Derivations of Free Joins of Algebras*

Dept. of Mathematics, Chungbuk National Univ.  
hjy@chungbuk.ac.kr  
Jae-young Han

Dept. of Mathematics, Chungbuk National Univ.  
nskz@chungbuk.ac.kr  
Sook-Ja Nam

Dept. of Mathematics, Chungbuk National Univ.  
kmhe@chungbuk.ac.kr  
Yeon-hee Kim

In this paper, we will prove that a free join algebra and a universal derivation module of its subalgebras have a universal derivation module induced by its subalgebras.

Key words: (universal) derivation module, tensor algebra, free join algebra, fractional extension

0. Introduction

Let $R$ be a commutative ring with identity and $A$ a unitary algebra over $R$ which is not necessarily commutative. For an $(A, A)$-bimodule $M$, an $R$-linear mapping $d : A \to M$ is called an $R$-derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. A pair $(M, d)$ of an $(A, A)$-bimodule $M$ and an $R$-derivation $d : A \to M$ is called a derivation module of $A$. An $(A, A)$-bimodule homomorphism $f : (M, d) \to (N, \delta)$ is a derivation module homomorphism if $f \cdot d = \delta$. A derivation module $(U, d)$ is called a universal derivation module if for any derivation module $(N, \delta)$ of $A$, there exists a unique derivation module homomorphism $f : (M, d) \to (N, \delta)$.

An $R$-algebra $A$ is called a tensor algebra of an $R$-module of $M$ over $R$ if for any $R$-algebra $C$ and an $R$-linear mapping $f : M \to C$, there exists a unique $R$-algebra homomorphism $g : A \to C$ extending $f$.

* 충북대학교 기초과학연구소 연구비에 의하여 연구되었음.
Every tensor algebra of an \( R \)-module \( M \) over \( R \) is generated by \( M \) and it is unique up to algebra isomorphism.

1. Free Joins of Algebras

An \( R \)-algebra \( A \) is called a free join of a family \( (A_a)_{a \in I} \) of its subalgebras if any algebra \( C \) and any family \( (f_a)_{a \in I}, \ f_a : A_a \to C \) of algebra homomorphisms, there exists a unique algebra homomorphism \( f : A \to C \) extending \( f_a \) for each \( a \in I \).

**Proposition 1.** Let \( A \) be a free join of a family \( (A_a)_{a \in I} \) of its subalgebras. If \( R \) is a direct summand of each \( A_a \) and there exists an \( R \)-module homomorphism \( g_a : A \to R \) for each \( a \in I \), then for any finite sequence \( \beta = (a_1, \ldots, a_k) \) where the \( a_i \) are all different, the mapping \( f : A_{a_1} \cdots A_{a_k} \to A_{a_1} \otimes \cdots \otimes A_{a_k} \) given by \( a_{a_1} \cdots a_{a_k} \to a_{a_1} \otimes \cdots \otimes a_{a_k} \) is an \( R \)-module homomorphism.

**Lemma 1.** Let \( A \) be a free join of a family \( (A_a)_{a \in I} \) of its subalgebras and \( T(A_a) \) and \( T(B) \) be the tensor algebras of \( A_a \) and \( B = \bigoplus_{a \in I} A_a \) respectively. If \( h_a : T(A_a) \to A_a \) is an algebra homomorphism extending the identity mapping \( i_{A_a} \) for each \( a \in I \) and \( h : T(B) \to A \) is an algebra homomorphism extending \( h_a \) for each \( a \in I \), then \( h \) is onto and \( \ker h \) is the ideal of \( T(B) \) generated by \( \sum_{a \in I} \ker h_a \).

**Theorem 1.** Let \( A \) be a free join of a family \( (A_a)_{a \in I} \) of its subalgebras and \( a_{a_1}, \ldots, a_{a_k} \) a finite sequence such that \( a_i \neq a_{i+1}, \ i = 1, \ldots, k-1 \), then the \( R \)-linear mapping \( f : A_{a_1} \otimes \cdots \otimes A_{a_k} \to A_{a_1} \cdots A_{a_k} \) given by \( a_{a_1} \otimes \cdots \otimes a_{a_k} \to a_{a_1} \cdots a_{a_k} \) is an \( R \)-module isomorphism.

**Proof.** Let \( h \) be the algebra homomorphism in Lemma 1. Let \( f = h \mid A_{a_1} \otimes \cdots \otimes A_{a_k} \).
\( \otimes A_a \). Since \( \ker f \subset \ker h_a \) and \( \ker h \cap (A_a \otimes \cdots \otimes A_a) = \emptyset \), \( \ker f \cap (A_a \otimes \cdots \otimes A_a) = \emptyset \). Hence this mapping is one to one. This implies that \( f \) is an isomorphism.

**Theorem 2.** Let \( A \) be a free join of a family \( (A_a)_{a \in I} \) of its subalgebras and \( (U_a, d_a) \) a universal derivation module of \( A_a \) for each \( a \in I \). If 
\[
U = \bigoplus_{a \in I} (A \otimes U_a \otimes A)
\]
and \( D: A \rightarrow U \) is the \( R \)-derivation defined by 
\[
\sum_{a \in I} a_i \cdots a_k 
\]
\[ \rightarrow \sum_{a \in I} (\sum_{i=1}^k a_i \cdots a_{i-1} \otimes d_{a_i}(a_i) \otimes a_{i+1} \cdots a_k) \] where \( a_i \in A_a \), \( a_i \in I \), then \( (U, D) \) is a universal derivation module of \( A \).

**Proof.** Let \( \phi_a: A_a \times \cdots \times A_a \rightarrow U \) be an \( R \)-multilinear mapping given by 
\[
(a_1, \ldots, a_k) \rightarrow a_1 \cdots a_{i-1} \otimes d_{a_i}(a_i) \otimes a_{i+1} \cdots a_k
\]
where \( a_i \in A_a \), and \( D_{a_i} \) the \( R \)-linearization of \( \phi_a \). Define a mapping 
\( D: A \rightarrow U \) by
\[
D(\sum_{a \in I} a_i \cdots a_k) = \sum_{a \in I} (\sum_{i=1}^k D_{a_i}(a_1 \cdots a_k)), \quad a_i \in A_a
\]
Then \( D \) is an \( R \)-derivation same as above. To show that \( (U, D) \) is a universal derivation module of \( A \), let \( (M, \delta) \) be any derivation module of \( A \) and \( \delta_{a_i} = \delta|A_a \).

Since each \( \delta_{a_i} \) is an \( R \)-derivation and \( (U_a, D_a) \) is a universal derivation module of \( A_a \), there exists a unique \( (A_a, \delta_a) \)-bimodule homomorphism \( f_a: A_a \rightarrow M \) such that \( f_a \cdot d_a = \delta_a \). Let \( g_a: A \times U_a \times A \rightarrow M \) be an \( A \)-multilinear mapping given by 
\[
(a, u_a, b) \rightarrow a f_a(u_a) b
\]
where \( a \in U_a, a, b \in A \). Let \( g_a \) be the \( A \)-liberalization of \( g_a \). Then each \( g_a \) is an \( (A, A) \)-bimodule homomorphism. Define a mapping \( g: U \rightarrow M \) by 
\[
\sum_a (a_a \otimes u_a \otimes b_a) \rightarrow \sum_a g_a(a_a \otimes u_a \otimes b_a)
\]
where \( a_a, b_a \in A \) and \( \sum_a (a_a \otimes u_a \otimes b_a) \in A \otimes U_a \otimes A \).

Then \( h \) is an \( R \)-derivation module homomorphism such that \( g \cdot D = \delta \). The uniqueness of such homomorphism from the fact that each \( A \otimes U_a \otimes A \) is generated by \( D_a(A_a) \) and hence \( U \) is generated by \( D(A) \). We proved that \( (U, D) \) constructed in this way is a universal derivation module of \( A \).
2. Extension of Algebras

Let $E$ be a unitary extension algebra of an $R$-algebra. An ideal $I$ of $A$ is said to be $E$-dense if $EI = IE = E$. An ideal $I$ of $A$ which contains an $E$-dense ideal $J$ is also $E$-dense, since $EI \supseteq EJ = E$, $IE \supseteq JE = E$. If $I$ and $J$ are $E$-dense ideals of $A$, then $IJ$ and $JI$ are $E$-dense.

An extension $E$ of an $R$-algebra $A$ is called a fractional extension of $A$ if for any $p, q \in A$, there exists $E$-dense ideals $I$ and $J$ such that $pI, Jq \in A$.

Proposition 2. Let $Q$ be a two-sided quotient algebra of an $R$-algebra $A$ with relative to a multiplicative subset of $S$ without zero divisor. Then $Q$ is a fractional extension of $A$.

An $(A, A)$-bimodule $M$ is said to be $E$-torsion free if for any $E$-dense ideal $I$ of $A$ and $x \in M$, $Ix = 0$ implies $x = 0$ and $xI = 0$ implies $x = 0$. For example every $(E, E)$-bimodule is an $E$-torsion free $(A, A)$-bimodule, when $E$ is a fractional extension of $A$.

Lemma 2. Let $E$ be a fractional extension of an $R$-algebra $A$.

(1) For any $E$-torsion free $(A, A)$-bimodule $M$, an $(A, A)$-bimodule homomorphism $f : M \to E \otimes_A M \otimes_A E$ given by $f(x) = 1 \otimes x \otimes 1$, $x \in M$ is one to one.

(2) Every $(A, A)$-bimodule homomorphism $f : M \to N$ of $(E, E)$-bimodules is an $(E, E)$-bimodules homomorphism.

Theorem 3. Let $E$ be a fractional extension of an $R$-algebra $A$, and $M$ an $(E, E)$-bimodule. If $R$-derivation $d, \delta : E \to M$ are equal on $A$, then $d = \delta$.

Proof. Let $I$ be an $E$-dense ideal of $A$ such that $Iq \subseteq A$, $q \in E$. Since $d - \delta$ is an $R$-derivation of $E$, $(d - \delta)(bq) = b(d - \delta)(q) + (d - \delta)(b)q$, $b \in I$. Since $bq \in A$, $b \in A$, we have $b(d - \delta)(q) = 0$ for all $q \in E$. Hence $I((d - \delta)q = 0$.

By the fact $M$ is an $E$-torsion free $(A, A)$-bimodule, $(d - \delta)(b)q = 0$, $q \in E$. ■

An $(A, A)$-bimodule $H$ is called an injective hull of an $(A, A)$-bimodule $M$ if $H$
is a left(right) injective hull of the left $A\otimes_A A^{op}$-module (right $A^{op}\otimes_A A$-module) $M$.

**Lemma 3.** Let $E$ be a fractional extension of an $R$-algebra $A$, and $M$ an $E$-torsion free $(A, A)$-bimodule. Then every injective hull of $M$ is $E$-torsion free.

**Lemma 4.** Let $E$ be a fractional extension of an $R$-algebra $A$, and $M$ an $E$-torsion free $(A, A)$-bimodule, and $I$ an $E$-dense ideal $A$. If $\phi: I \to M$ is an $(A, A)$-bimodule homomorphism, then there exists a unique $(A, A)$-bimodule homomorphism $f: A \to M$ extending $\phi$.

**Theorem 4.** Let $E$ be a fractional extension of an $R$-algebra $A$, and $f: M \to E \otimes_A M \otimes_A E$ an $(A, A)$-bimodule homomorphism given by $f(x) = 1 \otimes x \otimes 1$ for all $x \in M$. Then for any $R$-derivation $d: A \to M$, there exists a unique $R$-derivation $\delta: E \to E \otimes_A M \otimes_A E$ such that $\delta|A = f \cdot d$.

**Proof.** Let $q \in E$, and $I$ an $E$-dense ideal of $A$ such that $qI \subseteq A$. Define a mapping $f_{I, q}: I \to E \otimes_A M \otimes_A E$ by $b \to 1 \otimes d(ab) \otimes 1 - q \otimes d(b) \otimes 1$ for all $b \in I$. Then $f_{I, q}$ is a right $A$-module homomorphism. Indeed for any $a, b \in I, r, s \in R$, $f_{I, q}(ra + sb) = rf_{I, q}(a) + sf_{I, q}(b)$.

For all $c \in A$, $f_{I, q}(ra + sb) = rf_{I, q}(a) + sf_{I, q}(b)$ . By Lemma 4 there exists a unique $A$-module homomorphism $g_{I, q}: A \to E \otimes_A M \otimes_A E$ extending $f_{I, q}$ for all $q \in E$. To Show that $g_{I, q}$ is independent of the choice of an $E$-dense ideal of $A$, let $J$ be any $E$-dense ideal of $A$ such that $qJ \subseteq A$. Since $I \cap J$ is an $E$-dense ideal of $A$, there exists a unique right $A$-module homomorphism $g_{I \cap J, q}: A \to E \otimes_A M \otimes_A E$ extending the left $A$-module homomorphism $f_{I \cap J, q}: I \cap J \to E \otimes_A M \otimes_A E$ given by $c \to 1 \otimes d(qc) \otimes 1 - q \otimes d(c) \otimes 1$ for all $c \in I \cap J$. Then $f_{I, q}|I \cap J = f_{I \cap J, q}|I \cap J$. Hence $g_{I, q}$ and $g_{J, q}$ are right $A$-module homomorphism extending a right $A$-module homomorphism $f_{I \cap J, q}$. By the uniqueness of such $A$-module homomorphism, $g_{I, q} = g_{J, q}$. Let
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\[ g_q = g_{1,q} \text{ for all } q \in E. \] Define a mapping \( \delta : E \to E \otimes_A M \otimes_A E \) by
\[ \delta(q) = g_q(1), \quad q \in E. \] Then \( \delta \) is an \( R \)-derivation.

Let \( g_a = f_A, a : A \to E \otimes_A M \otimes_A E \) be a right \( A \)-module homomorphism given by
\[ f(ab) = 1 \otimes d(ab) \otimes 1 - a \otimes d(b) \otimes 1, \quad b \in A. \] Then \( \delta(a) = g_a(1) = 1 \otimes d(a) \otimes 1 - a \otimes d(1) \otimes 1, \quad b \in A. \) Hence \( \delta : E \to E \otimes_A M \otimes_A E \) is \( R \)-derivation such that \( \delta|A = f \cdot d \). By Theorem 3, \( \delta \) is a unique \( R \)-derivation of \( E \) satisfying the given condition.

**Theorem 5.** Let \( E \) be a fractional extension of an \( R \)-algebra \( A \), \( (U, d) \) a universal derivation module of \( A \), and \( \delta : E \to E \otimes_A M \otimes_A E \) an \( R \)-derivation such that \( \delta|A = f \cdot d \). Here \( f : U \to E \otimes_A U \otimes_A E \) is an \((A, A)\)-bimodule homomorphism given by \( f(x) = 1 \otimes x \otimes 1 \) for all \( x \in U \). Then \((E \otimes_A U \otimes_A E, \delta)\) is a universal derivation module of \( E \).

**Proof.** Let \((V, \tau)\) be any derivation module of \( A \), and let \( \tau' = \tau|A \). Since \( \tau' \) is an \( R \)-derivation of \( A \), there exists an \((A, A)\)-bimodule homomorphism \( g : U \to V \) such that \( g \cdot d = \tau' \). Let \( \phi \) be the \( A \)-liberalization of \( A \)-multilinear mapping \( \phi : E \times U \times E \to M \) given by \( (p, x, q) \to \rho g(x)q, \rho, q \in E \). Then \( \phi \) is an \((E, E)\)-bimodules homomorphism. Furthermore \( \phi \cdot \delta \) and \( \tau \) are \( R \)-derivation of \( E \) such that \( \phi \cdot \delta = \tau \) on \( A \). By Theorem 3, \( \phi \cdot \delta = \tau \) on \( E \). The uniqueness of such module homomorphism follows from the fact \((E \otimes_A U \otimes_A E, \delta)\) is a universal derivation module of \( E \).

**References**


다원환의 자유결합의 미분

충북대학교 수학과

한재영

충북대학교 수학과

남숙자

충북대학교 수학과

김연희

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