where $A : D(A) \subseteq E \to E$ is a closed linear operator, $F : J \times C(J_0, E) \to 2^E$ is a bounded, closed, and convex multivalued map, $\phi \in C(J_0, E)$ $(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, and $I_k \in C(E,\ E)$ $(k = 1, 2, \ldots, m)$ are bounded functions, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^-)$, $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

Equations of the type (1.1) or (1.2) arise in many areas of applied mathematics and such equations have received much attention in recent years. This paper is motivated by the recent papers of Adimy, Bouzahir and Ezzinbi [1], Benchora and Ntouyas [3] and Benchora and Henderson and Ntouyas [4]. In Adimy, Bouzahir and Ezzinbi [1], functional differential equation with infinite delay are studied, where $A : D(A) \subseteq E \to E$ has the Hille -Yosida condition while the domain is not dense. In Benchora and Ntouyas [3], Benchora and Henderson and Ntouyas [4], the neutral functional differential and integrodifferential inclusions and the impulsive semilinear neutral functional differential inclusions are studied. This paper is organized as follows. In section 2, we recall some basic definitions and preliminary facts which will be used throughout sections 3 and 4. In section 3, we establish existence theorems for the equation (1.1) and in section 4, we deal with equation (1.2). Our approaches are based on a fixed point theorem for condensing multivalued maps due to Martelli [7].

2. Preliminaries

In this section, we introduce notations, definitions, preliminary facts from multivalued analysis, lemmas and propositions which are used throughout this paper.

DEFINITION 2.1. Let E be a Banach space. A family $(S(t))_{t \geq 0} \subseteq L(E)$ is called an integrated semigroup if the following conditions are satisfied:

(i) $S(0) = 0$,

(ii) for any $x \in E$, $S(t)x$ is a continuous function of $t \geq 0$ with values in E,

(iii) for any $t, s \geq 0$, $S(s)S(t) = \displaystyle\int_0^s (S(t+\tau) - S(\tau))d\tau.$

If $(S(t))_{t \geq 0}$ is an integrated semigroup, exponentially bounded, i.e. for some $C,\ w > 0, \|S(t)\| \leq Ce^{wt},\ t \geq 0$, then the Laplace transform $R(\lambda) = \lambda \int_0^\infty e^{-\lambda t} S(t)dt$ exist for all λ with $Re(\lambda) > \omega$. $R(\lambda)$ is injective if and only if $(S(t))_{t\geq 0}$ is non-degenerate (i.e. if $S(t)x{=}0$, for all $t \geq 0$, implies that $x = 0$). $R(\lambda)$ satisfies the following expression

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$$

and in the case when $(S(t))_{t\geq 0}$ is non-degenerate, there exists a unique operator A satisfying $(\omega,\ \infty) \subset \rho(A)$ (the resolvent set of A) such that

$$R(\lambda) = R(\lambda,\ A) = (\lambda I - A)^{-1}$$

for all $Re(\lambda) > \omega$. This operator A is called the generator of $(S(t))_{t\geq 0}$.

DEFINITION 2.2. We say that a linear operator A satisfies the Hille -Yosida condition if there exist $M \geq 0$ and $\omega \in R$ such that $(\omega,\ +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \omega)^n \|R(\lambda,\ A)^n\|,\ n \in N,\ \lambda > \omega\} \leq M.$$

PROPOSITION 2.1. (Kellermann and Hieber [5]) *The followings are equivalent:*

(i) *A is the generator of a locally Lipschitz continuous integrated semigroup,*

(ii) *A satisfies the Hille -Yosida condition.*

In the sequel, we give results for the existence of solutions of the following Cauchy problem:

$$\begin{aligned} &\frac{dy}{dt}(t) = Ay(t) + f(t), \quad t \geq 0, \\ &y(0) = y_0 \in E, \end{aligned} \tag{2.1}$$

where A satisfies the Hille -Yosida condition, without being densely defined.

DEFINITION 2.3. Given $f \in L^1_{loc}$ $((0,\ +\infty)\ ; E)$ and $y_0 \in E$, we say that $y : [0,\ +\infty) \ \rightarrow\ E$ is an integral solution of (2.3) if the following assertions are true:

(i) $y \in C([0,\ +\infty)\ ;\ E)$,

(ii) $\int_0^t y(s)ds \in D(A) \quad$ for $t \geq 0$,

(iii) $y(t) = y_0 + A(\int_0^t y(s)ds) + \int_0^t f(s)ds \quad$ a.e. $t \geq 0$.

PROPOSITION 2.2. (Thieme [9]) *Let $A : D(A) \subseteq E \to E$ be a linear operator which satisfies the Hille -Yosida condition, $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A and $G : [0,T] \to E$, $T > 0$, be a Bochner-integrable function. Then, the function $K : [0,T] \to E$ defined by*

$$K(t) = \int_0^t S(t-s)G(s)ds$$

is continuously differentiable on $[0,T]$ and satisfies, for $\lambda > \omega$ and $t \in [0,T]$,

$$R(\lambda, A)K'(t) = \int_0^t S'(t-s)R(\lambda, A)G(s)ds.$$

This is suggestive to solve equation (2.1) by the variation of constants formula

$$y(t) = S'(t)y_0 + \frac{d}{dt}(\int_0^t S(t-s)f(s)ds), \qquad t \geq 0,$$

where $S(t)$ is the integrated semigroup generated by A.

Let $C(J,E)$ be a Banach space of continuous functions from J into E with the norm

$$||y|| = \sup\{ \ |y(t)| \ : \ t \in J \ \}.$$

Now we introduce a multivalued map. A multivalued map $G : E \to 2^E$ is convex(closed) valued, if $G(x)$ is convex(closed) for all $x \in E$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in E, for any bounded set B of E.

G is called upper semicontinuous(u.s.c.) on E if for each $x_* \in E$, the set $G(x_*)$ is a nonempty, closed subset of E, and if for each open set B of E containing $G(x_*)$, there exists an open neighborhood V of x_* such that $G(V) \subseteq B$. G is said to be completely continuous if $G(B)$ is relatively compact, for every bounded subset $B \subseteq E$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a close graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$). G has a fixed point if there is $x \in E$ such that $x \in Gx$. In the following $BCC(E)$ denotes the set of all nonempty bounded, closed and convex subsets of E. A multivalued map $G : J \to BCC(E)$ is said to be measurable if for each $x \in E$ the function $Y : J \to R$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x-z| : z \in G(t)\}$$

belongs to $L^1(J,R)$. An upper semicontinuous map $G : E \to 2^E$ is said to be condensing (Banas and Goebe [2]) if for any bounded subset

$B \subseteq E$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness.

LEMMA 2.1. (Martelli [7]) *Let E be a Banach space and $N : E \to BCC(E)$ a condensing map. If the set*

$$\Omega = \{y \in E : \lambda y \in Ny \text{ for some } \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Existence of solutions for functional differential inclusions

In this section, we consider the functional differential inclusion with nonlocal condition given by

$$\begin{aligned} &y'(t) \in Ay(t) + F(t, y_t), \qquad \text{a.e. } t \in J = [0, b], \\ &y(t) + (\xi(y_{t_1}, \dots, y_{t_p}))(t) = \phi(t), \quad t \in J_0 = [-r, 0], \end{aligned} \tag{3.1}$$

where $A : D(A) \subset E \to E$ is a closed linear operator, $F : J \times C(J_0, E) \to 2^E$ is a bounded, closed, convex multivalued map, $\phi \in C(J_0, E)$, $0 < t_1 < t_2 < \cdots < t_p \leq b$, $p \in \mathbb{N}$, $\xi : [C(J_0, E)]^p \to C(J_0, E)$.

In order to define the concept of the integral solution for (3.1), by comparison with the abstract Cauchy problem

$$\begin{aligned} &x'(t) = Ax(t) + f(t), \quad t \geq 0, \\ &x(0) = x_0, \end{aligned}$$

whose properties are well-known (see [3]), we associate the integral solution of (3.1)

$$\begin{aligned} y(t) &= S'(t)[\phi(0) - (\xi(y_{t_1}, \dots, y_{t_p}))(t)] \\ &\quad + \frac{d}{dt}\int_0^t S(t-s)g(s)ds, \ t \in J, \\ y(t) &= \phi(t) - \xi(y_{t_1}, \dots, y_{t_p})(t), \ t \in J_0, \end{aligned}$$

where $S(t)$ is an integrated semigroup and

$$g \in S_{F,y} = \{g \in L^1(J, E) \ : \ g(t) \in F(t, y_t) \text{ for a.e. } t \in J\}.$$

For the proof of theorem, we need the following hypotheses:

(H1) $A : D(A) \subset E \to E$ is a Hille -Yosida operator, i.e. there exist $M_1 \geq 0$ and $\omega \in R$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \omega)^n ||R(\lambda, A)^n||, \ n \in N, \ \lambda > \omega\} \leq M_1.$$

From Kellermann and Hieber [5], under this assumption, A is the generator of a locally Lipschitz-continuous integrated semigroup $\{S(t)\}_{t\geq 0}$ on E. In addition, $S^{'}(t)\ :\ \overline{D(A)}\ \rightarrow\ \overline{D(A)}$ is a C_0-semigroup satisfying

$$||S^{'}(t)y|| \leq M_1 e^{\omega t}||y|| \text{ for all } t \geq 0 \text{ and } y \in \overline{D(A)}.$$

(H2) The semigroup $(S^{'}(t))_{t\geq 0}$ is compact on $(\overline{D(A)}, \|\cdot\|)$ with $M_2 = \sup_{0\leq t\leq b}||S^{'}(t)||_{\overline{D(A)}}$.

(H3) ξ is completely continuous and there exists a constant Q such that $||\xi(y_{t_1}, \ldots, y_{t_p})(t)|| \leq Q$ for $y \in C(J_0, E)$.

(H4) $F : J \times C(J_0, E) \rightarrow BCC(E)\ ; (t, u) \rightarrow F(t, u)$ is measurable with respect to t for each $u \in C(J_0, E)$, u.s.c. with respect to u for each $t \in J$, and for each fixed $u \in C(J_0, E)$ the set $S_{F,u} = \{g \in L^1(J, E) : g(t) \in F(t, u)$ for a.e. $t \in J\}$ is nonempty.

(H5) $||F(t, u)|| = \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(||u||)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, R^+)$ and $\psi : R^+ \rightarrow (0, \infty)$ is continuous and increasing with $\int_0^b \hat{m}(s)ds < \int_c^{\infty} \frac{d\tau}{\psi(\tau)}$, where $c = M_1(||\phi|| + Q)$ and $\hat{m}(t) = M_1 M_2 p(t)$.

REMARK 3.1.

(i) If dim $E < \infty$, then for each $u \in C(J_0, E), S_{F,u} \neq \phi$ (see Lasota and Opial [6]).

(ii) $S_{F,u}$ is nonempty if and only if the function $Y : J \rightarrow R$ defined by $Y(t) = \inf\{|v| : v \in F(t, u)\}$ belongs to $L^1(J, R)$ (see Papageorgiou [8]).

LEMMA 3.1. (Lasota and Opial [6]). *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying* (H4) *and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S_F : C(I, X) \rightarrow BCC(C(I, X)),\ y \mapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y})$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

THEOREM 3.1. *Assume that hypotheses* (H1)-(H5) *hold. Then the equation* (3.1) *has at least one integral solution on $J_1 = [-r, b]$.*

Proof. Let $C = C(J_1, E)$ be the Banach space of continuous functions from J_1 into E endowed with the sup-norm

$$||y||_{\infty} = \sup\{|y(t)| : t \in [-r, b]\}, \text{ for } y \in C.$$

The main idea of the proof is to use the fixed point theorem. Consider the multivalued map, $N : C \to 2^C$ defined by

$$Ny = \left\{ \begin{array}{l} h \in C : \\ h(t) = \left\{ \begin{array}{ll} \phi(t) - (\xi(y_{t_1}, \dots, y_{t_p}))(t), & \text{if } t \in J_0 \\ S'(t)[\phi(0) - (\xi(y_{t_1}, \dots, y_{t_p}))(0)] & \\ \quad + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)g(s)ds, & \text{if } t \in J, \end{array} \right. \end{array} \right\},$$

where

$$g \in S_{F,y} = \{g \in L^1(J, E) \; : \; g(t) \in F(t, y_t) \quad \text{for a.e. } t \in J\}.$$

REMARK 3.2. It is clear that the fixed points of N are integral solutions to the equation (3.1).

We shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1. Ny is convex for each $y \in C$.

Indeed, let h_1, h_2 belong to Ny, then there exist g_1, $g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$h_1(t) = S'(t)[\phi(0) - (\xi(y_{t_1}, \dots, y_{t_p}))(0)] + \frac{d}{dt}\int_0^t S(t-s)g_1(s)ds$$

and

$$h_2(t) = S'(t)[\phi(0) - (\xi(y_{t_1}, \dots, y_{t_p}))(0)] + \frac{d}{dt}\int_0^t S(t-s)g_2(s)ds.$$

Let $0 \le \alpha \le 1$, then for each $t \in J$, we get

$$\begin{aligned}(\alpha h_1 + (1-\alpha)h_2)(t) &= S'(t)[\phi(0) - (\xi(y_{t_1}, \dots, y_{t_p}))(0)] \\ &\quad + \frac{d}{dt}\int_0^t S(t-s)[\alpha g_1(s) + (1-\alpha)g_2(s)]ds.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we obtain

$$\alpha h_1 + (1-\alpha)h_2 \in Ny.$$

We next will prove that N is a completely continuous operator. It suffices to show that the operator $N_1 : C \to 2^C$ defined by

$$N_1(y) = \left\{ h \in C \; : \; h(t) = \left\{ \begin{array}{ll} \phi(t) - (\xi(y_{t_1}, \dots, y_{t_p}))(t) & \text{if } t \in J_0, \\ \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)g(s)ds, & \text{if } t \in J, \end{array} \right. \right\}$$

is completely continuous.

Step 2. N_1 maps bounded sets into bounded sets in C.

It is enough to show that there exists a positive constant l such that for each $h \in N_1 y$, $y \in B_r = \{y \in C : ||y||_\infty \leq r\}$ one has $||h||_\infty \leq l$.

If $h \in N_1 y$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = \frac{d}{dt}\int_0^t S(t-s)g(s)ds.$$

From Proposition 2.2 and hypotheses (H1), (H2) and (H5) we obtain for $t \in J$,

$$\begin{aligned} |\lambda R(\lambda, A)h(t)| &= |\int_0^t S'(t-s)\lambda R(\lambda, A)g(s)ds| \\ &\leq \frac{\lambda}{\lambda - \omega} M_1 M_2 \sup_{y \in [0,r]} \psi(y)(\int_0^t p(s)ds), \qquad t \in J. \end{aligned}$$

Letting $\lambda \to \infty$, we see that

$$|h(t)| \leq M_1 M_2 \sup_{y \in [0,r]} \psi(y)(\int_0^t p(s)ds). \tag{3.2}$$

Then for each $h \in N_1(B_r)$ we have,

$$||h||_\infty \leq M_1 M_2 \sup_{y \in [0,r]} \psi(y)(\int_0^b p(s)ds) := \ell.$$

Step 3. N_1 maps bounded sets into equicontinuous sets of C.

Let $t_1, t_2 \in J$, $t_1 < t_2$ and $B_r = \{y \in C : ||y||_\infty \leq r\}$ be a bounded set of C. For each $y \in B_r$ and $h \in N_1 y$, there exists $g \in S_{F,y}$ such that

$$h(t) = \frac{d}{dt}\int_0^t S(t-s)g(s)ds, \qquad t \in J.$$

Thus we have

$$\begin{aligned} &\left|\lambda R(\lambda, A)(h(t_2) - h(t_1))\right| \\ = &\left|\int_{t_1}^{t_2} S'(t_2 - s)\lambda R(\lambda, A)g(s)ds\right| \\ &+ \left|\int_0^{t_1} (S'(t_2 - s) - S'(t_1 - s))\lambda R(\lambda, A)g(s)ds\right| \\ = &\left|\int_{t_1}^{t_2} S'(t_2 - s)\lambda R(\lambda, A)g(s)ds\right| \\ &+ \left|(S'(t_2 - t_1) - I)\int_0^{t_1} S'(t_1 - s)\lambda R(\lambda, A)g(s)ds\right|. \end{aligned}$$

Letting $\lambda \to +\infty$, by hypothesis (H1), we obtain,

$$\lim_{t_2 \to t_1} ||h(t_2) - h(t_1)|| = 0.$$

The equicontinuities for the case $t_1 < t_2 < 0$ and $t_1 \leq 0 \leq t_2$ are obvious. As a consequence of Step 2, Step 3, and Ascoli-Arzela theorem we can conclude that $N : C \to 2^C$ is a compact multivalued map, and therefore, a condensing map.

Step 4. N has a closed graph.

Let $y_n \to y_*, h_n \in Ny_n$ and $h_n \to h_*$. We shall prove that $h_* \in Ny_*$. $h_n \in Ny_n$ means that there exists $g_n \in S_{F,y_n}$ such that

$$\begin{aligned} h_n(t) = S^{'}(t)[\phi(0) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0)] \\ + \frac{d}{dt}\int_0^t S(t-s)g_n(s)ds, \quad t \in J. \end{aligned}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$\begin{aligned} h_*(t) = S^{'}(t)[\phi(0) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0)] \\ + \frac{d}{dt}\int_0^t S(t-s)g_*(s)ds, \quad t \in J. \end{aligned}$$

Clearly, we have that

$$\begin{aligned} &\Big\|(h_n - S^{'}(t)[\phi(0) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0)]) \\ &\quad -(h_* - S^{'}(t)[\phi(0) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0)])\Big\|_\infty \to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Consider the linear continuous operator

$$\Gamma : L^1(J, E) \to C^1(J, E), \quad g \to \Gamma(g)(t) = \frac{d}{dt}\int_0^t S(t-s)g(s)ds,$$

where $C^1(J, E)$ be the Banach space of continuous differential functions from J into E with the sup-norm.

From Lemma 3.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t) - S^{'}(t)[\phi(0) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0)] \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 3.1 that

$$h_*(t) - S^{'}(t)[\phi(0) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0)] = \frac{d}{dt}\int_0^t S(t-s)g_*(s)ds$$

for some $g_* \in \Gamma(S_{F,y_*})$.

Therefore N is a completely continuous multivalued map, u.s.c. with convex closed values. In order to prove that N has a fixed point, we need one more step.

Step 5. The set

$$\Omega = \{y \in C : \lambda y \in Ny, \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1} S'(t)(\phi(0) - (\xi(y_{t_1}, \ldots, y_{t_p}))(0)) \\ &+ \lambda^{-1} \frac{d}{dt} \int_0^t S(t-s) g(s) ds, \quad t \in J. \end{aligned}$$

This implies by (H1)-(H3), (H5) and (3.2) that for each $t \in J$ we have

$$|y(t)| \leq M_1[||\phi|| + Q] + M_1 M_2 \int_0^t p(s)\psi(\|y_s\|) ds.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(t)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality, we see that for $t \in J$

$$\begin{aligned} \mu(t) &\leq M_1[||\phi|| + Q] + M_1 M_2 \int_0^{t^*} p(s)\psi(\mu(s)) ds \\ &\leq M_1[||\phi|| + Q] + M_1 M_2 \int_0^t p(s)\psi(\mu(s)) ds. \end{aligned}$$

If $t^* \in J_0$, then $\mu(t) \leq ||\phi|| + Q$ and the previous inequality holds since $M_1 \geq 1$. Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$c = v(0) = M_1[||\phi|| + Q], \qquad \mu(t) \leq v(t),\ t \in J$$

and

$$v'(t) = M_1 M_2 p(t)\psi(\mu(t)),\ t \in J.$$

Using the nondecreasing characters of ψ, we get

$$\begin{aligned} v'(t) &\leq M_1 M_2 p(t)\psi(v(t)) \\ &= \hat{m}(t)[\psi(v(t))], \qquad t \in J. \end{aligned}$$

This implies for each $t \in J$, we obtain

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \int_0^b \hat{m}(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

This inequality implies that there exists a constant L, such that $v(t) \leq L$, $t \in J$, and hence $\mu(t) \leq L$, $t \in J$. Since for every $t \in J$, $||y_t|| \leq \mu(t)$, we have $||y||_\infty = \sup\{|y(t)| : -r \leq t \leq b\} \leq L$, where L depends only on b and on the functions p and ψ. This shows that Ω is bounded. Set $E = C$, then from Lemma 2.1 we deduce that N has a fixed point which is a solution of (3.1). □

4. Existence of solutions for impulsive functional differential inclusions

In this section we deal with the functional differential inclusions with impulsive effects given by

$$(4.1)\quad \begin{aligned} &y'(t) \in Ay(t) + F(t, y_t),\ t \in J = [0, b],\ t \neq t_k,\ k = 1, 2, \ldots, p,\\ &\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1, 2, \ldots, p,\\ &y(t) = \phi(t), \qquad t \in J_0 = [-r, 0], \end{aligned}$$

where $A : D(A) \subseteq E \to E$ is a closed linear operator, $F : J \times C(J_0, E) \to 2^E$ is a bounded, closed, and convex multivalued map, $\phi \in C([-r, 0], E)$ $(0 < r < \infty)$, $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = b$, and $I_k \in C(E, E)$ $(k = 1, 2, \ldots, p)$ are bounded functions, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$ respectively.

Similarly in Section 3, we obtain the integral solution of (4.1)

$$\begin{aligned} &y(t) = S'(t)\phi(0) + \frac{d}{dt}\int_0^t S(t-s)g(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)),\ t \in J,\\ &y(t) = \phi(t), \quad t \in J_0, \end{aligned}$$

where $g \in S_{F,y} = \{\ g \in L^1(J, E) \mid g(t) \in F(t, y_t) \text{ for a.e. } t \in J\ \}$.

We consider the space

$$Z = \{y : [-r, b] \to E\ :\ y_k \in C(J_k, E),\ k = 0, 1, \ldots, p$$

and there exist $y(t_k^-)$ and $y(t_k^+)$, with $y(t_k^-) = y(t_k), k = 1, 2, \ldots, p$,

$$y(t) = \phi(t),\ \forall t \in [-r, 0]\},$$

which is a Banach space with the norm

$$||y||_Z = \max\{||y_k||_{J_k} : k = 0, 1, \ldots, p\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}],\ k = 0, 1, \ldots, p$.

For the proof of theorem, we have some hypotheses.

(H6) There exist constants d_k such that $|I_k(y)| \leq d_k,\ k = 1, 2, \ldots, p$ for each $y \in E$.

(H7) $||F(t,u)|| = \sup\{|v| : v \in F(t,u)\} \leq p(t)\psi(||u||)$, for almost all $t \in J$ and all $u \in C([-r,0],E)$, where $p \in L^1(J, R^+)$ and $\psi : R^+ \to (0, +\infty)$ is continuous and increasing with

$$\int_0^b \hat{m}(s)ds < \int_{\bar{c}}^{\infty} \frac{d\tau}{\psi(\tau)},$$

where $\bar{c} = M_1||\phi|| + \sum_{k=1}^{p} d_k$, and $\hat{m}(t) = M_1 M_2 p(t)$.

THEOREM 4.1. *Assume that hypotheses* (H1), (H2), (H4), (H6) and (H7) *are satisfied. Then the equation* (4.1) *has at least one integral solution on* $J_1 = [-r, b]$.

Proof. Transform the problem into a fixed point problem. Consider the multivalued map, $N : Z \to 2^Z$ defined by

$$Ny = \left\{ \begin{array}{l} h \in C : \\ h(t) = \left\{ \begin{array}{ll} \phi(t), & \text{if } t \in J_0 \\ S'(t)\phi(0) + \dfrac{d}{dt}\displaystyle\int_0^t S(t-s)g(s)ds & \\ \quad + \displaystyle\sum_{0<t_k<t} I_k(y_{t_k}^-), & \text{if } t \in J, \end{array} \right. \end{array} \right\},$$

where $g \in S_{F,y} = \{g \in L^1(J,E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in J\}$.

REMARK 4.1. It is clear that the fixed points of N are solutions to (4.1).

As in Theorem 3.1 we can show that N is a completely continuous map, u.s.c. with convex closed values, and therefore a condensing map.

Here we repeat only the proof that the set

$$\Omega = \{y \in Z : \lambda y \in Ny, \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$ then $\lambda y \in Ny$ for some $\lambda > 1$. Thus for each $t \in J$,

$$y(t) = \lambda^{-1}S'(t)\phi(0) + \lambda^{-1}\frac{d}{dt}\int_0^t S(t-s)g(s)ds + \lambda^{-1}\sum_{0<t_k<t} I_k(y_{t_k}).$$

This implies by (H1), (H2), (H4), (H6) and (H7) that for each $t \in J$ we have

$$|y(t)| \leq M_1||\phi|| + M_1M_2\int_0^t p(s)\psi(||y_s||)ds + \sum_{k=1}^{p} d_k.$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\},\ 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$ that

$$\mu(t) \leq M_1||\phi|| + M_1M_2\int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^{p} d_k.$$

If $t^* \in J_0$ then $\mu(t) = ||\phi||$ and the previous inequality holds. Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$\bar{c} = v(0) = M_1||\phi|| + \sum_{k=1}^{p} d_k,\ \mu(t) \leq v(t),\ t \in J,$$

and

$$v'(t) = M_1M_2p(t)\psi(\mu(t)),\ t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq \hat{m}(t)\psi(v(t)),\ t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \int_0^b \hat{m}(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}\ .$$

This inequality implies that there exists a constant $\bar{L}$ such that $v(t) \leq \bar{L},\ t \in J$, and hence $\mu(t) \leq \bar{L},\ t \in J$. Since for every $t \in J,\ ||y_t|| \leq \mu(t)$, we have

$$||y||_Z \leq \max\{||\phi||, \bar{L}\} = \bar{L}'$$

where $\bar{L}'$ depends on b, on the functions p and ψ. This shows that Ω is bounded. Set $E = Z$. As a consequence of Lemma 2.1, we deduce that N has a fixed point which is a solution of (4.1). □

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