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A Cutting-plane Generation Method for a Variable-capacity (0,1)-Knapsack Problem with General Integer Variables*

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ABSTRACT

In this paper, we propose an effective cut generation method based on the Chvatal-Gomory procedure for a variable-capacity (0,1)-Knapsack problem with two general integer variables. We first derive a class of valid inequalities for the problem using Chvatal-Gomory procedure, then analyze the associated separation problem. Based on the results, we show that there exists a pseudo-polynomial time algorithm to solve the separation problem. By analyzing the theoretical strength of the inequalities which can be generated by the proposed cut generation method, we show that generated inequalties define facets under mild conditions. We also extend the result to the case in which a nontrivial upper bound is imposed on a general integer variable.

Keywords: Cutting-plane, Knapsack problem, Separation problem

1. INTRODUCTION

In this paper, we present a cut generation method for a specific knapsack polyhedron involving both binary and integer variables by customizing Chvatal-Gomory procedure [11]. Consider the following polyhedron Q.

$$Q = conv\{w \in B^{K}, y \in Z_{+}, z \in Z_{+} \mid \sum_{k \in K} r_k w_k - \lambda_1 y - \lambda_2 z \le 0\},$$

where $K = \{1,2,\cdots,\kappa\}$, $r_k, \forall k \in K$, λ_1 and λ_2 are positive integers, and $\lambda_2 / \lambda_1 = \lambda$

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for some positive integer λ. This type of constraints appears as sub-models in many network design applications [3, 8, 9].

Now, consider the following inequality which is valid to Q.

$$\sum_{k\in J} \lfloor u_0 r_k + u_k \rfloor w_k + \sum_{k\in K\setminus J} \lfloor u_0 r_k \rfloor w_k + \lfloor -u_0\lambda_1 \rfloor y + \lfloor -u_0\lambda_2 \rfloor z \le \left\lfloor \sum_{k\in J} u_k \right\rfloor, \quad (1)$$

where $J \subseteq K$, $u_0 \ge 0$, and $u_k \ge 0$, for all $k \in J$

For each $J \subseteq K$, the inequality (1) can be derived by using the Chvatal-Gomory procedure [11]. That is, we surrogate the upper bound constraints $w_k \le 1$, for all $k \in J$ with the constraint $\sum_{k\in K} r_k w_k - \lambda_1 y - \lambda_2 z \le 0$ using nonnegative surrogate multipliers u_k, for all $k \in J$, and u_0. Then by summing and rounding down the coefficients of the surrogate constraint, we obtained the inequality (1). From the derivation of (1), the validity of it is readily verified. For more details of Chvatal-Gomory procedure, refer to Nemhauser and Wolsey[11].

The set of valid inequalities of the type (1) include many strong valid inequalities for Q. We will give an example.

Example 1. Consider the following polyhedron.

$$conw\{w \in B^4, (y,z) \in Z_+^2 \mid 13w_1 + 11w_2 + 11w_3 + 10w_4 - 8y - 32z \le 0\}$$

with $J = \{1,4\}$ and $u_0 = 3/32$, $u_1 = 25/32$, and $u_4 = 2/32$. Then, the corresponding inequality of the type (1) is given by

$$2w_1 + w_2 + w_3 + w_4 - y - 3z \le 0,$$

which is a facet-defining inequality.

To use the inequalities of the type (1) in a branch-and-cut algorithm for some integer programs in which the defining constraint of Q is included, we should be able to generate inequalities of the type (1) which are violated by a given fractional solution to the corresponding LP relaxation. Lee and Park[10] proposed a special type of Chvatal-Gomory cuts which are valid to a polyhedron which is a special case of Q. They also proposed a separation algorithm for the inequalities. In this paper, we extend Lee and Park's results to more general cases. Before presenting the details of our cut generation method, we briefly introduce valid inequalities which can be obtained from existing cut generation methods.

First, Ceria et al. [4] proposed a minimal cover inequality for the knapsack problems with general integer variables. This inequality can be viewed as a natural generalization of minimal cover inequalities for the 0-1 knapsack polytope [1, 2, 11]. Now, let us define an integer knapsack polytope Q' associated with Q as follows:

$$Q' = conv\{w \in B^{K}, (y', z') \in Z_{+}^{2} \mid \sum_{k \in K} r_k w_k + \lambda_1 y' + \lambda_2 z' \le \lambda_1 U_1 + \lambda_2 U_2, y' \le U_1, z' \le U_2\},$$

where $y' = U_1 - y, z' = U_2 - z, U_1 = \lceil \sum_{k \in K} r_k / \lambda_1 \rceil$, and $U_2 = \lceil \sum_{k \in K} r_k / \lambda_2 \rceil$.

A cover defined in Ceria et al. [4] is a subset of variables that can not have values corresponding to their upper bounds simultaneously without violating the right-hand-side of the given knapsack constraint. A cover is minimal if all the variables in any proper subset of it can have values corresponding to their upper bounds simultaneously. From this definition, it can be readily verified that any minimal cover for Q' consists of w_k, for some $k \in K$, y', and z'. Then the corresponding minimal cover inequalities which are proposed by Ceria et al.[4] are as follows :

$$(1 - w_k) + (U_1 - y') + (U_2 - z') \ge 1, \text{ for all } k \in K.$$

For the details of the derivation of the minimal cover inequality, refer to Ceria et al. [4]. Now, the above inequalities can be rewritten as follows:

$$w_k - y - z \le 0, \text{ for all } k \in K.$$

These inequalities are valid to Q but trivial in the sense that they can be readily obtained from the definition of Q.

Brockmuller et al. [3] devised several classes of valid inequalities for a polyhedron which is similar to Q. Their idea which is used to derive those valid inequalities can be used to generate non-trivial valid inequalities for Q. If we use their idea, we can obtain the following valid inequalities (2) for Q .

$$\sum_{k \in J} \lceil r_k / \lambda_1 \rceil w_k + \sum_{k \in K \setminus J} \lfloor r_k / \lambda_1 \rfloor w_k - y - \lambda z \le \theta, \tag{2}$$

where $J \subseteq K$, $\lambda = \lambda_2 / \lambda_1$, and $\theta = \sum_{k \in J} \lceil r_k / \lambda_1 \rceil - \lceil \sum_{k \in J} r_k / \lambda_1 \rceil$. Note that λ is a positive integer as mentioned before.

We show later that the sets of valid inequalities (1) for Q obtained from our cut generation method include inequalities (2).

The remainder of this paper is organized as follows. In section 2, we analyze the separation problem associated with the inequalities of the type (1). In section 3, the theoretical strength of the inequalities of the type (1) is analyzed. We further extend our cut generation method to the case in which a nontrivial upper bound is imposed on y variable in section 4. Concluding remarks are given in section 5.

2. A CUTTING-PLANE GENERATION METHOD

Suppose that we are given a fractional solution $(\overline{w},\overline{y},\overline{z})$ such that $\overline{w}_k \le 1$, for all $k \in K$ and $\sum_{k\in K} r_k\overline{w}_k - \lambda_1\overline{y} - \lambda_2\overline{z} \le 0$. Then, if we let $J^* = \{k \in K \mid \overline{w}_k > 0\}$, the corresponding separation problem (MSEP) for identifying an inequality of the type (1) violated by $(\overline{w},\overline{y},\overline{z})$ can be formulated as follows:

$$\text{MSEP}: \max \sum_{k\in J^*} \lfloor u_0 r_k + u_k \rfloor \overline{w}_k + \lfloor -u_0\lambda_1 \rfloor \overline{y} + \lfloor -u_0\lambda_2 \rfloor \overline{z} - \lfloor \textstyle\sum_{k\in J^*} u_k \rfloor$$

$$\text{s.t.} \quad u_0 \ge 0, u_k \ge 0, \text{ for all } k \in J^*$$

Let $\hat{u}_0$ and $\hat{u}_k$, for all $k \in J^*$, be an optimal solution to MSEP. If the corresponding optimal objective value is less than or equal to 0, then we can conclude that there are none of the inequalities of the type (1) which are violated by the given solution $(\overline{w},\overline{y},\overline{z})$. Otherwise, the inequality of the type (1) obtained by using $\hat{u}_0$ and $\hat{u}_k$, for all $k \in J^*$, is the most violated inequality among those of the type (1).

To simplify the analysis of MSEP, let us suppose that we are interested in inequalities of the type (1) in which the coefficient of the variable z is equal to an integer in some prespecified range. To begin, suppose that we set $\lfloor -u_0\lambda_2 \rfloor = -p_0$, where p_0 is a positive integer. That is, $-p_0/\lambda_2 \le -u_0 < (-p_0+1)/\lambda_2$. Then, for any u_0 such that $-p_0/\lambda_2 \le -u_0 < (-p_0+1)/\lambda_2$, $-u_0\lambda_1 = -p_0/\lambda + l/\lambda$, where $0 \le l \le 1$ and a positive integer $\lambda = \lambda_2/\lambda_1$ so that $\lfloor -u_0\lambda_1 \rfloor = \lfloor -p_0/\lambda \rfloor$.

Then, the corresponding separation problem MSEP(p_0) for identifying an inequality of the type (1) in which the coefficient of z is equal to $-p_0$ can be formulated as follows :

$$\begin{aligned} \text{MSEP}(p_0): \max \ & \sum_{k\in J^*} \lfloor u_0 r_k + u_k \rfloor \overline{w}_k + \lfloor -u_0\lambda_1 \rfloor \overline{y} + \lfloor -u_0\lambda_2 \rfloor \overline{z} - \lfloor \textstyle\sum_{k\in J^*} u_k \rfloor \\ \text{s.t.} \ & p_0 - 1 < u_0\lambda_2 \le p_0 \\ & u_k \ge 0, \text{ for all } k \in J^* \end{aligned}$$

As mentioned above, for any u_0 such that $-p_0/\lambda_2 \le -u_0 < (-p_0+1)/\lambda_2$, $\lfloor -u_0\lambda_1 \rfloor = \lfloor -p_0/\lambda \rfloor$. Therefore, the following proposition can be easily verified, so we omit the proof of it.

Proposition 1. *There exists an optimal solution* u_0^* *and* u_k^*, *for all* $k \in J^*$ *to* MSEP(p_0) *such that* $u_0^* = p_0/\lambda_2$.

If we set $u_0 = p_0/\lambda_2$, $u_0 r_k$ can be expressed as $p_k + f_k$, where p_k is a non-negative integer and $0 \le (f_k = q_k/\lambda_2) < 1$ with a nonnegative integer q_k, for each $k \in J^*$. Then, by proposition 1, MSEP(p_0) can be transformed as follows :

$$\begin{aligned} \text{MSEP}(p_0): \max \ & \sum_{k\in J^*} p_k \overline{w}_k + \lfloor -p_0/\lambda \rfloor \overline{y} - p_0\overline{z} + \sum_{k\in J^*} \lfloor f_k + u_k \rfloor \overline{w}_k - \lfloor \textstyle\sum_{k\in J^*} u_k \rfloor \\ \text{s.t.} \ & u_k \ge 0, \text{ for all } k \in J^* \end{aligned}$$

Note that $C \equiv \sum_{k\in J^*} p_k\overline{w}_k + \lfloor -p_0/\lambda \rfloor \overline{y} - p_0\overline{z}$ is a constant. Since $\overline{w}_k \le 1$, for all $k \in J^*$, it can be easily shown that the following proposition holds.

Proposition 2. *There exits an optimal solution to* MSEP(p_0) *such that* $u_k \in \{0, 1-f_k\}$, *for all* $k \in J^*$.

By proposition 2, we can solve MSEP(p_0) by solving the following problem MSP(p_0) which is formulated by introducing a binary variable t_k, for each $k \in J^*$ such that $t_k = 1$ if and only if $u_k = 1 - f_k$ and $t_k = 0$ if and only if $u_k = 0$.

$$\begin{aligned} \text{MSP}(p_0): \quad \max \ & \sum_{k\in J^*} \overline{w}_k t_k - \left\lfloor \sum_{k\in J^*} (1-f_k) t_k \right\rfloor + C \\ \text{s.t.} \ & t_k \in \{0,1\}, \text{ for all } k \in J^* \end{aligned}$$

If we are to identify the most violated inequality of the type (1) with $l \le p_0 \le u$, where l and u are positive integers, we can do that by sequentially solving MSP(p_0) for all $l \le p_0 \le u$. Clearly, to prespecify the range of the value p_0 could restrict the ability to identify inequalities of the type (1) which are violated by the given fractional solution. Hence, we should be careful in specifying the range of p_0. We discuss this issue later in the next section. More important issue than this is whether we can solve MSP(p_0) efficiently for a fixed p_0. Unfortunately, MSP(p_0) is known to be NP-hard [10].

Though MSP(p_0) is NP-hard so that there might not exist a polynomial time algorithm, we have a pseudo-polynomial time algorithm to solve MSP(p_0). Lee and Park[10] give a pseudo-polynomial time algorithm to solve an optimization problem which is essentially the same as MSP(p_0). This algorithm is based on the dynamic programming approach [11]. If we adapt Lee and Park's algorithm to MSP(p_0), we can devise an algorithm whose number of steps required to get an optimal solution to MSP(p_0) is $O(n^2\lambda_2)$, where $n = |J^*|$. For more details, refer to Lee and Park [10].

Recall that we are interested in inequalities of the type (1) with $l \le p_0 \le u$, where l and u are positive integers. Hence, the overall complexity of our separation procedure for the inequalities of the type (1) is $O(n^2\lambda_2\mu)$, where $\mu = u - l + 1$.

3. STRENGTH OF THE GENERATED INEQUALITIES

In this section, we analyze the theoretical strength of the inequalities of the type (1). First, we mention that Q is full-dimensional, that is, the dimension of Q is $\kappa + 2$. Now, let us rewrite the inequalities of the type (1) which are obtained by solving MSP(p_0) for a given p_0 as follows :

$$\sum_{k\in J} \phi(p_0 r_k / \lambda_2) w_k + \sum_{k\in K\setminus J} \lfloor p_0 r_k / \lambda_2 \rfloor w_k + \lfloor -p_0/\lambda \rfloor y - p_0 z \le \lfloor \textstyle\sum_{k\in J}(1-f_k) \rfloor, \quad (3)$$

where $J \subseteq K$, $\phi(a) = \lfloor a \rfloor + 1, a \in R^+$ and $1 - f_k = \phi(p_0 r_k / \lambda_2) - p_0 r_k / \lambda_2$, for all $k \in J$.

If we let $p_0 = \lambda$, it can be easily shown that we can get inequalities of the type (2) by using the definition of $\phi(\bullet)$ and the following lemma due to Lee and Park [10]. For the details of the proof of the lemma, refer to Lee and Park [10].

Lemma 3. $\sum_{k\in J} \phi(p_0 r_k / \lambda_2) - \left\lfloor \sum_{k\in J}(1-f_k) \right\rfloor = \left\lceil \sum_{k\in J} p_0 r_k / \lambda_2 \right\rceil$, *for every* $J \subseteq K$ *and for every positive integer* p_0.

For a given inequality of the type (3), let Ω be the set of integral points in Q which satisfy (3) at equalities. By proposition 6.6 in Nemhauser and Wolsey [11](p.108), the inequality is a facet-defining inequality of Q if and only if there exist $\kappa + 2$ affinely independent points in Ω. Moreover, if $p_0 = \lambda$ and $\left\lceil \sum_{k\in J} r_k / \lambda_1 \right\rceil \geq \lambda$, then the inequalities of the type (3) define facets of Q under mild conditions. The following proposition states the result. Since the proof of the proposition is somewhat straightforward, we omit it.

Proposition 4. *Suppose that* $p_0 = \lambda$ *and* $\left\lceil \sum_{k\in J} r_k / \lambda_1 \right\rceil \geq \lambda$, *for a given* $J \subseteq K$. *Then (3) defines a facet of Q if and only if the following conditions hold:*

i) $\left\lfloor \sum_{k\in J\setminus\{u\}}(1-f_k) \right\rfloor = \left\lfloor \sum_{k\in J}(1-f_k) \right\rfloor$, *for all* $u \in J$ *and*

ii) $\left\lfloor \sum_{k\in J\cup\{v\}}(1-f_k) \right\rfloor > \left\lfloor \sum_{k\in J}(1-f_k) \right\rfloor$, *for all* $u \in K \setminus J$.

Now, we discuss the effective range of p_0. First, it can be easily shown that if $p_0 > \lambda_2$, the inequalities of the type (3) can not define a facet of Q. Also, if $p_0 = \lambda_2$, the inequality of the type (3) is equivalent to or dominated by the defining inequality of Q. Therefore, the effective range of p_0 is within $[1, \lambda_2 - 1]$. For more insights on the strength of the inequality of the type (3), further in-depth analysis is required.

4. AN EXTENSION

Now suppose that a non-trivial upper bound τ on the value of y is given. Then, the upper bound constraint $y \leq \tau$ can be incorporated in generating stronger inequalities. If we incorporate the constraint with a surrogate multiplier $u_y \geq 0$ in the derivation of Chvatal-Gomory inequalities, we have, for $J \subseteq K$,

$$\sum_{k\in J} \lfloor u_0 r_k + u_k \rfloor w_k + \sum_{k\in K\setminus J} \lfloor u_0 r_k \rfloor w_k + \lfloor -u_0\lambda_1 + u_y \rfloor y + \lfloor -u_0\lambda_2 \rfloor z \leq \left\lfloor \sum_{k\in J} u_k + \tau u_y \right\rfloor \tag{4}$$

If we set $u_y = 0$, the above inequalities are the same as those of the type (1). One of the inequalities of the type (4) is given in the following example.

Example 2. Consider the following polyhedron.

$$conv\{w_1 \in B, (y,z) \in Z_+^2 \mid 7w_1 - y - 4z \le 0, y \le 1\}$$

We can obtain the following inequality of the type (4) with $u_0 = 1/4$, $u_y = 0$, and $u_1 = 1/4$, which is not a facet-defining inequality.

$$2w_1 - y - z \le 0$$

We can also get stronger one with $u_0 = 1/4$, $u_y = 1/4$, and $u_1 = 1/4$,

$$2w_1 - z \le 0$$

which defines a facet. □

Suppose also that we are to generate inequalities of the type (4) in which the coefficient of z is equal to $-p_0$. We mention that the corresponding separation problem might not have an optimal solution in which $u_0 = p_0 / \lambda_2$. This is because the coefficient of y is no longer a constant. Hence, to set $u_0 = p_0 / \lambda_2$ might weaken the strength of generated inequalities of the type (4). However, we know that the generated inequalities of the type (4) are at least as strong as those of the type (1) even if we set $u_0 = p_0 / \lambda_2$. Also, in order to use the algorithm for MSEP(p_0), we set $u_0 = p_0 / \lambda_2$. Then the corresponding MSEP'(p_0) is formulated as follows :

$$\text{MSEP'}(p_0): \quad \max \sum_{k \in J^*} \lfloor f_k + u_k \rfloor \overline{w}_k + \lfloor u_y - f_y \rfloor \overline{y} - \lfloor \textstyle\sum_{k \in J^*} u_k + \tau u_y \rfloor + C$$

$$\text{s.t.} \quad u_y \ge 0, u_k \ge 0, \text{ for all } k \in J^*$$

where $f_y = p_0 / \lambda - \lfloor p_0 / \lambda \rfloor$ and $C \equiv \sum_{k \in J^*} p_k \overline{w}_k + \lfloor -p_0 / \lambda \rfloor \overline{y} - p_0 \overline{z}$.

Note that $f_y = q_y / \lambda_2$, for some nonnegative integer $0 \le q_y < \lambda_2$. Also, note that there exists an optimal solution to MSEP'(p_0) in which $u_y \in \{0, f_y\}$ and $u_k \in \{0, 1 - f_k\}$, for all $k \in J^*$. Therefore, we can solve MSEP'(p_0) by solving the following MSP'(p_0) which is essentially the same problem as MSP(p_0) :

$$\text{MSP'}(p_0): \quad \max \sum_{k\in J^*} \overline{w}_k t_k - \overline{y}(1-t_y) - \left\lfloor \sum_{k\in J^*}(1-f_k)t_k + \tau f_y t_y \right\rfloor + C$$

$$\text{s.t. } t_y \in \{0,1\},\ t_k \in \{0,1\},\ \text{for all } k \in J^*,$$

where t_y is defined such that $t_y = 1$ if and only if $u_y = f_y$. The above MSP'(p_0) can be solved by using the algorithm for MSP(p_0) with little modification.

5. CONCLUDING REMARKS

In this paper, we propose an effective cut generation method based on the Chvatal-Gomory procedure for a variable-capacity (0,1)-Knapsack problem with two general integer variables. We derive a class of valid inequalities for the problem using Chvatal-Gomory procedure, then analyze the associated separation problem. Based on the results, we show that there exists a pseudo-polynomial time algorithm to solve the separation problem. However, the cut generation method presented in section 2 needs quite much computational efforts. This observation naturally suggests several interesting research topics. First, an effective heuristic algorithm to approximately solve the separation problem for the inequality will be needed if the coefficient of z variable becomes larger. Further strengthening procedure would be needed to strengthen the heuristically separated inequalities. Second, more theoretical investigations on the strength of the inequalities are needed. Those results may be used to improve our cut generation method.

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