PRECONDITIONING FOR THE p-VERSION BOUNDARY ELEMENT METHOD IN THREE DIMENSIONS WITH TRIANGULAR ELEMENTS

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ABSTRACT. A preconditioning algorithm is developed in this paper for the iterative solution of the linear system of equations resulting from the p-version boundary element approximation of the three dimensional integral equation with hypersingular operators. The preconditioner is derived by first making the nodal and side basis functions locally orthogonal to the element internal bases, and then by decoupling the nodal and side bases from the internal bases. Its implementation consists of solving a global problem on the wire-basket and a series of local problems defined on a single element. Moreover, the condition number of the preconditioned system is shown to be of order $O((1 + \ln p)^7)$. This technique can be applied to discretization with triangular elements and with general basis functions.

1. Introduction

It is well-known that for the solution of the linear algebraic systems arising from the finite element and boundary element methods iterative methods can be very efficient if suitable preconditioning is incorporated. Indeed, over the past decades, the preconditioned iterative methods based on domain decompositions have been developed and applied widely for the finite element methods. For boundary element methods, extensive studies have also been made recently. For instance, Tran and

Stephan [20], Steinbach and Wendland [19], Heuer, Stephan and Tran [13] studied the preconditioning methods for the h- and h-p versions BEM for two dimensional boundary integral equations. More recently, Heuer [11], Heuer and Stephan [12] also extended their study to the three dimensional problems. In particular, in [11] an additive Schwarz method is developed for the p-version boundary element approximation of the hypersingular operator in three dimensions. Guo and Ainsworth [1] presented a rigorous analysis of this method, and proved strictly that the condition number of the preconditioned system can grow at most polylogarithmically in p, where p is the polynomial degree used in the p-version approximation. However, in the above work for three dimensional problems, the preconditioners are built on special basis functions that are tensorial products of one dimensional polynomials. Therefore, the algorithm is applicable only to the discretization with quadrilateral meshes. This limitation restricts greatly the applicability of the BEM, since partition with triangular elements is one of the most popular choices in practice, and an analysis of preconditioning for the p-version of the boundary element method with triangular elements is not available in the literature.

In this paper, we intend to remove the aforementioned restriction on the partition for building preconditioners in the case of the p-version approximation of boundary integral equations with hypersingular operators. We develop an algorithm that works with general partitions and with general basis functions. The basic idea comes from the well-known substructuring methods and additive Schwarz methods developed in, e.g., [6, 7, 10, 15]. We group the basis functions into two types: those associated with the element nodes and sides, or the so-called wire-basket, and those associated with element interiors. We first minimize the coupling between the two groups by a local orthogonalization procedure. Then the preconditioner is designed to retain only the coupling between the wire-basket components and between the internal components of each element. To multiply the preconditioner to a residual vector, one needs to solve a global problem, which is associated with the L^2-projection on the wire-basket, and a series of local problems defined only on a single element. The stiffness matrix of the global problem on the wire-basket is in general tridiagonal, and the matrices of the local problems have already been made available when the linear system of equations for the BEM is formed. Thus the preconditioner is not expensive to implement. As for the effectiveness of the algorithm, we prove

that the condition number of the preconditioned system can grow at most at the rate of $O((1 + \ln p)^7)$.

The paper is organized as follows: in Section 2, we describe the model problem and the p-version boundary element approximation of the model problem. In Section 3, we introduce the preconditioner and its matrix representation. In Section 4, a number of lemmas are presented, and finally in Section 5 the condition number of the preconditioned system is analyzed rigorously.

2. BEM for model problem

We first introduce some notation. Let $\Omega \in \mathbf{R}^d$, $d = 1$ or 2, be a Lipschitz domain. Denote by $L^2(\Omega)$ the usual space of Lebesgue square integrable functions. Its inner product and norm are denoted by $(\cdot, \cdot)_{L^2(\Omega)}$ and $\| \cdot \|_{L^2(\Omega)}$, respectively. Similarly, $H^1(\Omega)$ denotes the usual Sobolev space. $H_0^1(\Omega) \subset H^1(\Omega)$ consists functions vanishing on $\partial\Omega$. The spaces $H^{\frac{1}{2}}(\Omega)$ and $\tilde{H}^{\frac{1}{2}}(\Omega)$ are defined as the half-way interpolation between $L^2(\Omega)$ and $H^1(\Omega)$, $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively, refer to, e.g., [4]. According to [14], an equivalent norm for $H^{\frac{1}{2}}(\Omega)$ can be expressed as

$$\|v\|_{H^{\frac{1}{2}}(\Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \int_\Omega \int_\Omega \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{d+1}} d\boldsymbol{x}\, d\boldsymbol{y},$$

and the norm of $\tilde{H}^{\frac{1}{2}}(\Omega)$ is equivalent to

$$(1) \qquad \|v\|_{\tilde{H}^{\frac{1}{2}}(\Omega)}^2 = \|v\|_{H^{\frac{1}{2}}(\Omega)}^2 + \int_\Omega \frac{|v(\boldsymbol{x})|^2}{dist(\boldsymbol{x}, \partial\Omega)} d\boldsymbol{x}.$$

Model problem: Let Γ be an open or closed surface in $\mathbf{R}^3$. Let

$$Du(\boldsymbol{x}) = \frac{1}{4\pi} \frac{\partial}{\partial \boldsymbol{n}_x} \int_\Gamma u(\boldsymbol{y}) \frac{\partial}{\partial \boldsymbol{n}_y} (\frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}) \, dS_y,$$

and define

$$(2) \qquad B(u, v) = \,< Du, v >, \qquad \forall u, v \in \tilde{H}^{\frac{1}{2}}(\Gamma).$$

For a given data $f \in H^{-\frac{1}{2}}(\Gamma)$, the boundary integral problem with the hypersingular operator D is to find $u \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ such that

$$B(u, v) = f(v), \qquad \forall v \in \tilde{H}^{\frac{1}{2}}(\Gamma).$$

Basis functions: For simplicity, we consider only that Γ is a polygon, which can be partitioned into triangles. Assume that $\Gamma = \cup_{i=1}^M K_i$, where K_i's are the triangular elements. In particular, in the p-version

approximation all elements are assumed shape regular and of diameter $O(1)$.

Now we describe the approximation subspace of $\tilde{H}^{\frac{1}{2}}(\Gamma)$. Let p be a positive integer. Denote by $V_p(\Gamma)$ the subspace of $\tilde{H}^{\frac{1}{2}}(\Gamma)$ consisting of all the piecewise continuous polynomials of total degree $\leq p$. Suppose further that $V_p(\Gamma)$ can be spanned by the following three types of basis functions:

Nodal bases. For each node n_i, $\phi^{[n_i]}$ takes value 1 at the vertex n_i and 0 on all the elements not connected to n_i;

Side bases: For each side s, $\phi_j^{[s]}$, $1 \leq j \leq p-1$, are non-zero on side s and vanish on all the elements not connected to s;

Internal bases: For each element K, $\phi_\ell^{[K]}$, $1 \leq \ell \leq p(p-1)/2$, are 0 on all the elements except on K.

These basis functions are locally supported. Typically on each element they are defined as the pull-back of standard shape functions on a master element. We do not specify the choice of the basis functions, as long as they can be grouped into the above three types. Most of the commonly used shape functions satisfy this assumption [3].

Approximation subspaces: For the purpose of describing the preconditioner later, we divide the basis functions according to their geometric association. Let $\mathcal{W}$, the so-called wire-basket, denote the union of all the vertices and sides in the triangulation of Γ. Define

$$V^{[\mathcal{W}]} = \text{span}\{\text{all the nodal and side basis functions } \phi^{[n_i]} \text{ and } \phi_j^{[s]}\}$$
$$V^{[K]} = \text{span}\{\text{all the internal basis functions } \phi_\ell^{[K]} \text{ associated with element } K\}.$$

Then the approximation subspace can be expressed as

$$V_p(\Gamma) = V^{[\mathcal{W}]} + \sum_{\forall K} V^{[K]}.$$

Galerkin approximation: The Galerkin approximation of (2) is to find $u_p \in V_p(\Gamma)$ such that

$$(3) \qquad B(u_p, v) = f(v), \qquad \forall v \in V_p(\Gamma).$$

If we expand u_p as a linear combination of the basis functions, and arrange the unknowns according to their geometric associations, then

$$u_p = \mathbf{u}^{[\mathcal{W}]}\{\phi^{[\mathcal{W}]}\} + \sum_{\forall K} \mathbf{u}^{[K]}\{\phi^{[K]}\},$$

where $\mathbf{u}^{[\mathcal{W}]}$ and $\mathbf{u}^{[K]}$ are the unknowns associated with the wire-basket $\mathcal{W}$ and the element K, respectively. Then we get the following linear system of equations for the unknown $\boldsymbol{u}^T = (\mathbf{u}^{[\mathcal{W}]T}, \mathbf{u}^{[K_1]T}, \ldots, \mathbf{u}^{[K_M]T})$,

$$(4) \qquad\qquad B\boldsymbol{u} = \boldsymbol{f},$$

where

$$B = \begin{bmatrix} B(\phi^{[\mathcal{W}]}, \phi^{[\mathcal{W}]}), & B(\phi^{[\mathcal{W}]}, \phi^{[K_1]}), & \cdots, & B(\phi^{[\mathcal{W}]}, \phi^{[K_M]}) \\ & B(\phi^{[K_1]}, \phi^{[K_1]}), & \cdots, & B(\phi^{[K_1]}, \phi^{[K_M]}) \\ & \text{symmetric} & \ddots & \cdots \\ & & & B(\phi^{[K_M]}, \phi^{[K_M]}) \end{bmatrix}$$

$$\boldsymbol{f} = \begin{bmatrix} f(\phi^{[\mathcal{W}]}) \\ f(\phi^{[K_1]}) \\ \vdots \\ f(\phi^{[K_M]}) \end{bmatrix}$$

3. Preconditioning

To introduce the preconditioning algorithm for the iterative solution of the linear system (4), we follow the ideas of the well-known substructuring methods and Additive Schwarz Methods (ASM) for the finite element methods, see [6, 7, 10, 15]. Under the framework of ASM, to define a preconditioner it suffices to define a decomposition of the approximation space and a set of inner products associated with each subspace in the decomposition.

New bases and decomposition of $V_p(\Gamma)$: First we introduce a set of new basis functions associated with the wire-basket.

New side basis: Let s be an element side in the partition, and let K_1 and K_2 be the two elements sharing side s. For each $j = 1, \ldots, p-1$, we define the new side basis functions $\psi_j^{[s]}$ as follows:

$$(5) \qquad \begin{cases} \text{(i)} & \psi_j^{[s]} = \phi_j^{[s]}, & \text{on } s; \\ \text{(ii)} & \psi_j^{[s]} = 0, & \text{on } \Gamma \setminus (K_1 \cup K_2); \\ \text{(iii)} & B(\psi_j^{[s]}, v) = 0, & \forall v \in V^{[K_1]} \cup V^{[K_2]}. \end{cases}$$

Note that $\psi_j^{[s]}$ is uniquely determined by its value on side s.

New nodal basis: Let n_i be an element vertex in the partition. Let S_{n_i} be the set of all the element sides connected to n_i, and let I_{n_i} be the

index set for the elements connected to n_i. We define the new nodal basis function $\psi^{[n_i]}$ associated with vertex n_i as follows:

$$(6) \quad \begin{cases} \text{(i)} & \psi^{[n_i]} = \phi^{[n_i]}, & \text{at } n_i; \\ \text{(ii)} & \psi^{[n_i]} = 0, & \text{on } \Gamma \setminus (\cup_{\ell \in I_{n_i}} K_\ell); \\ \text{(iii)} & (\psi^{[n_i]}, v)_{L^2(\gamma)} = 0, & \forall v \in \mathcal{P}_p^0(\gamma), \quad \text{on } \forall \gamma \in S_{n_i}; \\ \text{(iv)} & B(\psi_j^{[s]}, v) = 0, & \forall v \in \cup_{\ell \in I_{n_i}} V^{[K_\ell]}, \end{cases}$$

where $\mathcal{P}_p^0(\gamma)$ is the set of polynomials on γ of degree $\leq p$ vanishing at both ends of γ. The above definition of the new basis is similar to a process of orthogonalization in the finite element methods [2, 5]. In FEM the new nodal and side bases are orthogonal to all the internal bases with respect to the inner product on the whole domain. This property leads to the decoupling of the interface unknowns from the internal ones. However, in BEMs the bilinear form $B(\cdot, \cdot)$ is a double integral. The bilinear product of two functions is not necessarily zero even though their supports do not overlap. Therefore, the new nodal and side bases are not orthogonal to the internal bases except those associated with the elements next to the underlining side or node.

Define a new subspace associated with the wire-basket as follows:
(7)
$$\tilde{V}^{[\mathcal{W}]} = \text{span}\{\text{all new nodal and side basis functions } \psi^{[n_i]} \text{ and } \psi_j^{[s]}\}.$$

We then decompose the approximation space $V_p(\Gamma)$ as follows:

$$V_p(\Gamma) = \tilde{V}^{[\mathcal{W}]} + \sum_{\forall K} V^{[K]}.$$

Preconditioner: Suppose that $r = (r_1, r_2, \ldots, r_N)^T$ is a residual vector in the iterative solution procedure of the linear system (4). It defines uniquely a residual function $r(x) = \sum_{i=1}^{N} r_i \phi_i(x)$ by the inner product relation $r_i = (r, \phi_i)_{L^2(\Gamma)}$, where ϕ_i and r_i are the i-th basis function and the i-th component of r, respectively. To define a preconditioner C, it suffices to describe the multiplication of C^{-1} with r. To this end, we define a correction function $u(x) \in V_p(\Gamma)$ (see below) and set $C^{-1}r$ as the coefficients of the linear combination of $u(x)$ in the basis $\{\phi_i\}$.

First, we define an inner product $C(\cdot, \cdot)$ on $V_p(\Gamma)$ as follows: for any $u, v \in V_p(\Gamma)$, let

$$u = u_{\mathcal{W}} + \sum_{\forall K} u_K, \qquad \text{and} \qquad v = v_{\mathcal{W}} + \sum_{\forall K} v_K,$$

with $u_\mathcal{W}, v_\mathcal{W} \in \tilde{V}^{[\mathcal{W}]}$ and $u_K, v_K \in V^{[K]}$. We define a bilinear form on $V_p(\Gamma) \times V_p(\Gamma)$ as

$$(8) \qquad C(u, v) = (u_\mathcal{W}, v_\mathcal{W})_{L^2(\mathcal{W})} + \sum_{\forall K} B(u_K, v_K).$$

The correction $u(\boldsymbol{x})$ is defined as the solution of

$$(9) \qquad C(u, v) = (r, v), \qquad \forall v \in V_p(\Gamma).$$

Let the solution $u(\boldsymbol{x})$ be expressed as $u(\boldsymbol{x}) = \sum_{i=1}^{N} u_i \phi_i(\boldsymbol{x})$. Then we set $C^{-1}\boldsymbol{r} = \boldsymbol{u} = (u_1, u_2, \ldots, u_N)^T$.

We now consider the implementation of the above preconditioner. We may solve (9) in terms of a number of sub-problems as follows:

First, compute the values of $u_\mathcal{W} \in \tilde{V}^{[\mathcal{W}]}$ on the wire-basket $\mathcal{W}$ by

$$(10) \qquad (u_\mathcal{W}, v)_{L^2(\mathcal{W})} = (r, v), \qquad \forall v \in \tilde{V}^{[\mathcal{W}]}.$$

This problem involves only the inversion of the matrix associated with the L^2-inner product on $\mathcal{W}$. For popular choices of the basis functions in practice, e.g., the anti-derivative of Legendre polynomials, or the Lagrange basis of the Gauss-Lobatto interpolation, the corresponding matrix is almost tridiagonal. Therefore, this subproblem can be solved easily. Since functions in $\tilde{V}^{[\mathcal{W}]}$ are uniquely defined by their values on $\mathcal{W}$, $u_\mathcal{W}$ is well defined by (10).

Next, on each element K, solve for $u_K \in V^{[K]}$ from

$$(11) \qquad B(u_K, v) = (r, v), \qquad \forall v \in V^{[K]}.$$

Finally, set $u = u_\mathcal{W} + \sum_{\forall K} u_K$, and expand it as a linear combination of the original basis functions $\{\phi_i\}$, with $\boldsymbol{u}$ as the expansion coefficients. Then $C^{-1}\boldsymbol{r} = \boldsymbol{u}$.

It is worth mentioning that the stiffness matrices in (11) is simply the local stiffness matrix on element K, which has already been calculated when the system (4) is formed. The inner products on the right hand side of (11) are also the linear combination of inner products between the original basis functions, which have been computed in forming (4).

REMARK 3.1. Although the preconditioner is described only for triangular elements, it is also applicable to partitions with quadrilateral elements. Indeed, with quadrilateral elements, the basis functions can also be grouped into nodal, side and internal types. Therefore, the preconditioner and the condition number analysis are the same as for triangular elements.

REMARK 3.2. In the bilinear form $C(\cdot, \cdot)$, the L^2-inner product for the wire-basket component $u_\mathcal{W}$ can be replaced by equivalent ones, e.g., the discrete L^2-inner product based on Gauss-Lobatto quadrature formulas. All the conclusions will remain unchanged. Moreover, such a treatment can lead to a diagonal system when solving (10) to calculate the wire-basket components.

Matrix representation of C: It is helpful to describe the matrix representation of the preconditioner defined above by the bilinear form $C(\cdot, \cdot)$. Denote by $\{\phi^{[\mathcal{W}]}\}$ and $\{\psi^{[\mathcal{W}]}\}$ the set of all the original and new basis functions associated with the wire-basket, respectively, and by $\{\phi^{[K]}\} = \{\ \{\phi^{[K_1]}\}, \ldots, \{\phi^{[K_M]}\}\ \}$ the set of the internal basis functions on all the elements. Then we may express the change of basis functions from $\{\phi^{[\mathcal{W}]}\}$ to $\{\psi^{[\mathcal{W}]}\}$ as

$$\{\psi^{[\mathcal{W}]}\} = R_\mathcal{W}\,\{\phi^{[\mathcal{W}]}\} + R_K\,\{\phi^{[K]}\},$$

where $R_\mathcal{W}$ and R_K are rectangular transformation matrices. Let the solution $u_p(\boldsymbol{x})$ of the p-version approximation (3) be expressed as

$$u_p(\boldsymbol{x}) = \mathbf{u}^{[\mathcal{W}]}\{\phi^{[\mathcal{W}]}\} + \mathbf{u}^{[K]}\{\phi^{[K]}\} = \tilde{\mathbf{u}}^{[\mathcal{W}]}\{\psi^{[\mathcal{W}]}\} + \tilde{\mathbf{u}}^{[K]}\{\phi^{[K]}\}.$$

Then the unknown $\tilde{\boldsymbol{u}}^T = (\tilde{\mathbf{u}}^{[\mathcal{W}]^T}, \tilde{\mathbf{u}}^{[K]^T})$ is related to the unknown $\boldsymbol{u}^T = (\mathbf{u}^{[\mathcal{W}]^T}, \mathbf{u}^{[K]^T})$ by $\boldsymbol{u} = Q^T\tilde{\boldsymbol{u}}$ with

$$Q = \left[\begin{array}{cc} R_\mathcal{W} & R_K \\ 0 & I \end{array}\right].$$

The linear system of equations for the unknown $\tilde{\boldsymbol{u}}$ is

(12) $$\tilde{B}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{f}},$$

where

$$\tilde{B} = QBQ^T, \qquad \text{and} \qquad \tilde{\boldsymbol{f}} = Q\boldsymbol{f}.$$

$\tilde{B}$ is indeed the stiffness matrix under the basis functions $\{\psi^{[\mathcal{W}]}\}$ and $\{\phi^{[K]}\}$. The preconditioner defined by the bilinear form $C(\cdot, \cdot)$ using the new basis functions induces the following stiffness matrix

$$\tilde{C} = \left[\begin{array}{cc} C_{\mathcal{W}\mathcal{W}} & 0 \\ 0 & C_{KK} \end{array}\right]$$

with

$$C_{KK} = \left[\begin{array}{ccc} B(\phi^{[K_1]}, \phi^{[K_1]}) & & 0 \\ & \ddots & \\ 0 & & B(\phi^{[K_M]}, \phi^{[K_M]}) \end{array}\right],$$

where $C_{\mathcal{W}\mathcal{W}}$ is the stiffness matrix associated with the new basis functions $\{\psi^{[\mathcal{W}]}\}$ and the $L^2(\mathcal{W})$-inner product. Now multiplying $\tilde{C}^{-1}$ to (12) is identical to multiplying C^{-1} to (4), with

$$
C = Q^{-T}\tilde{C}Q^{-1} = \begin{bmatrix} R_{\mathcal{W}}^{-T} & 0 \\ -R_K^T R_{\mathcal{W}}^{-T} & I \end{bmatrix} \tilde{C} \begin{bmatrix} R_{\mathcal{W}}^{-1} & -R_{\mathcal{W}}^{-1} R_K \\ 0 & I \end{bmatrix}
$$

This is the matrix representation of the preconditioner.

Comparing the matrices $\tilde{C}$ and $\tilde{B}$, it is clear that the preconditioner is essentially designed by deleting all the coupling between the wire-basket components and the internal components, and all the coupling between the internal components in the different elements. But this has been done by using the new bases $\{\psi^{[\mathcal{W}]}\}$, instead of directly the original bases $\{\phi^{[\mathcal{W}]}\}$. The reason is that for general basis functions, the coupling between the wire-basket components and the internal components is strong, a straightforward decoupling will not be effective. On the other hand, the coupling between the internal bases in the different elements is weak, therefore, the decoupling between different element internals is viable.

A condition number estimate:

THEOREM. *There exist positive constants c_1 and c_2 depending on the partition of Γ, but not on p, such that for all $u \in V_p(\Gamma)$,*

$$
c_1(1 + \ln p)^{-2} B(u, u) \le C(u, u) \le c_2(1 + \ln p)^5 B(u, u),
$$

which implies that $\kappa(C^{-1}B) \le c(1 + \ln p)^7$.

REMARK 3.3. For a preconditioner proposed in [1] for the BEM of (2), it is shown that the condition number of the preconditioned system can be bounded by $(1 + \ln p)^2$. We emphasize that the estimate in the above theorem is for a preconditioner working with general partitions and general bases, while the result in [1] is with a special choice of the basis and for quadrilateral elements. Technically, the extra power of $\ln p$ results from an estimate of the $\tilde{H}^{\frac{1}{2}}$-norm of the nodal and side functions, refer to lemmas 4.10 and 4.11 below. In the case of quadrilateral elements, the nodal and side basis function are given explicitly by the tensor products of special one-dimensional bases. Their $\tilde{H}^{\frac{1}{2}}$-norms can be estimated precisely by interpolating their L^2- and H^1-norms, which are calculated exactly; see Lemma 7 and Lemma 8 in [1]. For triangular elements, it is not clear how to estimate precisely the $\tilde{H}^{\frac{1}{2}}$-norm of the nodal and side functions. We conjecture that the estimates given in lemmas 4.10 and 4.11 are not optimal, and that therefore the conclusion of this theorem can be improved.

4. Notation and basic lemmas

Let K be a triangular element, and let p be a positive integer. Let $\mathcal{P}_p(K)$ be the set of polynomials of total degree $\leq p$, and let $\mathcal{P}_p^0(K) = \{v \in \mathcal{P}_p(K),$ and $v = 0$ on $\partial K\}$. Furthermore, let γ be a side of element K, and let $d(x, \gamma)$ be the distance from a point $x \in K$ to γ. We introduce the following subspace of $H^{\frac{1}{2}}(K)$:

$$H_\gamma^{\frac{1}{2}}(K) = \{v \in H^{\frac{1}{2}}(K) \mid$$

$$\|v\|_{H_\gamma^{\frac{1}{2}}(K)} = [\|v\|_{H^{\frac{1}{2}}(K)}^2 + \|(d(\cdot, \gamma))^{-\frac{1}{2}} v\|_{L^2(K)}^2]^{\frac{1}{2}} < \infty\}.$$

By the equivalent norm on $\tilde{H}^{\frac{1}{2}}$ space (cf. (1)),

$$(13) \qquad\qquad \tilde{H}^{\frac{1}{2}}(K) = \cup_{\gamma \in \partial K} H_\gamma^{\frac{1}{2}}(K).$$

Also, let A be a vertex of the element K, and let $d(x, A)$ be the distance from $x \in K$ to A. We introduce

$$H_A^{\frac{1}{2}}(K) = \{v \in H^{\frac{1}{2}}(K) \mid$$

$$\|v\|_{H_A^{\frac{1}{2}}(K)} = [\|v\|_{H^{\frac{1}{2}}(K)}^2 + \|(d(\cdot, A))^{-\frac{1}{2}} v\|_{L^2(K)}^2]^{\frac{1}{2}} < \infty\}.$$

We first present two lemmas about different norms on a single element. To simplify the notation, we shall use "$\simeq$" to denote that two quantities are equivalent with each other with constants independent of the functions involved. We shall also use "c" to denote a generic positive constant independent of the polynomial degree p.

LEMMA 4.1. *Let K be an element in the triangulation, and let γ be a side of K. For any $v \in \mathcal{P}_p(K)$ vanishing on γ, there exists a positive constant c independent of v and p such that*

$$\|v\|_{H_\gamma^{\frac{1}{2}}(K)} \leq c(1 + \ln p)\|v\|_{H^{\frac{1}{2}}(K)}.$$

Proof. See Lemma 5.9 of [17] or Lemma 4.7 of [5]. $\qquad\square$

LEMMA 4.2. *Let K be a triangular element, and let A be a vertex of K. For any $v \in \mathcal{P}_p(K)$ vanishing at A, there exists a positive constant c independent of v and p such that*

$$\|v\|_{H_A^{\frac{1}{2}}(K)} \leq c(1 + \ln p)\|v\|_{H^{\frac{1}{2}}(K)}.$$

Proof. Without loss of generality, we assume that $K = \{(x, y) \mid 0 < x < 1, x < y < 1\}$ and that the vertex $A = (0, 0)$. Then it is easy to see

$$
(14) \quad
\begin{aligned}
\|v\|^2_{H_A^{\frac{1}{2}}(K)} &= \|v\|^2_{H^{\frac{1}{2}}(K)} + \int_K \frac{|v(x, y)|^2}{\sqrt{x^2 + y^2}} dx dy \\
&\le c \|v\|^2_{H^{\frac{1}{2}}(K)} + \int_0^{\frac{1}{2}} \int_0^y \frac{|v(x, y)|^2}{y} dx dy.
\end{aligned}
$$

Now, we proceed in the same way as for the proof of Lemma 4.7 of [5]. We first study the following function

$$
F(y) = \int_0^y |v(x, y)|^2 \, dx.
$$

By using Lemma 6.4 of [2] and a scaling argument, we have for any $y \in (0, \frac{1}{2})$ that

$$
F(y) \le \int_0^y \max_{y \in (x, 1)} |v(x, y)|^2 \, dx \le c(1 + \ln p) \int_0^y \|v(x, \cdot)\|^2_{H^{\frac{1}{2}}(x, 1)} \, dx.
$$

Furthermore, by using the equivalent norm over $H^{\frac{1}{2}}(K)$ (cf. [9]), we have

$$
(15) \quad \max_{0 < y < 1/2} F(y) \le c(1 + \ln p) \|v\|^2_{H^{\frac{1}{2}}(K)}.
$$

Now, we may split the last integral on the right hand side of (14) into

$$
(16) \quad \int_{1/p^2}^{1/2} \frac{F(y)}{y} dy + \int_0^{1/p^2} \frac{F(y)}{y} dy.
$$

Clearly, the first term of the above expression can be bounded by

$$
\int_{1/p^2}^{1/2} \frac{F(y)}{y} dy \le \max_{0 < y < 1/2} F(y) \int_{1/p^2}^{1/2} \frac{1}{y} dy \le c(1 + \ln p)^2 \|v\|^2_{H^{\frac{1}{2}}(K)}
$$

For the second term in (16), we note that $F(0) = 0$. It follows from the mean value theorem that

$$
\frac{1}{y} F(y) \le \max_{0 < y < 1/2} |\frac{dF}{dy}(y)| \le c(2p + 1)^2 \max_{0 < y < 1/2} F(y),
$$

where we have used the classical Markov's inequality and the fact that F is a polynomial of degree $\leq 2p + 1$. Consequently,

$$
\int_0^{1/p^2} \frac{F(y)}{y} dy \leq c(2p+1)^2 \max_{0 < y < 1/2} F(y) \int_0^{1/p^2} dy
$$
$$
\leq c \max_{0 < y < 1/2} F(y)
$$
$$
\leq c(1 + \ln p)\|v\|^2_{H^{\frac{1}{2}}(K)}.
$$

$\square$

LEMMA 4.3. *Let* $v \in \mathcal{P}_p(K)$, *then there exists a positive constant* c *independent of* v *and* p, *such that*

(17) $$\|v\|^2_{L^2(\partial K)} \leq c(1 + \ln p)\|v\|^2_{H^{\frac{1}{2}}(K)}.$$

Proof. We consider only the case $K = \{(x, y) \mid -1 < x < 1, -1 < y < -x\}$, and we prove only that

$$\|v\|^2_{L^2(\gamma)} \leq c(1 + \ln p)\|v\|^2_{H^{\frac{1}{2}}(K)},$$

where $\gamma = \{(x, y) \mid -1 < x < 0, \ y = -1\}$ is half of the bottom side of K. The $L^2(\partial K)$-norm on the left hand side of (17) can be decomposed and reduced to this case by using regular mappings.

Let $I_x = \{-1 < x < 0\}$ and $I_y = \{-1 < y < 0\}$. For each $x \in I_x$, it follows from Lemma 6.4 of [2] that

$$|v(x, -1)|^2 \leq \|v(x, \cdot)\|^2_{L^\infty(I_y)} \leq c(1 + \ln p)\|v(x, \cdot)\|^2_{H^{\frac{1}{2}}(I_y)}.$$

Therefore,

$$
\|v\|^2_{L^2(\gamma)} \leq \int_{-1}^0 |v(x, -1)|^2 dx \leq c(1 + \ln p) \int_{-1}^0 \|v(x, \cdot)\|^2_{H^{\frac{1}{2}}(I_y)} dx
$$
$$
= c(1 + \ln p)\|v\|^2_{L^2(I_x, H^{\frac{1}{2}}(I_y))}.
$$

By the fact that (see Proposition 2.1, page 7 of [14])

$$H^{\frac{1}{2}}(I_x \times I_y) = L^2(I_x, H^{\frac{1}{2}}(I_y)) \cap H^{\frac{1}{2}}(I_x, L^2(I_y)),$$

we have easily

$$\|v\|^2_{L^2(\gamma)} \leq c(1 + \ln p)\|v\|^2_{H^{\frac{1}{2}}(I_x \times I_y)} \leq c(1 + \ln p)\|v\|^2_{H^{\frac{1}{2}}(T)}.$$

Now we consider the relation between the $H^{\frac{1}{2}}$-norm of a function over a union of elements and its norms over each element. Let ω be the union of an arbitrary set of n elements, $n \leq M$, denoted without loss of generality as $\omega = \cup_{i=1}^{n} K_i$. By the definition of the spaces, it is easy to see that

$$(18) \qquad \|v\|^2_{\tilde{H}^{\frac{1}{2}}(\omega)} \leq \sum_{i=1}^{n} \|v\|^2_{\tilde{H}^{\frac{1}{2}}(K_i)}$$

and

$$(19) \qquad \sum_{i=1}^{n} \|v\|^2_{H^{\frac{1}{2}}(K_i)} \leq \|v\|^2_{H^{\frac{1}{2}}(\omega)}.$$

Note that the other direction of the above inequalities is not true (see [1] for a counter example). However, it is possible to establish such results for functions in the finite dimensional subspace $V_p(\omega) = \{v|_\omega, \forall v \in V_p(\Gamma)\}$ with some constants dependent of p. To this end, we first introduce an auxiliary mapping between two neighboring elements.

Affine-mirror mapping: Let K_1 and K_2 be two triangular elements, and let $\bar{K}_1 \cap \bar{K}_2 \neq \emptyset$. Then there exists an affine mapping $F_{12} : K_1 \to K_2$ such that $F_{12} = I$ on $\bar{K}_1 \cap \bar{K}_2$. Indeed, there are only two cases as shown in Figure 1. In the left case, the affine mapping maps $A \to A'$, $B \to B$, and $C \to C$; in the right case, it maps $A \to A$, $B \to B'$, and $C \to C'$. We shall call this type of mapping *affine-mirror mapping*. Clearly, we can also define the affine-mirror mapping F_{21} from K_2 to K_1, and $F_{12} = (F_{21})^{-1}$.

Affine-mirror transform: Suppose $\bar{K}_1 \cap \bar{K}_2 \neq \emptyset$. Let v_1 be defined on $\bar{K}_1$. We may define $v_2(P) = v_1(F_{12}(P))$ for all $P \in K_2$. We call v_2 the affine-mirror transform of v_1 and denote $v_2 = \mathcal{F}_{21}(v_1)$. Similarly, we may define the affine-mirror transform $\mathcal{F}_{21}$ for functions on K_2. It is obvious that $\mathcal{F}_{12} = (\mathcal{F}_{21})^{-1}$.
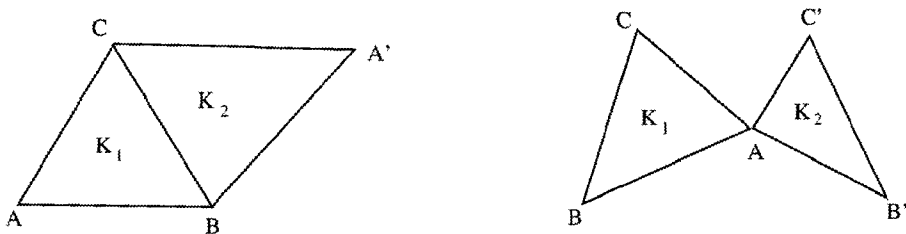

Figure 1. Affine-mirror mapping between K_1 and K_2

LEMMA 4.4. *Let S be a compact set, and let $f(P,Q) > 0$ be continuous on $S \times S$ except possibly when $P = Q$. Assume that*

$$\lim_{|P-Q|\to 0} f(P,Q) > 0.$$

Then $\inf_{\forall P,Q} f(P,Q) > 0$.

Proof. Suppose that $\inf_{\forall P,Q} f(P,Q) = 0$. Then there exist sequences $\{P_n\}$ and $\{Q_n\}$ in S such that $f(P_n, Q_n) \to 0$. Since S is compact, we may choose subsequences (still denoted by $\{P_n\}$ and $\{Q_n\}$) such that they converge to P^* and Q^* (both in S), respectively. Clearly, $P^* \neq Q^*$. Therefore by continuity, $f(P^*, Q^*) = \lim_{n\to\infty} f(P_n, Q_n) = 0$, which is in contradiction to $f > 0$ on $S \times S$. $\qquad\square$

LEMMA 4.5. *Assume that $\bar{K}_1 \cap \bar{K}_2 \neq \emptyset$ and $v_1 \in H^{\frac{1}{2}}(K_1)$. Let v_2 be the affine-mirror transform of v_1, i.e., $v_2 = \mathcal{F}_{12}(v_1)$. Define $v = v_i$ on $\bar{K}_i$, $i = 1, 2$. Then*

$$(20) \qquad \|v\|_{H^{\frac{1}{2}}(K_1 \cup K_2)} \simeq \|v_1\|_{H^{\frac{1}{2}}(K_1)} \simeq \|v_2\|_{H^{\frac{1}{2}}(K_2)}.$$

Proof. By $v_2 = \mathcal{F}_{21}(v_1)$, and the fact that in the p-version all elements are assumed fixed, we have readily from the definition of $H^{\frac{1}{2}}$-norm that

$$\|v_1\|_{H^{\frac{1}{2}}(K_1)} \simeq \|v_2\|_{H^{\frac{1}{2}}(K_2)}.$$

Now we prove the first equivalence relation in (20). Obviously,

$$\|v\|_{H^{\frac{1}{2}}(K_1 \cup K_2)} \geq \|v_1\|_{H^{\frac{1}{2}}(K_1)}.$$

On the other hand,
(21)

$$\|v\|^2_{H^{\frac{1}{2}}(K_1 \cup K_2)} = \|v\|^2_{L^2(K_1 \cup K_2)} + \left(\int_{K_1}\int_{K_1} + \int_{K_2}\int_{K_2} \right.$$
$$\left. + \int_{K_1}\int_{K_2} + \int_{K_2}\int_{K_1} \right) \left(\frac{|v(P) - v(Q')|^2}{|P - Q'|^3} \right) dS_{Q'} dS_P.$$

Clearly the first two integrals on the right hand side of (21) are bounded by $\|v_1\|^2_{H^{\frac{1}{2}}(K_1)}$. We consider the third one. Change the integration variable $Q' \to Q = F_{21}(Q')$. Then $dS_{Q'} = \frac{|K_2|}{|K_1|} dS_Q$, where $|K_i|$ denotes the area of K_i, $i = 1, 2$. By $v_2 = \mathcal{F}_{12}(v_1)$, we have $v(Q') = v_2(Q') = v_1(Q)$ for all $Q' \in K_2$. Thus

$$\int_{K_1}\int_{K_2} \frac{|v(P) - v(Q')|^2}{|P - Q'|^3} dS_{Q'} dS_P = \int_{K_1}\int_{K_1} \frac{|v_1(P) - v_1(Q)|^2}{|P - Q'|^3} dS_Q dS_P.$$

Furthermore, define for each $(P, Q) \in \bar{K}_1 \times \bar{K}_1$

$$f(P, Q) = \frac{|P - Q'|}{|P - Q|} = \frac{|P - F_{12}(Q)|}{|P - Q|}.$$

Obviously, $f(P, Q) > 0$ is continuous on $\bar{K}_1 \times \bar{K}_1$ except when $P = Q$. If $P \to Q \in \bar{K}_1 \cap \bar{K}_2$, we have $Q' = Q$ and $f(P, Q) = 1$; while if $P \to Q \notin \bar{K}_1 \cap \bar{K}_2$, $f(P, Q) \to +\infty$. Therefore, by Lemma 4.4 there exists a positive constant c depending only on the shape of the elements K_1 and K_2, such that

$$\inf_{\forall P, Q} \frac{|P - Q'|}{|P - Q|} \geq c,$$

which implies

$$|P - Q'| \geq c|P - Q|, \qquad \forall P, Q \in K_1.$$

Hence,

$$\int_{K_1} \int_{K_1} \frac{|v_1(P) - v_1(Q)|^2}{|P - Q'|^3} dS_Q dS_P \leq c \int_{K_1} \int_{K_1} \frac{|v_1(P) - v_1(Q)|^2}{|P - Q|^3} dS_Q dS_P$$
$$\leq c\|v_1\|^2_{H^{\frac{1}{2}}(K_1)}.$$

We may bound the last term on the right hand side of (21) similarly. Hence the lemma follows. $\qquad \square$

LEMMA 4.6. *Let K_1 and K_2 be two elements sharing a common side γ. For any $v \in H^{\frac{1}{2}}(K_1 \cup K_2)$ vanishing on γ, there is a positive constant c depending only on the shape of K_1 and K_2, but not on v, such that*

$$(22) \qquad \|v\|_{H^{\frac{1}{2}}(K_1 \cup K_2)} \leq c(\|v_1\|_{H^{\frac{1}{2}}_\gamma(K_1)} + \|v_2\|_{H^{\frac{1}{2}}_\gamma(K_2)}),$$

where $v_i = v|_{K_i}$, $i = 1, 2$.

Proof. By (21), we need only to estimate

$$\int_{K_1} \int_{K_2} \frac{|v(P) - v(Q')|^2}{|P - Q'|^3} dS_{Q'} dS_P.$$

It is obvious that

$$(23) \qquad \begin{aligned} & \int_{K_1} \int_{K_2} \frac{|v(P) - v(Q')|^2}{|P - Q'|^3} dS_{Q'} dS_P \\ \leq\ & 2 \int_{K_1} \int_{K_2} \frac{|v_1(P)|^2}{|P - Q'|^3} dS_{Q'} dS_P + 2 \int_{K_1} \int_{K_2} \frac{|v_2(Q')|^2}{|P - Q'|^3} dS_{Q'} dS_P. \end{aligned}$$

We estimate the first term on the right hand side. Let

$$G(P) = \int_{K_2} \frac{1}{|P - Q'|^3} dS_{Q'}.$$

Then

$$\int_{K_1} \int_{K_2} \frac{|v_1(P)|^2}{|P - Q'|^3} dS_{Q'} dS_P = \int_{K_1} G(P)|v_1(P)|^2 dS_P.$$

Set up a local polar coordinate system (ρ, θ) with the origin at P, we have (see the left diagram in Figure 2)

$$(24) \qquad G(P) \le \int_0^{\theta_1} \int_{d(P,\gamma)}^{d(P,\gamma)+H} \frac{1}{\rho^2} \, d\rho d\theta$$

where $d(P, \gamma)$ is the distance from P to the side γ, H is the diameter of K_2, and θ_1 is the angle of the sector containing K_2. Since in the p-version H is bounded from both above and below, and obviously $\theta_1 \le \pi$, we have for all $P \in K_1$ that

$$(25) \qquad G(P) \le \frac{c}{d(P, \gamma)},$$

which leads to

$$\int_{K_1} \int_{K_2} \frac{|v_1(P)|^2}{|P - Q'|^3} dS_{Q'} dS_P \le c \int_{K_1} \frac{|v_1(P)|^2}{d(P, \gamma)} dS_P \le c\|v_1\|^2_{H^{\frac{1}{2}}_{\gamma}(K_1)}.$$

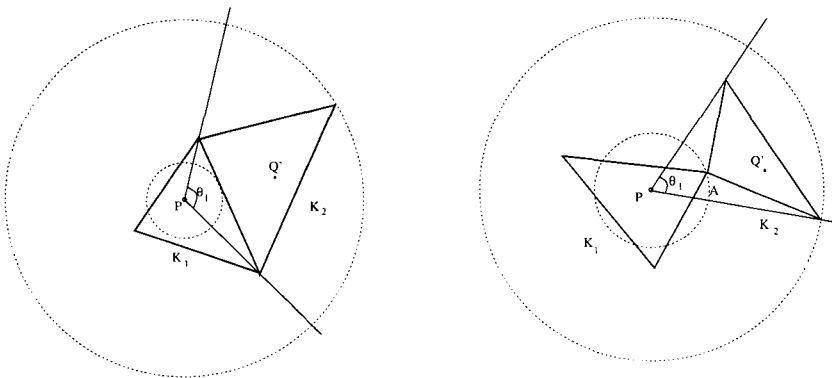The second term in (23) is treated similarly.   □

Figure 2. Diagrams for estimating $G(P)$

LEMMA 4.7. *Let K_1 and K_2 be two elements sharing a common vertex A. For any $v \in H^{\frac{1}{2}}(K_1 \cup K_2)$ vanishing at A, there is a positive constant c depending only on the shape of K_1 and K_2, but not on v, such that*

$$(26) \qquad \|v\|_{H^{\frac{1}{2}}(K_1 \cup K_2)} \leq c(\|v_1\|_{H_A^{\frac{1}{2}}(K_1)} + \|v_2\|_{H_A^{\frac{1}{2}}(K_2)}),$$

where $v_i = v|_{K_i}$ on $\bar{K}_i$, $i = 1, 2$.

Proof. We may proceed in the same way as in the previous lemma. Here instead of (24) and (25), we have

$$G(P) \leq \int_0^{\theta_1} \int_{d(P,K_2)}^{d(P,K_2)+H} \frac{1}{\rho^2} \, d\rho d\theta \leq \frac{c}{d(P, K_2)},$$

where $d(P, K_2)$ is the distance from P to K_2, refer to the right diagram of Figure 2. Note that in this case $d(P, A) \leq cd(P, K_2)$, hence

$$G(P) \leq \frac{c}{d(P, A)}.$$

The conclusion (26) follows readily from the above inequality and the definition of $H_A^{\frac{1}{2}}$-norm.   □

LEMMA 4.8. *Let $K_1, K_2, \ldots, K_n$ be an arbitrary set of n elements, and let $\omega = \cup_{i=1}^n K_i$. Then for any $v \in V_p(\Gamma)$, there exists a positive constant c depending on the shape of K_i, $1 \leq i \leq n$, but not on v, such that*

$$(27) \qquad \|v\|_{H^{\frac{1}{2}}(\omega)}^2 \leq c(1 + \ln p)^2 \sum_{i=1}^n \|v\|_{H^{\frac{1}{2}}(K_i)}^2.$$

In addition, if v vanishes on $\partial\omega$, then

$$(28) \qquad \|v\|_{\tilde{H}^{\frac{1}{2}}(\omega)}^2 \leq c(1 + \ln p)^2 \sum_{i=1}^n \|v\|_{H^{\frac{1}{2}}(K_i)}^2.$$

Proof. By the definition of $H^{\frac{1}{2}}$-norm,

$$\|v\|_{H^{\frac{1}{2}}(\omega)}^2 = \|v\|_{L^2(\omega)}^2 + \sum_{\ell,m=1}^n \int_{K_\ell} \int_{K_m} \frac{|v(P) - v(Q)|^2}{|P - Q|^3} \, dS_Q dS_P.$$

Now we estimate each of the integrals on the right hand side of the above formula.

Case (i): $K_\ell = K_m$. In this case the integral can be bounded from above by $\|v\|_{H^{\frac{1}{2}}(K_\ell)}^2$.

Case (ii): $\bar{K}_\ell \cap \bar{K}_m = \emptyset$. Then we can bound the integral as follows

$$
\int_{K_\ell}\int_{K_m} \frac{|v(P) - v(Q)|^2}{|P - Q|^3}\, dS_Q dS_P
$$

$$
\leq 2\int_{K_\ell}\int_{K_m} \frac{|v(P)|^2}{|P - Q|^3}\, dS_Q dS_P + 2\int_{K_\ell}\int_{K_m} \frac{|v(Q)|^2}{|P - Q|^3}\, dS_Q dS_P
$$

$$
\leq c_1\int_{K_\ell} |v(P)|^2\, dS_P + c_2\int_{K_m} |v(Q)|^2\, dS_Q,
$$

where

$$
c_1 = \int_{K_m} \frac{2}{[dist(Q, K_\ell)]^3}\, dS_Q \quad \text{and} \quad c_2 = \int_{K_\ell} \frac{2}{[dist(P, K_\ell)]^3}\, dS_P.
$$

Both of the above two constants can be bounded from above by constants, since in the p-version the triangulation is assumed fixed. Therefore the integral can be bounded from above by $c\|v\|_{L^2(K_\ell \cup K_m)}^2$.

Case (iii): K_ℓ and K_m share a common edge or a vertex. Obviously,

$$
\int_{K_\ell}\int_{K_m} \frac{|v(P) - v(Q)|^2}{|P - Q|^3}\, dS_Q dS_P \leq \|v\|_{H^{\frac{1}{2}}(K_\ell \cup K_m)}^2.
$$

To bound the norm on the right hand side, define $v_1 = v|_{\bar{K}_\ell}$ and $v_2 = v|_{\bar{K}_m}$. Furthermore, let $\tilde{v}_1 = \mathcal{F}_{\ell m}(v_1)$ and $\tilde{v}_2 = \mathcal{F}_{m\ell}(v_2)$, where $\mathcal{F}_{m\ell}$ and $\mathcal{F}_{\ell m}$ are the affine-mirror transforms described before. Then we introduce

$$
v_{ev} = \begin{cases} \frac{1}{2}(v_1 + \tilde{v}_2), & \text{on } K_\ell; \\ \frac{1}{2}(v_2 + \tilde{v}_1), & \text{on } K_m; \end{cases} \quad \text{and} \quad v_{od} = \begin{cases} \frac{1}{2}(v_1 - \tilde{v}_2), & \text{on } K_\ell; \\ \frac{1}{2}(v_2 - \tilde{v}_1), & \text{on } K_m. \end{cases}
$$

Clearly $v = v_{ev} + v_{od}$ on $K_\ell \cup K_m$. We estimate separately the norms of v_{ev} and v_{od}. Note that $\frac{1}{2}(v_2 + \tilde{v}_1)$ is the affine-mirror transform of $\frac{1}{2}(v_1 + \tilde{v}_2)$. It follows from Lemma 4.5 that

$$
\begin{aligned}
\|v_{ev}\|_{H^{\frac{1}{2}}(K_\ell \cup K_m)} &\leq c\|v_1 + \tilde{v}_2\|_{H^{\frac{1}{2}}(K_\ell)} \\
&\leq c(\|v_1\|_{H^{\frac{1}{2}}(K_\ell)} + \|\tilde{v}_2\|_{H^{\frac{1}{2}}(K_\ell)}) \\
&\leq c(\|v_1\|_{H^{\frac{1}{2}}(K_\ell)} + \|v_2\|_{H^{\frac{1}{2}}(K_m)}).
\end{aligned}
$$

If K_ℓ and K_m share a common side γ, we have by the fact that $v_{od} = 0$ on γ, Lemma 4.6, Lemma 4.1, and Lemma 4.5

$$
\begin{aligned}
\|v_{od}\|_{H^{\frac{1}{2}}(K_\ell \cup K_m)} & \\
\leq\ & c(\|v_{od}\|_{H_\gamma^{\frac{1}{2}}(K_\ell)} + \|v_{od}\|_{H_\gamma^{\frac{1}{2}}(K_m)}) \\
\leq\ & c(1 + \ln p)(\|v_{od}\|_{H^{\frac{1}{2}}(K_\ell)} + \|v_{od}\|_{H^{\frac{1}{2}}(K_m)}) \\
\leq\ & c(1 + \ln p)(\|v_1 - \tilde{v}_2\|_{H^{\frac{1}{2}}(K_\ell)} + \|v_2 - \tilde{v}_1\|_{H^{\frac{1}{2}}(K_m)}) \\
\leq\ & c(1 + \ln p)(\|v_1\|_{H^{\frac{1}{2}}(K_\ell)} + \|v_2\|_{H^{\frac{1}{2}}(K_m)}).
\end{aligned}
\tag{29}
$$

If K_ℓ and K_m share a common vertex A, we have similarly by $v_{od}(A) = 0$, Lemma 4.7, Lemma 4.2, and Lemma 4.5 that

$$
\begin{aligned}
\|v_{od}\|_{H^{\frac{1}{2}}(K_\ell \cup K_m)} &\leq c(\|v_{od}\|_{H_A^{\frac{1}{2}}(K_\ell)} + \|v_{od}\|_{H_A^{\frac{1}{2}}(K_m)}) \\
&\leq c(1 + \ln p)(\|v_{od}\|_{H^{\frac{1}{2}}(K_\ell)} + \|v_{od}\|_{H^{\frac{1}{2}}(K_m)}) \\
&\leq c(1 + \ln p)(\|v_1\|_{H^{\frac{1}{2}}(K_\ell)} + \|v_2\|_{H^{\frac{1}{2}}(K_m)}).
\end{aligned}
\tag{30}
$$

Hence

$$
\|v\|_{H^{\frac{1}{2}}(K_\ell \cup K_m)} \leq c(1 + \ln p)(\|v\|_{H^{\frac{1}{2}}(K_\ell)} + \|v\|_{H^{\frac{1}{2}}(K_m)}).
\tag{31}
$$

The conclusion (27) follows from the bounds in the above three cases.

In addition, if $v = 0$ on $\partial \omega$, then it follows from (1) that

$$
\|v\|_{\tilde{H}^{\frac{1}{2}}(\omega)}^2 \simeq \|v\|_{H^{\frac{1}{2}}(\omega)}^2 + \sum_{i=1}^I \|d_i^{-\frac{1}{2}} v\|_{L^2(\omega)}^2,
$$

where I is the total number of element sides of $\partial \omega$, and d_i is the distance to the i-th side s_i of ω. Let K_{s_i} be the element in ω which contains s_i. It follows from Lemma 4.1 that

$$
\begin{aligned}
\sum_{i=1}^I \|d_i^{-\frac{1}{2}} v\|_{L^2(\omega)}^2 &\leq c(\|v\|_{L^2(\omega)}^2 + \sum_{i=1}^I \|d_i^{-\frac{1}{2}} v\|_{L^2(K_{s_i})}^2) \\
&\leq c(1 + \ln p)^2 \sum_{i=1}^I \|v\|_{H^{\frac{1}{2}}(K_{s_i})}^2,
\end{aligned}
$$

which leads to the second conclusion of this lemma. $\qquad \square$

Finally, we present two lemma relating the $\tilde{H}^{\frac{1}{2}}(\Gamma)$-norm of a function in $\tilde{V}^{[\mathcal{W}]}(\Gamma)$ to its $L^2(\mathcal{W})$-norm. The following polynomial extension theorem from [16] plays an essential role.

LEMMA 4.9. *Let $f \in \mathcal{P}_p(\partial K)$. There exists an extension $\mathcal{E}(f) \in \mathcal{P}_p(K)$ such that $\mathcal{E}(f) = f$ on ∂K, and*

$$\|\mathcal{E}(f)\|_{H^{\frac{1}{2}}(K)} \le c\|f\|_{L^2(\partial K)},$$

where c is independent of f and p.

LEMMA 4.10. *Let s be an arbitrary side in the partition. For any $v \in \mathrm{Span}\{\psi_j^{[s]}, \quad 1 \le j \le p - 1\}$, there is a constant c independent of v and p such that*

$$(32) \qquad \|v\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)}^2 \le c(1 + \ln p)^2 \|v\|_{L^2(s)}^2.$$

Proof. By Theorem 1 and Theorem 2 in [8],

$$(33) \qquad B(v, v) \simeq \|v\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)}^2.$$

Let K_1 and K_2 be the elements sharing side s. For any $z \in V^{[K_1]} \cup V^{[K_2]}$, it follows from (5) and (18) that

$$B(v, v) = B(v, v + z) \le c\|v\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)}\|v + z\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)}$$
$$\le c\|v\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)}\|v + z\|_{\tilde{H}^{\frac{1}{2}}(K_1 \cup K_2)}.$$

Hence it follows from Lemma 4.8 that

$$\|v\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \le c\|v + z\|_{\tilde{H}^{\frac{1}{2}}(K_1 \cup K_2)}$$
$$\le c(1 + \ln p)(\|v + z\|_{H^{\frac{1}{2}}(K_1)} + \|v + z\|_{H^{\frac{1}{2}}(K_2)}).$$

Choosing $z = \mathcal{E}(v|_{\partial K_i}) - v$ on each element K_i, $i = 1, 2$, where $\mathcal{E}$ is the extension operator defined in Lemma 4.9, we have easily from Lemma 4.9

$$\|v\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \le c(1 + \ln p)\|v\|_{L^2(s)}.$$

$\square$

LEMMA 4.11. *Let n_i be an arbitrary vertex in the partition, and let S_{n_i} be the set of all the element sides connected to n_i. Then there is a constant c independent of p such that*

$$(34) \qquad \|\psi^{[n_i]}\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)}^2 \le c(1 + \ln p)^2 \sum_{\gamma \in S_{n_i}} \|\psi^{[n_i]}\|_{L^2(\gamma)}^2.$$

Proof. It can be proved in the same way as the previous lemma. $\square$

5. Proof of the main theorem

By the definition of $C(\cdot, \cdot)$ (see (8)) and the fact that $B(v, v) \simeq \|v\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)}$ for any $v \in \tilde{H}^{\frac{1}{2}}(\Gamma)$ (see theorems 1 and 2 in [8]), the conclusion of Theorem 1 is equivalent to the following statement: for any $u = u_{\mathcal{W}} + \sum_{\forall K} u_K$ with $u_{\mathcal{W}} \in \tilde{V}^{[\mathcal{W}]}$ and $u_K \in V^{[K]}$,

$$
(35) \qquad
\begin{aligned}
c_1 (1 + \ln p)^{-2} \|u\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} &\leq \|u_{\mathcal{W}}\|^2_{L^2(w)} + \sum_{\forall K} \|u_K\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \\
&\leq c_2 (1 + \ln p)^5 \|u\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)}.
\end{aligned}
$$

We now verify both of the above inequalities.

Lower bound: By the triangle inequality and the fact that the number of elements in a p-version approximation is fixed,

$$
(36) \qquad
\begin{aligned}
\|u\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} &\leq c \big(\|u_{\mathcal{W}}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} + \| \sum_{\forall K} u_K \|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \big) \\
&\leq c \big(\|u_{\mathcal{W}}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} + \sum_{\forall K} \|u_K\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \big).
\end{aligned}
$$

We use a standard coloring argument to bound the term $\|u_{\mathcal{W}}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)}$, see, e.g., [1]. Let $u_{\mathcal{W}} = u_N + u_E$ with u_N and u_E being the combinations of nodal and side components, respectively. Then

$$
\|u_{\mathcal{W}}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} = \|u_N + u_E\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \leq c \big(\|u_N\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} + \|u_E\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \big).
$$

Note that the set E of all sides of the partition can be decomposed into disjoint subsets as follows

$$
E = E_1 \cup E_2 \cup \cdots \cup E_L,
$$

where for each pair of sides $\gamma, \gamma' \in E_m$, the elements adjacent to γ do not overlap with the elements adjacent to γ'. By the triangle inequality

$$
\|u_E\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} = \| \sum_{\ell=1}^{L} u_{E_\ell} \|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \leq L \sum_{\ell=1}^{L} \|u_{E_\ell}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)}.
$$

Note that the supports of u_γ and $u_{\gamma'}$ are disjoint for $\gamma, \gamma' \in E_\ell$. Therefore, we have from Lemma 4.10 that

$$
\begin{aligned}
\|u_{E_\ell}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} = \| \sum_{\forall \gamma \in E_\ell} u_\gamma \|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} &\leq c \sum_{\forall \gamma \in E_\ell} \|u_\gamma\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \\
&\leq c (1 + \ln p)^2 \sum_{\forall \gamma \in E_\ell} \|u_\gamma\|^2_{L^2(\gamma)},
\end{aligned}
$$

which implies further that

$$\|u_E\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \leq c(1 + \ln p)^2 \sum_{\ell=1}^{L} \sum_{\forall \gamma \in E_\ell} \|u_\gamma\|^2_{L^2(\gamma)}$$
$$\leq c(1 + \ln p)^2 \sum_{\forall \gamma \in E} \|u_\gamma\|^2_{L^2(\gamma)}.$$

Treating u_N in a similar way, we can prove by using Lemma 4.11 that

$$\|u_N\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \leq c(1 + \ln p)^2 \sum_{i=1}^{I} \|u_{n_i}\|^2_{L^2(\mathcal{W})},$$

where u_{n_i} is the component in u_N associated with the node n_i. Combining the above two estimates and using the fact that u_N is $L^2(\mathcal{W})$-orthogonal to u_E, we have

$$(37) \qquad \begin{aligned} \|u_{\mathcal{W}}\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} &\leq c(1 + \ln p)^2 (\sum_{i=1}^{I} \|u_{n_i}\|^2_{L^2(\mathcal{W})} + \sum_{\forall \gamma \in E} \|u_\gamma\|^2_{L^2(\gamma)}) \\ &\leq c(1 + \ln p)^2 \|u\|^2_{L^2(\mathcal{W})}. \end{aligned}$$

Hence, the left hand side of (35) follows from the above inequality and (36).

Upper bound: First, note that $u_{\mathcal{W}} = u$ on $\mathcal{W}$. It follows from Lemma 4.3 that

$$(38) \qquad \begin{aligned} \|u_{\mathcal{W}}\|^2_{L^2(\mathcal{W})} &\leq \sum_{\forall K} \|u\|^2_{L^2(\partial K)} \leq c(1 + \ln p) \sum_{\forall K} \|u\|^2_{H^{\frac{1}{2}}(K)} \\ &\leq c(1 + \ln p) \|u\|^2_{H^{\frac{1}{2}}(\Gamma)}. \end{aligned}$$

On the other hand, for any $u_K \in V^{[K]}$, (18) implies

$$\|u_K\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \leq \|u_K\|^2_{\tilde{H}^{\frac{1}{2}}(K)},$$

since $u_K = 0$ on $\Gamma \setminus K$. It follows from Lemma 4.1 and (19) that

$$\begin{aligned} \sum_{\forall K} \|u_K\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} &\leq \sum_{\forall K} \|u_K\|^2_{\tilde{H}^{\frac{1}{2}}(K)} \\ &\leq c(1 + \ln p)^2 \sum_{\forall K} \|u_K\|^2_{H^{\frac{1}{2}}(K)} \\ &\leq c(1 + \ln p)^2 \|\sum_{\forall K} u_K\|^2_{H^{\frac{1}{2}}(\Gamma)} \\ &= c(1 + \ln p)^2 \|u - u_{\mathcal{W}}\|^2_{H^{\frac{1}{2}}(\Gamma)} \\ &\leq c(1 + \ln p)^2 (\|u\|^2_{H^{\frac{1}{2}}(\Gamma)} + \|u_{\mathcal{W}}\|^2_{H^{\frac{1}{2}}(\Gamma)}). \end{aligned}$$

Furthermore, it follows from (37) and (38) that

$$\|u_{\mathcal{W}}\|^2_{H^{\frac{1}{2}}(\Gamma)} \leq c(1 + \ln p)^2 \|u\|^2_{L^2(\mathcal{W})} \leq c(1 + \ln p)^3 \|u\|^2_{H^{\frac{1}{2}}(\Gamma)},$$

which implies

$$\sum_{\forall K} \|u_K\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} \leq c(1 + \ln p)^5 \|u\|^2_{H^{\frac{1}{2}}(\Gamma)}.$$

The above estimate and (38) lead to the upper bound of (35).

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