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ON GENERALIZED FINSLER STRUCTURES WITH A VANISHING hv-TORSION

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ABSTRACT. A canonical Finsler connection $N\Gamma$ is defined by a generalized Finsler structure called a (G, N)-structure, where G is a generalized Finsler metric and N is a nonlinear connection given in a differentiable manifold, respectively. If $N\Gamma$ is linear, then the (G, N)-structure has a linearity in a sense and is called *Berwaldian*. In the present paper, we discuss what it means that $N\Gamma$ is with a vanishing hv-torsion: $P^i{}_{jk} = 0$ and introduce the notion of a stronger type for linearity of a (G, N)-structure. For important examples, we finally investigate the cases of a Finsler manifold and a Rizza manifold.

1. Introduction

Let M be a differentiable manifold and $T(M)$ its tangent bundle. We assume the zero vectors to be excluded from $T(M)$. A coordinate system (x^i) in M induces a canonical coordinate system (x^i, y^i) in $T(M)$. We put $\partial_k = \partial/\partial x^k$ and $\dot{\partial}_k = \partial/\partial y^k$.

A Finsler tensor $G_{ij}(x, y)$ defined on a domain of $T(M)$ is called a *generalized Finsler metric* in M, if it is symmetric, non-degenerate, and positively homogeneous of degree zero for y. A differentiable manifold M is said to admit a *(G, N)-structure* ([1], [5]) or simply called a *(G, N)-manifold* if M admits a generalized Finsler metric $G_{ij}(x, y)$ and a non-linear connection $N^i{}_j(x, y)$. A non-linear connection $N^i{}_j(x, y)$ is assumed to be positively homogeneous of degree one for y.

Now we put

$$L^*(x, y) = \sqrt{G_{ij}(x, y) y^i y^j}, \tag{1.1}$$

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$$g^*_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^{*2}(x,y), \tag{1.2}$$

and we assume

$$\det(g^*_{ij}) \neq 0. \tag{1.3}$$

A (G,N)-structure satisfying (1.3) is called a *regular* (G,N)-structure, and the above $g^*_{ij}(x,y)$ becomes a Finsler metric and we call this *a Finsler metric induced by* G_{ij}.

Here, we denote by $B\Gamma^* = (\dot{\partial}_k N^{*i}{}_j, N^{*i}{}_j, 0)$ the Berwald connection and by $C\Gamma^* = (\Gamma^{*i}_j{}_k, N^{*i}{}_j, C^{*i}_j{}_k)$ the Cartan connection determined by g^*_{ij}, respectively.

In a (G,N)-structure, we put

$$\begin{aligned} F_j{}^i{}_k &= \frac{1}{2}G^{im}(X_j G_{mk} + X_k G_{jm} - X_m G_{jk}), \\ C_j{}^i{}_k &= \frac{1}{2}G^{im}(\dot{\partial}_j G_{mk} + \dot{\partial}_k G_{jm} - \dot{\partial}_m G_{jk}), \end{aligned} \tag{1.4}$$

where $X_k = \partial_k - N^m{}_k\dot{\partial}_m$. The triplet $N\Gamma = (F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ is naturally a Finsler connection [8]. The above Finsler connection $N\Gamma$ is said to be the *Finsler connection associated with a* (G,N)*-structure*, in short, the (G,N)-connection.

Denoting the h- and v- covariant differentiations by ∇_k and $\dot{\nabla}_k$ with respect to $N\Gamma$, respectively, we have directly $\nabla_k G_{ij} = 0$ and $\dot{\nabla}_k G_{ij} = 0$.

With respect to the torsions and curvatures of the connection $N\Gamma$, we have

$$\begin{aligned} &R^i{}_{jk} = X_k N^i{}_j - X_j N^i{}_k, \quad P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j, \\ &T^i{}_{jk} = F_j{}^i{}_k - F_k{}^i{}_j, \quad S^i{}_{jk} = C_j{}^i{}_k - C_k{}^i{}_j, \\ &R_h{}^i{}_{jk} = K_h{}^i{}_{jk} + C_h{}^i{}_m R^m{}_{jk}, \quad P_h{}^i{}_{jk} = \dot{\partial}_k F_h{}^i{}_j - Q_h{}^i{}_{jk}, \\ &S_h{}^i{}_{jk} = \dot{\partial}_k C_h{}^i{}_j - \dot{\partial}_j C_h{}^i{}_k + C_m{}^i{}_k C_h{}^m{}_j - C_m{}^i{}_j C_h{}^m{}_k, \end{aligned} \tag{1.5}$$

where we put

$$\begin{aligned} K_h{}^i{}_{jk} &= X_k F_h{}^i{}_j - X_j F_h{}^i{}_k + F_m{}^i{}_k F_h{}^m{}_j - F_m{}^i{}_j F_h{}^m{}_k, \\ Q_h{}^i{}_{jk} &= \nabla_j C_h{}^i{}_k - C_h{}^i{}_m P^m{}_{jk}. \end{aligned} \tag{1.6}$$

A (G,N)-structure is said to be *Berwaldian* if the associated (G,N)-connection $N\Gamma$ is linear, that is, $F_j{}^i{}_k$ depend on position alone. With respect to the Berwaldian (G,N)-structure, the following theorem was known:

THEOREM 1 ([5], Theorem 1). *A (G,N)-manifold is Berwaldian if and only if*

$$Q_{hijk} + Q_{ihjk} = 0, \tag{1.7}$$

where $Q_{hijk} = G_{im}Q_h{}^m{}_{jk}$.

Furthermore, we denote the deflection tensor by $D^i{}_j$, that is,

$$D^i{}_j = F_m{}^i{}_j y^m - N^i{}_j = -P^i{}_{jm} y^m, \tag{1.8}$$

where the homogeneity of $N^i{}_j$ is essentially used. As it is remarked by Remark 1 in [6], we have

THEOREM 2. *In an associated (G,N)-connection $N\Gamma$, the condition $P^i{}_{jk} = 0$ implies $D^i{}_j = 0$. In the case where $N\Gamma$ is linear, the converse is also true.*

The purpose of the present paper is to discuss the meaning of the condition $P^i{}_{jk} = 0$ for $N\Gamma$. In the second section, we obtain Theorem 3 for a regular (G,N)-structure, and in the following section, we introduce a stronger notion for linearity of a (G,N)-structure (Definition and Theorem 4), and proceed further with some related discussions (Theorems 5, 6, and 7). Finally, in the fourth and fifth sections, we investigate the cases of a Finsler manifold (Theorems 9 and 10) and a Rizza manifold (Theorems 11, 12 and 13), respectively.

2. Regular (G,N)-structures with a vanishing hv-torsion

Corresponding to a (G,N)-connection $N\Gamma = (F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$, we put $N\Gamma' = (F_j{}^i{}_k, N^i{}_j, 0)$. Then $N\Gamma'$ is also a Finsler connection in a (G,N)-manifold and the torsions and curvatures for $N\Gamma'$ are as follows:

$$S'^i{}_{jk} = 0, \quad S'_h{}^i{}_{jk} = 0, \quad Q'_h{}^i{}_{jk} = 0, \tag{2.1}$$

and $R'^i{}_{jk}$, $P'^i{}_{jk}$, $T'^i{}_{jk}$, $K'_h{}^i{}_{jk}$ of $N\Gamma'$ coincide with $R^i{}_{jk}$, $P^i{}_{jk}$, $T^i{}_{jk}$, $K_h{}^i{}_{jk}$ of $N\Gamma$, respectively. It is trivial that

$$P'_h{}^i{}_{jk} = \dot\partial_k F_h{}^i{}_j, \quad R'_h{}^i{}_{jk} = K_h{}^i{}_{jk},$$

and $\nabla' = \nabla$.

Here, we assume that the hv-torsion vanishes, that is,

$$P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j = 0 \tag{2.2}$$

for $N\Gamma'$(or $N\Gamma$). Then we have from Theorem 2

$$F_j{}^i{}_k = F_k{}^i{}_j, \quad \nabla_k G_{ij} = 0, \quad \nabla_k y^i = D^i{}_k = 0. \tag{2.3}$$

Therefore, we obtain

$$\nabla_k L^{*2} = \nabla_k (G_{ij} y^i y^j) = 0. \tag{2.4}$$

So, we get $\nabla_k L^* = 0$. That is, if $P^i{}_{jk} = 0$, then $D^i{}_j = 0$, $\nabla_k L^* = 0$ and $F_j{}^i{}_k = F_k{}^i{}_j$ hold good. Therefore, if a (G, N)-structure is regular, then the connection $N\Gamma'$ satisfies Okada's axiom which uniquely determines the Berwald connection $B\Gamma^*$ of a Finsler metric L^*. Hence, by Okada's Theorem [9], the connection $N\Gamma'$ is just $B\Gamma^*$ of L^*, that is, $N^i{}_j = N^{*i}{}_j$, $F_k{}^i{}_j = \dot{\partial}_k N^{*i}{}_j$. The converse is easily verified, so we have

THEOREM 3. *In a regular (G, N)-manifold, $P^i{}_{jk} = 0$ if and only if $N\Gamma' = B\Gamma^*$:*

$$N^i{}_j = N^{*i}{}_j, \quad F_k{}^i{}_j = \dot{\partial}_k N^{*i}{}_j.$$

This theorem gives a meaning of $P^i{}_{jk} = 0$ in a regular (G, N)-structure. It is noteworthy that the condition $P^i{}_{jk} = 0$ determines a non-linear connection $N^i{}_j$.

3. Strongly Berwaldian structures

Suggested by Theorem 1 and Theorem 3, we shall define the notion of a stronger type for linearity of a (G, N)-structure as follows:

DEFINITION. A (G, N)-structure is called *strongly Berwaldian*, if it satisfies

$$Q_{hijk} + Q_{ihjk} = 0, \quad P^i{}_{jk} = 0. \tag{3.1}$$

By virtue of Theorem 1, a strongly Berwaldian (G, N)-structure is Berwaldian, that is, $N\Gamma$ is linear as follows: $F_j{}^i{}_k = F_j{}^i{}_k(x)$. It is noted from Theorem 2 that (3.1) is equivalent to

$$Q_{hijk} + Q_{ihjk} = 0, \quad D^i{}_j = 0. \tag{3.2}$$

We shall here assume that a (G, N)-structure is regular. By Theorem 1 and Theorem 3, the condition (3.1) is satisfied if and only if $N\Gamma$ is linear and $N\Gamma' = B\Gamma^*$, that is, L^* is a Berwald metric and $N\Gamma$ is determined by L^* as $N^i{}_j = N^{*i}{}_j$ and $F_k{}^i{}_j = \dot{\partial}_k N^{*i}{}_j$. For the Berwald metric L^*, these coefficients are also expressed as $N^i{}_j = \Gamma^*_m{}^i{}_j y^m$ and $F_k{}^i{}_j = \Gamma^*_k{}^i{}_j$. Thus for the condition (3.1), we have the following characterization in terms of L^*.

THEOREM 4. *The regular (G, N)-structure is strongly Berwaldian if and only if the induced Finsler metric L^* is a Berwald metric and*

$$N^i{}_j = N^{*i}{}_j = \Gamma^*_m{}^i{}_j(x) y^m, \quad F_k{}^i{}_j(x) = \dot{\partial}_k N^{*i}{}_j = \Gamma^*_k{}^i{}_j(x).$$

Especially, if a regular (G, N)-structure satisfies $\nabla_h C_j{}^i{}_k = 0$, $P^i{}_{jk} = 0$, or equivalently $Q_{hijk} = 0$, $P^i{}_{jk} = 0$, then it is strongly Berwaldian. This condition is considered to be interesting as a more strongly Berwaldian condition than (3.1).

Now, we shall further modify the condition (3.1) and treat the cases where the induced Finsler metric L^* becomes a Landsberg metric and locally Minkowskian metric.

A (G, N)-structure satisfying the following conditions

$$(\dot{\partial}_k G_{ij}) y^i y^j = 0, \quad \det(A^i{}_j) \neq 0, \tag{3.3}$$

where $A^i{}_j = \delta^i_j + G^{im}(\dot{\partial}_j G_{pm}) y^p$, is called a *regular (G, N)-structure in the sense of Miron* and (3.3) is called the *regular condition in the sense of Miron* ([7]).

Here, we assume that a (G, N)-structure is regular in the sense of Miron. In this case, we have $g^*_{ij} = G_{im} A^m{}_j$ and $\det(g^*_{ij}) \neq 0$ by virtue of (1.1), (1.2) and (3.3). Therefore, the structure (G, N) is regular and g^*_{ij} is a Finsler metric. It holds that

$$g^*_{im} y^m = G_{im} y^m. \tag{3.4}$$

If a regular (G, N)-structure in the sense of Miron satisfies $P^i{}_{jk} = 0$, then, from Theorem 3, we have $N\Gamma' = B\Gamma^*$, $D^i{}_j = 0$ and $\nabla_k L^{*2} = 0$. Since $L^{*2} = g^*_{pq} y^p y^q$ follows from (3.4) or directly from (1.2), we have $(\nabla_k g^*_{pq}) y^p y^q = 0$, that is,

$$\{\partial_k g^*_{pq} - N^{*m}{}_k \dot{\partial}_m g^*_{pq} - (\dot{\partial}_k N^{*m}{}_p) g^*_{mq} - (\dot{\partial}_k N^{*m}{}_q) g^*_{pm}\} y^p y^q = 0. \tag{3.5}$$

Since $\dot{\partial}_k N^{*i}{}_j = \dot{\partial}_j N^{*i}{}_k$, (3.5) is reduced to

$$(\partial_k g^*_{pq}) y^p y^q - 2N^{*m}{}_k g^*_{mp} y^p = 0. \tag{3.6}$$

Differentiating (3.6) partially with respect to y^i, we have

$$(\partial_k g^*_{ip}) y^p - (\dot{\partial}_i N^{*m}{}_k) g^*_{mp} y^p - N^{*m}{}_k g^*_{im} = 0. \tag{3.7}$$

Furthermore, differentiating (3.7) partially with respect to y^j, we have

$$\begin{aligned} \partial_k g^*_{ij} - (\dot{\partial}_i \dot{\partial}_j N^{*m}{}_k) g^*_{mp} y^p - (\dot{\partial}_i N^{*m}{}_k) g^*_{mj} \\ - (\dot{\partial}_j N^{*m}{}_k) g^*_{im} - N^{*m}{}_k \dot{\partial}_j g^*_{im} = 0. \end{aligned} \tag{3.8}$$

Since $N\Gamma' = B\Gamma^*$, (3.8) is reduced to

$$\nabla_k g^*_{ij} = (\dot{\partial}_i \dot{\partial}_j N^{*m}{}_k) g^*_{mp} y^p. \tag{3.9}$$

By the regularity in the sense of Miron, we have (3.4), so (3.9) is written as

$$\nabla_k g^*_{ij} = (\dot{\partial}_i \dot{\partial}_j N^{*m}{}_k) G_{mp} y^p. \tag{3.10}$$

With respect to the Berwald connection $N\Gamma' = B\Gamma^*$, the condition that L^* be a Landsberg metric is given by $\nabla_k g^*_{ij} = 0$. Thus we have

THEOREM 5. *In a regular (G, N)-structure in the sense of Miron satisfying $P^i{}_{jk} = 0$, the necessary and sufficient condition that L^* be a Landsberg metric is $(\dot{\partial}_i \dot{\partial}_j N^{*m}{}_k) G_{mp} y^p = 0$.*

Moreover, if $\dot{\partial}_i \dot{\partial}_j N^{*h}{}_k = 0$, then $\dot{\partial}_k N^{*i}{}_j$ is a function of x alone. Therefore, L^* is a Berwaldian metric. In this case, the condition (3.4) is not necessary, since $\nabla_k g^*_{ij} = 0$ follows from (3.9). Therefore, from Theorem 1, we have

THEOREM 6. *In a regular (G, N)-structure satisfying $P^i{}_{jk} = 0$, L^* is a Berwaldian metric if and only if $Q_{hijk} + Q_{ihjk} = 0$.*

The above theorem is also followed from Theorem 4.

Next, in a regular (G, N)-structure satisfying $P^i{}_{jk} = 0$, from Theorem 3 we have $F_j{}^i{}_k = \dot{\partial}_k N^{*i}{}_j$ and $N^i{}_j = N^{*i}{}_j$. If L^* is a locally Minkowskian metric, then L^* becomes a Berwaldian metric properly. Therefore, $Q_{hijk} + Q_{ihjk} = 0$ holds good. Further, the manifold is covered by the coordinate neighborhood system such that $\partial_k g^*_{ij} = 0$, because g^*_{ij} is a locally Minkowski metric. Therefore, $N^{*i}{}_j = 0$ and hence $F_j{}^i{}_k = 0$. Thus we obtain $K_h{}^i{}_{jk} = 0$.

Conversely, it is known [6] that if $P^i{}_{jk} = 0$, $Q_{hijk} + Q_{ihjk} = 0$ and $K_h{}^i{}_{jk} = 0$, then the (G, N)-structure is strongly flat and L^* becomes a locally Minkowskian metric. Thus we have

THEOREM 7. *In a regular (G, N)-structure satisfying $P^i{}_{jk} = 0$, L^* is a locally Minkowskian metric if and only if*

$$Q_{hijk} + Q_{ihjk} = 0, \quad K_h{}^i{}_{jk} = 0.$$

4. (L, N)-structures with a vanishing hv-torsion

In this section, we consider the case where the G_{ij} of a (G, N)-structure is a Finsler metric $g_{ij} = (\dot{\partial}_i \dot{\partial}_j L^2)/2$, where L is the fundamental function. This structure is called a *(L, N)-structure.* Let us consider the Finsler connection $N\Gamma^* = (F_j{}^i{}_k, N^i{}_j, C_j^{*i}{}_k)$ associated with a (L, N)-structure, where we put

$$(4.1) \qquad \begin{aligned} F_j{}^i{}_k &= g^{im}(X_j g_{mk} + X_k g_{jm} - X_m g_{jk})/2, \\ C_j^{*i}{}_k &= g^{im} \dot{\partial}_m g_{jk}/2. \end{aligned}$$

It is noted that $N\Gamma^*$ is the (G, N)-connection $N\Gamma$. Since we have $L^* = L$ and then $g^*_{ij} = g_{ij}$, the (G, N)-structure is regular. In a (L, N)-structure, if $F_j{}^i{}_k$ is a function of x alone, then it is called *Berwaldian.* It is known [3] that

THEOREM 8. *In a (L, N)-structure, the following three conditions are mutually equivalent:*

(1) *(L, N)-structure is Berwaldian,*

(2) $Q_h{}^i{}_{jk} = 0$,

(3) *L is Berwaldian and $C_j^{*i}{}_m P^m{}_{kr} y^r = 0$.*

Here, if a (L, N)-structure satisfies $P^i{}_{jk} = 0$, then, from Theorem 3 the non-linear connection $N^i{}_j$ of $N\Gamma = N\Gamma^*$ coincides with the one $N^{*i}{}_j$ of the Cartan connection $C\Gamma^*$ induced from L. Therefore, $F_j{}^i{}_k$ given by (4.1) become the coefficients $F_j^{*i}{}_k$ of $C\Gamma^*$. Hence we have $N\Gamma^* = C\Gamma^*$. Since $P^i{}_{jk} = 0$ means that L is a Landsberg metric, we have

THEOREM 9. *In a (L, N)-structure, $P^i{}_{jk} = 0$ if and only if L is a Landsberg metric and $N^i{}_j = N^{*i}{}_j$. Then $N\Gamma^* = C\Gamma^*$.*

Furthermore, from Theorem 4, we have

THEOREM 10. *The necessary and sufficient condition that a (L, N)-structure be strongly Berwaldian is that L is a Berwaldian metric and $N^i{}_j = N^{*i}{}_j$. Then $N\Gamma^* = C\Gamma^*$.*

5. Rizza manifolds whose intrinsic (G, N)-structures are with a vanishing hv-torsion

In this section, we shall discuss the notion of a (G, N)-structure in a Rizza manifold.

We suppose that a $2n$-dimensional manifold M admits an almost complex structure $f^i{}_j(x)$ and a Finsler metric $g_{ij}(x,y)$ which satisfy the so-called Rizza condition as follows:

$$(g_{ij}(x,y) - g_{pq}(x,y) f^p{}_i(x) f^q{}_j(x)) y^j = 0. \tag{5.1}$$

Then the triplet $(M, g_{ij}(x,y), f^i{}_j(x))$ is called a *Rizza manifold* [10] and investigated in details by Y. Ichijō [2]. As it is well known, the tensor

$$G_{ij} = (g_{ij} + g_{pq} f^p{}_i f^q{}_j)/2 \tag{5.2}$$

is not a Finsler metric but a generalized Finsler metric if g_{ij} is not a Riemannian metric. Moreover, we can see easily

$$G_{pq} f^p{}_i f^q{}_j = G_{ij}. \tag{5.3}$$

On the other hand, differentiating the both sides of (5.1) with respect to y^k, we have

$$g_{jk} - (\dot{\partial}_k g_{pq}) f^p{}_j f^q{}_r y^r - g_{pq} f^p{}_j f^q{}_k = 0. \tag{5.4}$$

From $g_{jk} = g_{kj}$ and (5.4), we get

$$(\dot{\partial}_k g_{pq}) f^p{}_j f^q{}_r y^r - (\dot{\partial}_j g_{pq}) f^p{}_k f^q{}_r y^r = 0. \tag{5.5}$$

Differentiating (5.2) with respect to y^k and transvecting the obtained equation by y^i, we get

$$y^i \dot{\partial}_k G_{ij} = \{\dot{\partial}_k g_{ij} + (\dot{\partial}_k g_{pq}) f^p{}_i f^q{}_j\} y^i/2 = (\dot{\partial}_j g_{pq}) f^p{}_i f^q{}_k y^i/2 = y^i \dot{\partial}_j G_{ik}$$

by virtue of (5.5). Therefore, by the homogeneity of G_{ij}, we have

$$(\dot{\partial}_k G_{ij}) y^i y^j = 0. \tag{5.6}$$

Furthermore, from (5.4) and (5.2), we get

$$g_{jr} y^r = g_{pq} f^p{}_j f^q{}_r y^r = (2G_{jr} - g_{jr}) y^r.$$

Hence we have $g_{jr}y^r = G_{jr}y^r$. Differentiating the last equation with respect to y^k, we have

$$g_{jk} = (\dot{\partial}_k G_{jr})y^r + G_{jk}. \tag{5.7}$$

Here, from $L^{*2} = G_{pq}y^p y^q$, we have

$$g^*_{ij} = (\dot{\partial}_i \dot{\partial}_j L^{*2})/2 = (\dot{\partial}_i G_{jp})y^p + G_{ij} = g_{ij} \tag{5.8}$$

by virtue of (5.6) and (5.7). Therefore, L^* is nothing but a Finsler metric $L = (g_{ij}y^i y^j)^{1/2}$.

In a Rizza manifold, there exists also in [4] a generalized Chern's non-linear connection $N^i{}_j(x, y)$ which is given by

$$\begin{aligned} N^i{}_j = \frac{1}{2}(&G^{ih}\partial_j G_{hs} - f^i{}_h G^{hr} f^t{}_j \partial_t G_{rs} + B^i{}_{sj} - G^{ih} G_{ms} B^m{}_{jh} \\ &- G^{ih}(\dot{\partial}_r G_{hm})y^m B^r{}_{sj} + G^{ih} f^m{}_h G_{rs} f^r{}_t B^t{}_{mj})y^s, \end{aligned} \tag{5.9}$$

where $B^i{}_{hj} = (\partial_h f^i{}_r) f^r{}_j$.

Now, the pair $(G_{ij}, N^i{}_j)$ given by (5.2) and (5.9) defines a regular (G, N)-structure in a Rizza manifold, which is called the *intrinsic (G, N)-structure* of a Rizza manifold. The intrinsic (G, N)-structure of a Rizza manifold is not only regular but regular in the sense of Miron by virtue of (5.8). Therefore, the results of the section 3 can be applied to the intrinsic (G, N)-structure of a Rizza manifold. Hence, from Theorem 3, Theorem 4 and Theorem 5, we have the following theorems immediately.

THEOREM 11. *In a Rizza manifold, the necessary and sufficient condition that the intrinsic (G, N)-structure satisfy $P^i{}_{jk} = 0$ is that*

(1) *the generalized Chern's non-linear connection $N^i{}_j$ coincides with the one $N^{*i}{}_j$ of the Berwald connection $B\Gamma^*$ of the given Finsler metric g_{ij},*
(2) *the associated coefficient $F_k{}^i{}_j$ coincides with $\dot{\partial}_k N^{*i}{}_j$ of $B\Gamma^*$.*

THEOREM 12. *In a Rizza manifold, the necessary and sufficient condition that the intrinsic (G, N)-structure be strongly Berwaldian is that*

(1) *the given Finsler metric g_{ij} is a Berwald metric,*
(2) *the generalized Chern's non-linear connection $N^i{}_j$ coincides with the one $N^{*i}{}_j$ of the Cartan connection $C\Gamma^*$ of g_{ij},*
(3) *the associated coefficient $F_k{}^i{}_j$ coincides with $\Gamma^{*i}_j{}_k$ of $C\Gamma^*$.*

THEOREM 13. *In the intrinsic (G, N)-structure of a Rizza manifold satisfying $P^i{}_{jk} = 0$, the necessary and sufficient condition that L^* be a Landsberg metric is*

$$(\dot{\partial}_i \dot{\partial}_j N^m{}_k) G_{mp} y^p = 0,$$

where $N^i{}_j$ is the generalized Chern's non-linear connection.

References

[1] M. Hashiguchi, *On generalized Finsler spaces*, An Ştiinţ. Univ. "Al. I. Cuza"Iasi **30** (1984), 69–73.

[2] Y. Ichijyō, *Finsler metrics on almost complex manifolds*, Rev. Mat. Univ. Parma (4) **14** (1988), 1–28.

[3] ———, *Conformally flat Finsler structures*, J. Math. Tokushima Univ. (1991), 13–25.

[4] ———, *Kaehlerian Finsler manifolds of Chern type*, Res. Bull. Toku-shima Bunri Univ. **57** (1999), 9–16.

[5] ———, *On the flatness of generalized Finsler manifolds and Kaehlerian Finsler manifolds of Chern type*, Res. Bull. Toku-shima Bunri Univ. **59** (2000), 11–18.

[6] Y. Ichijyō and M. Hashiguchi, *On generalized Finsler structures I*, Res. Bull. Toku-shima Bunri Univ. **61** (2001), 49–62.

[7] R. Miron, *Metrical Finsler structures and metrical Finsler connections*, J. Math. Kyoto Univ. **23** (1983), 219–224.

[8] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Press, Saikawa, Ohtsu-shi, Japan (1986).

[9] T. Okada, *Minkowskian product of Finsler spaces and Berwald connection*, J. Math. Kyoto Univ. **22** (1982), 323–332.

[10]G. B. Rizza, *Strutture di Finsler sulle varietā quasi complesse*, Atti Acad. Naz. Lincei Rend. **33** (1962), 271–275.

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