ON THE GENUS OF $S^m \times S^n$

PAOLA CRISTOFORI

Abstract. By using a recursive algorithm, we construct edge-coloured graphs representing products of spheres and consequently we give upper bounds for the regular genus of $S^m \times S^n$, for each $m, n > 0$.

1. Introduction

Throughout this paper we shall work in the PL category. In the following the term “manifold” will denote a closed, connected one and “graph” a finite connected multigraph (i.e. without loops).

An $(n + 1)$-coloured graph (without boundary) is a pair $(\Gamma, \gamma)$, where $\Gamma = (V(\Gamma), E(\Gamma))$ is a graph, regular of degree $n + 1$, and $\gamma : E(\Gamma) \to \Delta_n = \{0, 1, \ldots, n\}$ a map such that $\gamma(e) \neq \gamma(f)$, for each pair $e, f$ of adjacent edges of $\Gamma$. For each $B \subseteq \Delta_n$, the $B$-residues of $(\Gamma, \gamma)$ are the connected components of the graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$. For each $c \in \Delta_n$, we set $\dot{c} = \Delta_n \setminus \{c\}$ and we shall write $\Gamma_{\odot d}$ instead of $\Gamma_{\{c,d\}}$.

An $(n + 1)$-coloured graph is called contracted if and only if for every $c \in \Delta_n$, $\Gamma_{\dot{c}}$ is connected.

From now on we often drop the edge-colorations, writing $\Gamma$ instead of $(\Gamma, \gamma)$.

Let $K$ be an $n$-dimensional pseudocomplex, the disjoint star $\text{std}(s, K)$ of a simplex $s$ in $K$ is the disjoint union of the $n$-simplexes containing $s$, with re-identification of the $(n - 1)$-simplexes containing $s$ and of all their faces; the disjoint link of $s$ in $K$ is the complex $\text{lkd}(s, K) = \{t \in \text{std}(s, K) | s \cap t = \emptyset\}$.

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A coloured $n$-complex is a homogeneous pseudocomplex $K$ together with a “coloration” of its vertices by $\Delta_n$, which is injective on every simplex.

Given an $(n + 1)$-coloured graph $\Gamma$, we can construct a coloured $n$-complex $K(\Gamma)$ in the following way:

- take an $n$-simplex $s(v)$ for each $v \in V(\Gamma)$ and label its vertices by $\Delta_n$;
- for each $c \in \Delta_n$ and each pair $v, w$ of $c$-adjacent vertices in $\Gamma$, identify the $(n - 1)$-faces of $s(v)$ and $s(w)$ opposite to the vertices labelled $c$, so that equally labelled vertices coincide.

The above construction can be easily reversed in order to associate an $(n + 1)$-coloured graph $\Gamma(K)$ to each coloured $n$-complex $K$. Therefore these constructions give rise to a correspondence between $(n + 1)$-coloured graphs and coloured $n$-complexes.

It is easy to see that $\Gamma(K(\Gamma)) = \Gamma$; conversely $K(\Gamma(K)) = K$ if and only if the disjoint star of every simplex in $K$ is strongly connected. In this case $|K|$ is said to be represented by $\Gamma$.

A contracted $(n + 1)$-coloured graph representing a manifold $M$ is called a crystallization of $M$.

By results in [8] and [3], every $n$-manifold admits crystallizations.

The above definitions, together with a general survey on edge-coloured graphs, can be found in [4].

Given an $(n + 1)$-coloured graph $\Gamma$, each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ of $\Delta_n$ defines a particular imbedding (called regular) of $\Gamma$ into a closed surface $F_\varepsilon$, whose Euler characteristic is (see [5] and [6]):

\[
\chi(F_\varepsilon) = \sum_{i \in \mathbb{Z}_{n+1}} g_{\varepsilon_i, \varepsilon_{i+1}}(\Gamma) + \frac{1}{2}(1 - n)p(\Gamma)
\]

where $g_{ij}(\Gamma)$ is the number of connected components of $\Gamma_{ij}$ and $p(\Gamma)$ is the number of vertices of $\Gamma$.

$F_\varepsilon$ is orientable or non-orientable according to $\Gamma$ being bipartite or not.

The regular genus $\rho(\Gamma)$ of $\Gamma$ is defined as:

\[
\rho(\Gamma) = \min\{\rho_\varepsilon(\Gamma) | \varepsilon \text{ is a cyclic permutation of } \Delta_n\}
\]

where $\rho_\varepsilon(\Gamma)$ denotes the genus of $F_\varepsilon$.

Given an $n$-manifold $M$ the regular genus of $M$ is the minimum among the regular genera of the graphs representing $M$. 
In the following we shall describe a construction, introduced in [7], which, starting from two coloured graphs representing two polyhedra, produces a coloured graph representing their product.

If we apply this construction to the product of spheres, we get several simplifications which, given \( m, n > 0 \), allow us to build, by inductive steps, a graph representing \( S^m \times S^n \). Furthermore we obtain some relations among the numbers of coloured cycles in the resulting graphs, by which we can find a “minimal” permutation (i.e. a cyclic permutation defining a regular imbedding of minimal genus) and we can compute the genera of these graphs in a recursive way. We also give direct formulas in the particular cases of \( m = 2, 3 \).

2. Representing products by edge-coloured graphs

We briefly outline the construction introduced in [7], to obtain “products” of coloured graphs.

Let \( \sigma^m \) (resp. \( \tau^n \)) be an \( m \)-dimensional (resp. \( n \)-dimensional) simplex, whose vertices are labelled by \( \{v_0, \ldots, v_m\} \) (resp. by \( \{w_0, \ldots, w_n\} \)); then the set of the vertices of the product ball complex \( \sigma^m \times \tau^n \) is \( \{(v_r, w_s) | r \in \Delta_m, s \in \Delta_n\} \).

Let \( A(\sigma^m, \tau^n) (m, n > 0) \) be the matrix with \( (m + n + 1) \) columns, whose \( \binom{m+n}{n} \) rows are sequences of elements of

\[
\{v_0, \ldots, v_m\} \times \{w_0, \ldots, w_n\}
\]

of the following type:

\[
(v_m, w_n) = (v_{r_m}, w_{s_n}), \ldots, (v_{r_0}, w_{s_0}) = (v_0, w_0)
\]

\[0 \leq r_0 \leq r_1 \leq \cdots \leq r_m = m, 0 \leq s_0 \leq s_1 \leq \cdots \leq s_n = n.\]

These elements can be thought as “words” of length \( m+n+1 \) in the alphabet \( \{v_0, \ldots, v_m\} \times \{w_0, \ldots, w_n\} \), lexicographically ordered, where each “letter” is obtained by decreasing by one, at each step, the index of one and only one of the two components \( v_r \) and \( w_s \).

The sequences represent the \( (m+n+1) \) vertices of \( \{v_0, \ldots, v_m\} \times \{w_0, \ldots, w_n\} \) which span the maximal simplexes of a simplicial triangulation \( \sigma^m \times \tau^n \) of \( \sigma^m \times \tau^n \) (see [2], [9]).
The matrix \( A(\sigma^m, \tau^n) \) can be constructed according to the following scheme:

\[
A(\sigma^m, \tau^n) = \begin{pmatrix}
(v_m, w_n) \\
\vdots \\
\vdots \\
(v_m, w_n)
\end{pmatrix}
\]

where \( A(\sigma^{m-1}, \tau^n) \) (resp. \( A(\sigma^m, \tau^{n-1}) \)) represents the simplicial complex \( \sigma^{m-1} \boxtimes \tau^n \) (resp. \( \sigma^m \boxtimes \tau^{n-1} \)), obtained by deleting the vertex \( v_m \) (resp. \( w_n \)) from \( \sigma^m \) (resp. \( \tau^n \)) and has \( \binom{m+n-1}{n} \) (resp. \( \binom{m+n-1}{m} \)) rows.

Let \( \Gamma' \) (resp. \( \Gamma'' \)) be an \((m+1)\)-coloured (resp. \((n+1)\)-coloured) graph, an \((m+n+1)\)-coloured graph \( \Gamma' \boxtimes \Gamma'' \) representing \(|K(\Gamma') \times K(\Gamma'')|\) can be obtained in the following way:

- for each pair \((\alpha^i, \beta_j)\) of vertices of \( V(\Gamma') \times V(\Gamma'') \), consider the \( \binom{m+n}{n} \) vertices \( \delta_j^i(k) \) which are in one-to-one correspondence with the rows of the matrix \( A(\sigma_i^m, \tau_j^n) \), where \( \sigma_i^m \) (resp. \( \tau_j^n \)) is the \( m \)-simplex (resp. \( n \)-simplex) of \( K(\Gamma') \) (resp. \( K(\Gamma'') \)) represented by \( \alpha^i \) (resp. \( \beta_j \));
- set \( V(\Gamma' \boxtimes \Gamma'') = \{ \delta_j^i(k) | i = 1, \ldots, \#V(\Gamma'), j = 1, \ldots, \#V(\Gamma''), k = 1, \ldots, \binom{m+n}{n} \} \);
- for each vertex \( \delta_j^i(k) \in V(\Gamma' \boxtimes \Gamma'') \) \( (i = 1, \ldots, \#V(\Gamma'), j = 1, \ldots, \#V(\Gamma''), k = 1, \ldots, \binom{m+n}{n}) \), let us denote by \( \omega_k \) its corresponding row of \( A(\sigma_i^m, \tau_j^n) \); then:
  a) for each \( d \in \Delta_{m+n} \), delete from \( \omega_k \) the unique element \((v_r, w_s)\) such that \( r + s = d \), yielding a sequence \( \omega_k(d) \). If there exists another row \( \omega_h \) of \( A(\sigma_i^m, \tau_j^n) \) such that \( \omega_k(d) = \omega_h(d) \), then the way the matrix is constructed guarantees that it is unique; in this case join \( \delta_j^i(k) \) and \( \delta_j^i(h) \) by a \( d \)-coloured edge;
  b) if \( v_r \) (resp. \( w_s \)) appears exactly once in a pair \((v_r, w_s')\) (resp. \((v_r', w_s)\)) of \( \omega_k \) for some \( r \in \Delta_m \) (resp. for some \( s \in \Delta_n \), let \( \alpha^i \) (resp. \( \beta_j \)) be the vertex of \( \Gamma' \) (resp. \( \Gamma'' \)) \( r \)-adjacent with \( \alpha^i \) (resp. \( s \)-adjacent with \( \beta_j \)). Join \( \delta_j^i(k) \) and \( \delta_j^i(k) \) (resp. \( \delta_j^i(k) \) and \( \delta_j^i(k) \)) by a \( d \)-coloured edge, with \( d = r + s' \) (resp. \( d = r' + s \)).

In the particular case of products of spheres, we can simplify the above procedure by using the standard \((p+1)\)-coloured graph \( \Gamma^{(p)} \) representing \( S^p \) and having two vertices joined by \( p+1 \) edges.

Starting from \( \Gamma^{(m)} \) and \( \Gamma^{(n)} \), we construct \( \Gamma^{(m)} \boxtimes \Gamma^{(n)} \) as follows:

- \( \#V(\Gamma^{(m)} \boxtimes \Gamma^{(n)}) = 4 \binom{m+n}{n} \);
- if $\omega_k(\hat{d}) = \omega_h(\hat{d})$, join $\delta^i_j(k)$ and $\delta^i_j(h)$ ($i, j = 1, 2$) by a $d$-coloured edge;
- if $v_r$ (resp. $w_s$) appears exactly once in a pair $(v_r, w_s)$ (resp. $(v_{r'}, w_{s'})$) of $\omega_k$, join $\delta^1_1(k)$ with $\delta^2_1(k)$ (resp. with $\delta^1_2(k)$) and $\delta^2_2(k)$ with $\delta^1_2(k)$ (resp. with $\delta^1_1(k)$) by a $d$-coloured edge, with $d = r + s'$ (resp. $d = r' + s$).

It is clear that the structure of this “product” graph depends only on the structure of the matrix $A(\sigma^m, \tau^n)$; moreover, the inductive construction of $A(\sigma^m, \tau^n)$ allows us to describe a method to build $\Gamma^{(m)} \boxtimes \Gamma^{(n)}$, starting from $\Gamma^{(m-1)} \boxtimes \Gamma^{(n)}$ and $\Gamma^{(m)} \boxtimes \Gamma^{(n-1)}$, without further reference to $A(\sigma^m, \tau^n)$. Construct an $(m+n+1)$-coloured graph $\Gamma^{(m,n)}$ as follows:

- $V(\Gamma^{(m,n)}) = V(\Gamma^{(m-1)} \boxtimes \Gamma^{(n)}) \cup V(\Gamma^{(m)} \boxtimes \Gamma^{(n-1)}) = \{ \delta^i_j(k) | i, j = 1, 2, k = 1, \ldots, \binom{m+n-1}{n} \} \cup \{ \tilde{\delta}^i_j(k) | i, j = 1, 2, k = 1, \ldots, \binom{m+n-1}{n} \}$;

- for each $k = 1, \ldots, \binom{m+n-1}{n}$ (resp. $k = 1, \ldots, \binom{m+n-1}{n-1}$) join $\delta^1_1(k)$ with $\delta^2_1(k)$ (resp. $\tilde{\delta}^1_1(k)$ with $\tilde{\delta}^2_1(k)$) and $\delta^2_2(k)$ with $\delta^1_2(k)$ (resp. $\tilde{\delta}^2_2(k)$ with $\tilde{\delta}^1_2(k)$) by an $(m+n)$-coloured edge;

- for each $k = \binom{m+n-2}{n} + 1, \ldots, \binom{m+n-2}{n-1} + \binom{m+n-2}{n-1}$ join $\delta^i_j(k)$ and $\tilde{\delta}^i_j(k - \binom{m+n-2}{n-1})$ (resp. $i, j = 1, 2$) by an $(m+n-1)$-coloured edge, for the remaining vertices of $\Gamma^{(m,n)}$ re-establish the edges as they are in $\Gamma^{(m-1)} \boxtimes \Gamma^{(n)}$ and $\Gamma^{(m)} \boxtimes \Gamma^{(n-1)}$.

**Proposition 1.** $\Gamma^{(m,n)} = \Gamma^{(m)} \boxtimes \Gamma^{(n)}$.

**Proof.** Note that, for each $d \neq m+n-1$, if two rows of the submatrix $B$ (resp. $C$) of $A$ corresponding to $A(\sigma^{m-1}, \tau^n)$ (resp. $A(\sigma^m, \tau^{n-1})$), say $\omega_k$ and $\omega_h$, lead to equal sequences $\omega_k(\hat{d})$ and $\omega_h(\hat{d})$ in $B$ (resp. in $C$) they also lead to equal sequences in $A$; furthermore if $v_r \neq w_n$, (resp. $w_s$ or $v_s \neq v_m$) appears exactly once in a row of $B$ (resp. of $C$), then it appears exactly once in the same row of $A$. Thus all $d$-coloured edges ($d \neq m+n-1$) of $\Gamma^{(m-1,n)}$ and $\Gamma^{(m,n-1)}$ remain unchanged in $\Gamma^{(m,n)}$.

Furthermore, following the more detailed scheme below for the matrix $A(\sigma^m, \tau^n)$, it is easy to see that:

$$A = A(\sigma^m, \tau^n) = \begin{pmatrix} (v_m, w_n) & (v_{m-1}, w_n) & B'' = A(\sigma^{m-2}, \tau^n) \\ \vdots & (v_{m-1}, w_n) & B' = A(\sigma^{m-1}, \tau^{n-1}) \\ \vdots & (v_m, w_{n-1}) & C' = A(\sigma^{m-1}, \tau^{n-1}) \\ (v_m, w_n) & (v_m, w_{n-1}) & C'' = A(\sigma^m, \tau^{n-2}) \end{pmatrix}$$

a) $w_n$ (resp. $v_m$) appears once in all rows of the submatrix $B'$ (resp. $C'$) corresponding to $A(\sigma^{m-1}, \tau^{n-1})$, but twice in all the corresponding rows of $A$, i.e. all the $(m+n-1)$-coloured edges of
\( \Gamma^{(m-1,n)} \) and \( \Gamma^{(m,n-1)} \) joining the vertices corresponding to \( B' \) and 
\( C' \) disappear in \( A \);

b) each row of \( B' \), with the element \((v_{m-1}, w_n)\) deleted, is equal to 
a row of \( C' \), with the element \((v_m, w_{n-1})\) deleted, therefore the 
 corresponding vertices are joined by \((m + n - 1)\)-coloured edges;

c) \( v_m \) (resp. \( w_n \)) appears once in the first \( \binom{m+n-1}{n} \) (resp. in the 
 last \( \binom{m+n-1}{m} \)) rows of \( A \), therefore the corresponding vertices are 
 joined by \((m + n)\)-coloured edges.

\( \square \)

Using the above construction and starting from the \((r + 2)\)-coloured 
graphs \( \Gamma^{(r,1)} \) and \( \Gamma^{(r,1)} \) \((r \geq 1)\), it is possible to build by successive steps, the 
\((m + n + 1)\)-coloured graph \( \Gamma^{(m,n)} \), for each \( m, n > 0 \).

**Remark 1.** Note that all \( \Gamma^{(m,n)} \) have a double symmetry. In fact, for 
each \( k = 1, \ldots, \binom{m+n}{n} \), each edge between the vertices \( \delta^1(k) \) and \( \delta^2(k) \) 
(resp. \( \delta^1(k) \) and \( \delta^2(k) \)) has a corresponding edge, with the same colour, 
between \( \delta^2(k) \) and \( \delta^1(k) \) (resp. \( \delta^2(k) \) and \( \delta^1(k) \)).

An easily implemented program allows us to build \( \Gamma^{(m,n)} \) for each 
\( m, n > 0 \).

As an example, figure 1 shows \( \Gamma^{(3,3)} \). Since its number of vertices is 
too big (= 80) to fit the picture, we only drew part of the graph, which, 
because of the symmetries, is sufficient to represent the whole of it.

3. The genus of \( \Gamma^{(m,n)} \)

Let us denote by \( g_{cd} \), where \( c, d \in \Delta_{m+n} \) (resp. \( g_{cd} \) where \( c, d \in \Delta_{m+n-1} \)) 
the number of connected components of \( \Gamma^{(m,n)} \) (resp. \( \Gamma^{(m-1,n)} \)) (resp. \( \Gamma^{(m,n-1)} \)). Moreover, let 
\( \alpha^c_{m,n} \) (resp. \( \beta^c_{m,n} \)) \((c \in \Delta_{m+n-2})\) denote the number of \( \{c, m + n\}\)-
residues of length two of \( \Gamma^{(m,n)} \), whose vertices correspond to rows of 
the submatrix \( A(\sigma^m, \tau^n-1) \) (resp. \( A(\sigma^m-1, \tau^n) \)) of \( A(\sigma^m, \tau^n) \) (see the 
scheme above).
Figure 1
Lemma 2. We have the following equalities:

\[ g_{cd} = \bar{g}_{cd} + \tilde{g}_{cd} \quad \text{for each } c, d \in \Delta_{m+n-2} \]
\[ g_{c \ m+n} = \bar{g}_c m+n-1 + \tilde{g}_c m+n-1 \quad \text{for each } c \in \Delta_{m+n-2} \]
\[ g_{m+n-1 \ m+n} = \binom{m + n - 2}{n - 1} + 2 \binom{m + n - 2}{n} + 2 \binom{m + n - 2}{n - 2} \]
\[ g_{c \ m+n-1} = \bar{g}_c m+n-1 + \tilde{g}_c m+n-1 - \frac{1}{2} \left( \alpha_{m-1,n}^c + \beta_{m,n-1}^c \right) \]
for each \( c \in \Delta_{m+n-3} \)
\[ g_{m+n-2 \ m+n-1} = \bar{g}_{m+n-2} m+n-1 + \tilde{g}_{m+n-2} m+n-1 - \binom{m + n - 2}{n - 1}. \]

Proof. By the construction of section 2 it is clear that all \( c \)-coloured edges \( (c \in \Delta_{m+n-2}) \) of \( \Gamma^{(m,n)} \) are the same as in \( \Gamma^{(m-1,n)} \) and \( \Gamma^{(m,n-1)} \), while the \( (m+n) \)-coloured edges in \( \Gamma^{(m,n)} \) take the places of the \( (m+n-1) \)-coloured edges of \( \Gamma^{(m-1,n)} \) and \( \Gamma^{(m,n-1)} \); therefore we obtain equalities 1) and 2).

To prove the third equality, recall the scheme for \( A(\sigma^m, \tau^n) \) in the proof of Proposition 1.

Note that, for each row \( \omega_k \) of \( B'' \) (resp. \( C'' \)), we have two \( \{m+n-1, m+n\} \)-residues, whose sets of vertices are \( \{\delta_1^1(k), \delta_1^2(k)\} \) and \( \{\tilde{\delta}_2(k), \tilde{\delta}_1(k)\} \) (resp. \( \{\bar{\delta}_1^1(k), \bar{\delta}_1^2(k)\} \) and \( \{\bar{\tilde{\delta}}_2(k), \bar{\tilde{\delta}}_1(k)\} \)).

Furthermore, for each \( k = \binom{m+n-2}{n} + 1, \ldots, \binom{m+n-2}{n-2} + \binom{m+n-2}{n-1} \), we have only one \( \{m+n-1, m+n\} \)-residue, whose set of vertices is \( \{\bar{\delta}_1^1(k), \bar{\delta}_1^2(k), \bar{\tilde{\delta}}_2(k), \bar{\tilde{\delta}}_1(k), \bar{\tilde{\delta}}_1^1(h), \bar{\tilde{\delta}}_1^2(h), \bar{\tilde{\delta}}_2^1(h), \bar{\tilde{\delta}}_2^2(h)\} \), where \( h = k - \binom{m+n-2}{n} \) (see figure 2). Equality (3) follows.

Let us now consider the \( \{c, m+n-1\} \)-residues of \( \Gamma^{(m-1,n)} \) and \( \Gamma^{(m,n-1)} \) \( (c \in \Delta_{m+n-2}) \); note that those having all vertices corresponding to rows of \( B'' \) or \( C'' \) don’t change in \( \Gamma^{(m,n)} \).

For \( c \neq m+n-2 \), we have the following situations:

(i) for every pair of length two \( \{c, m+n-1\} \)-residues of \( \Gamma^{(m-1,n)} \) (resp. \( \Gamma^{(m,n-1)} \)) corresponding to a row \( \omega_k \) of \( B' \) (resp. \( C' \)), there exists exactly one \( \{c, m+n-1\} \)-residue of \( \Gamma^{(m,n-1)} \) (resp. \( \Gamma^{(m-1,n)} \)) of length four, whose vertices correspond to the row \( \omega_k \) of \( C' \), with \( h = k - \binom{m+n-2}{n} \) (resp. of \( B' \) with \( h = k + \binom{m+n-2}{n} \)) and conversely;

(ii) for every pair of length four \( \{c, m+n-1\} \)-residues of \( \Gamma^{(m-1,n)} \), whose sets of vertices are \( \{\delta_i^1(k), \delta_i^2(h)|i = 1, 2\} \) and \( \{\tilde{\delta}_i^1(k), \tilde{\delta}_i^2(h)|i = 1, 2\} \), corresponding to the rows \( \omega_k \) and \( \omega_h \) of \( B' \), there exists
exactly two \( \{c, m+n-1\} \)-residues of \( \Gamma^{(m,n-1)} \) of length four, whose sets of vertices are \( \{ \delta_1^i(k'), \tilde{\delta}_1^i(h') \mid i = 1, 2 \} \) and \( \{ \delta_2^i(k'), \tilde{\delta}_2^i(h') \mid i = 1, 2 \} \), corresponding to the rows \( \omega_{k'} \) and \( \omega_{h'} \) of \( C' \), with \( k' = k - \binom{m+n-2}{n} \) and \( h' = h - \binom{m+n-2}{n} \) and conversely.

These are the only \( \{c, m+n-1\} \)-residues which change in \( \Gamma^{(m,n)} \). It is easy to see that in case (ii) the number of the residues doesn’t change and in case (i) the three residues produce two of length four in \( \Gamma^{(m,n)} \).

Finally, let us consider the case \( c = m + n - 2 \). The only \( \{m+n-2, m+n-1\} \)-residues changing in \( \Gamma^{(m,n)} \), are as follows:

(iii) for each \( k = \binom{m+n-2}{n} + 1, \ldots, \binom{m+n-2}{n} + \binom{m+n-3}{n-1} \) (resp. \( k = \binom{m+n-3}{n-1} + 1, \ldots, \binom{m+n-3}{n-1} \)), there is exactly one \( \{m+n-2, m+n-1\} \)-residue of length eight in \( \Gamma^{(m-1,n)} \) (resp. in \( \Gamma^{(m,n-1)} \)), whose set of vertices is \( \{ \delta_1^i(k), \tilde{\delta}_1^i(h) \mid i, j = 1, 2 \} \), \( h = k - \binom{m+n-3}{n-1} \) (resp. \( \{ \tilde{\delta}_1^i(k), \delta_1^i(h) \mid i, j = 1, 2 \} \), \( h = k + \binom{m+n-3}{n-2} \)), to which corresponds a pair of length two \( \{m+n-2, m+n-1\} \)-residues of \( \Gamma^{(m,n-1)} \) (resp. of \( \Gamma^{(m-1,n)} \)), whose sets of vertices are \( \{ \delta_1^i(k'), \tilde{\delta}_2^i(k') \} \) and \( \{ \tilde{\delta}_2^i(k'), \delta_1^i(k') \} \), with \( k' = k - \binom{m+n-2}{n} \) (resp. \( \delta_1^i(k'), \tilde{\delta}_1^i(k') \)) and \( \{ \delta_2^i(k'), \tilde{\delta}_1^i(k') \} \), with \( k' = k + \binom{m+n-2}{n} \).

Since, as can be directly seen, every three residues which correspond, yield two of length six in \( \Gamma^{(m,n)} \), equality 5) easily follows. \( \square \)

Let us now consider the graphs \( \Gamma^{(1,n)} \) \( (n = 1, 2, \ldots) \), which are shown in figure 3.

An easy calculation gives:

\[ g_{01} = g_{02} = \cdots = g_{0n} = 2n - 1 \]
\[ g_{n+1} = 2n \]
\[ g_{1n+1} = g_{2n+1} = \cdots = g_{nn+1} = 2n - 1 \]
\[ g_{cd} = 2(n-1) \quad \text{for each} \ c = 1, \ldots, n - 1 \quad \text{and for each} \ d = 1, \ldots, n. \]

The following result guarantees that similar relations hold among the number of residues \( g_{cd} \) of \( \Gamma^{(m,n)} (m, n > 0) \):

\textbf{Proposition 2.} For each \( m, n > 0 \), there exist constants \( r_{m,n}, s_{m,n}, t_{m,n}, u_{m,n} \) such that

\[ g_{0c} = g_{0c} m+n = r_{m,n} \quad \text{for each} \ c = 1, \ldots, m + n - 1 \]
\[ g_{0m+n} = s_{m,n} \]
\[ g_{c c+1} = t_{m,n} \quad \text{for each} \ c = 1, \ldots, m + n - 2 \]
\[ g_{cd} = u_{m,n} \quad \text{for each} \ c, d = 1, \ldots, m + n - 1 \quad \text{and} \ d \neq c + 1. \]
Furthermore, if \( m > 1 \) and \( n > 1 \)

\[
r_{m,n} = \binom{m+n-2}{n-1} + 2\binom{m+n-2}{n} + 2\binom{m+n-2}{n-2}
\]

and

\[
r_{m,n} = r_{m-1,n} + r_{m,n-1}
\]

\[
s_{m,n} = s_{m-1,n} + s_{m,n-1}
\]

\[
t_{m,n} = t_{m-1,n} + t_{m,n-1}
\]

\[
u_{m,n} = u_{m-1,n} + u_{m,n-1} \quad \text{(if } m > 2 \text{ or } n > 2)\]

with \( t_{m,n} \leq u_{m,n} \leq r_{m,n} \leq s_{m,n} \), for each \( \{m,n\} \neq \{1,2\} \).

**Proof.** If \( c, d \neq m + n - 1 \) or \( c, d \in \{m + n - 2, m + n - 1\} \), it follows easily by induction on \( m \) and \( n \), and making use of equalities 1) - 3) and 5). An easy calculation shows that \( r_{m,n} = r_{m-1,n} + r_{m,n-1} \). Furthermore, it is easy to see that, for each \( c \neq m + n - 2 \), we have:

\[
\alpha_{m-1,n}^c = \alpha_{m-2,n}^c + \alpha_{m-1,n-1}^c \quad \text{and} \quad \beta_{m,n-1}^c = \beta_{m,n-2}^c + \beta_{m-1,n-1}^c.
\]
Therefore, by applying induction to equality 4), we complete the proof.

Let us consider now a cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{m+n})$ of $\Delta_{m+n}$. We can always suppose that $\varepsilon_{m+n} = m + n$. It is clear, by formula (*), that for an $\varepsilon$ corresponding to a surface $F_\varepsilon$ of minimal genus for $\Gamma^{(m,n)}$, the sum $\sum_{i \in \varepsilon_{m+n}} g_{\varepsilon_i \varepsilon_{i+1}}$ must be maximal.

First note that, by Proposition 2, $g_{i \ i+1} \leq g_{ij}$ for each $i, j \neq 0, m+n$ and $i \neq m+n-1$.

Therefore it is sufficient to consider permutations which have all pairs $\varepsilon_i, \varepsilon_{i+1}$ (with $\varepsilon_i, \varepsilon_{i+1} \not\in \{0, m+n\}$) made by non-consecutive numbers (i.e. $\varepsilon_{i+1} \neq \varepsilon_i + 1$ and conversely). There are essentially two types of
such permutations:

\[ e^{(1)} = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1}, 0, \varepsilon_{k+1}, \ldots, \varepsilon_{m+n-1}, m + n) \]

if 0 is not “near” \((m + n)\)

\[ e^{(2)} = (0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m+n-2}, \varepsilon_{m+n-1}, m + n) \]

if 0 is “near” \((m + n)\)

where all pairs \(e_i, e_{i+1}\) are non-consecutive numbers.

If \(m + n > 4\) we can always build such permutations in the following way:

\(e^{(1)}:\) if \((m + n)\) is even (resp. odd) put all even (resp. odd) numbers after 0 and all odd (resp. even) before 0;

\(e^{(2)}:\) if \((m + n)\) is even (resp. odd) put first all the odd (resp. even) numbers and then the even (resp. odd) ones, all in increasing order.

From now on we suppose \(m + n > 4\). Let us compute \(\sum_{i\in\mathbb{Z}_{m+n}} g_{e_i e_{i+1}}\) for \(e^{(1)}\) and \(e^{(2)}\):

\[ e^{(1)} : g_{e_0 e_1} + \cdots + g_{e_{k-2} e_{k-1}} + g_{e_{k-1} 0} + g_{0 e_{k+1}} + \cdots \]

\[ + g_{e_{m+n-1} m+n} + g_{m+n 0} \]

\[ = (k - 1)u_{m,n} + r_{m,n} + r_{m,n} + (m + n - k - 2)u_{m,n} + r_{m,n} + r_{m,n} \]

\[ = 4r_{m,n} + (m + n - 3)u_{m,n} \]

\[ e^{(2)} : g_{0 e_1} + g_{e_1 e_2} + \cdots + g_{e_{m+n-2} e_{m+n-1}} + g_{e_{m+n-1} m+n} + g_{m+n 0} \]

\[ = r_{m,n} + (m + n - 2)u_{m,n} + r_{m,n} + s_{m,n} \]

\[ = 2r_{m,n} + s_{m,n} + (m + n - 2)u_{m,n}. \]

It is easy to see, by using induction, that \(2r_{m,n} = s_{m,n} + t_{m,n}\). Since \(t_{m,n} \leq u_{m,n}\) we have \(2r_{m,n} \leq s_{m,n} + u_{m,n}\).

Comparing the above inequalities with the formulas just found, we have:

\[ \sum_{i\in\mathbb{Z}_{m+n}} g_{e_i} g_{e_{i+1}}^{(1)} \leq \sum_{i\in\mathbb{Z}_{m+n}} g_{e_i} g_{e_{i+1}}^{(2)}. \]

Hence, by applying formula (*) to \(\Gamma^{(m,n)}\) and \(e^{(2)}\), we can state the following result for the genus of the “product” graphs:

\textbf{Proposition 3.} For each \(m, n > 0\), \(m + n > 4\), we have:

\[ \rho(\Gamma^{(m,n)}) = 1 - r_{m,n} - \frac{1}{2} s_{m,n} - \frac{1}{2} (m + n - 2)u_{m,n} + (m + n - 1) \binom{m + n}{n}. \]
Remark 2. If \( m + n = 4 \), the only interesting case for the genus is for \( m = n = 2 \) (since all \( \Gamma^{(1,n)} \) have genus 1 (see [7])). We can’t find a permutation of type \( \varepsilon^{(2)} \) for \( \Delta_4 \), since we always have at least two consecutive numbers, therefore we must compare the sum of the \( g_{\varepsilon_i \varepsilon_{i+1}} \)'s for the two permutations: \((3,1,0,2,4)\) and \((0,1,3,2,4)\). The calculation shows that both permutations are minimal and the genus of \( \Gamma^{(2,2)} \) turns out to be 4. Actually this is the regular genus of \( S^2 \times S^2 \), as proved in [7].

Let us consider some particular cases:

Proposition 4. For each \( n \geq 3 \), \( \rho(\Gamma^{(2,n)}) = n^2 - 1 \).

Proof.

\[
\begin{align*}
r_{2,n} &= 2n - 1 + r_{2,n-1} \\
s_{2,n} &= 2n + s_{2,n-1} \\
t_{2,n} &= 2(n - 1) + t_{2,n-1} \\
u_{2,n} &= 2(n - 1) + u_{2,n-1}.
\end{align*}
\]

Moreover \( t_{2,n} = s_{2,n-1} \) for each \( n \geq 1 \). In fact \( t_{2,2} = s_{2,1} = 4 \) (see figure 3) and supposing that \( t_{2,n-1} = s_{2,n-2} \), it follows:

\[
t_{2,n} = 2(n - 1) + t_{2,n-1} = 2(n - 1) + s_{2,n-2} = s_{1,n-1} + s_{2,n-2} = s_{2,n-1}
\]

Similar calculations give: \( r_{2,n} = n + s_{2,n-1} \) and \( u_{2,n} = 2 + s_{2,n-1} \). Furthermore:

\[
s_{2,n} = 2n + s_{2,n-1} = 2n + 2(n - 1) + s_{2,n-2} = 2n + 2(n - 1) + 2(n - 2) + \cdots + 4 + 4 = 2(n + (n - 1) + (n - 2) + \cdots + 2 + 1) - 2 + 4 = n(n + 1) + 2.
\]

Applying the equalities above, we have:

\[
\begin{align*}
r_{2,n} &= n + n(n - 1) + 2 = n^2 + 2 \\
u_{2,n} &= 2 + n(n - 1) + 2 = n^2 - n + 4.
\end{align*}
\]

Suppose now \( n > 2 \) and compute the genus of \( \Gamma^{(2,n)} \) using Proposition 3.

\[
\rho(\Gamma^{(2,n)}) = 1 - r_{2,n} - \frac{1}{2} s_{2,n} - \frac{1}{2} n u_{2,n} + (n + 1) \binom{n + 2}{n}
\]

\[
= 1 - n^2 - 2 - \frac{1}{2} (n(n + 1) + 2) - \frac{1}{2} n(n^2 - n + 4)
\]

\[
+ \frac{1}{2} (n + 1)^2 (n + 2) = n^2 - 1.
\]

\( \square \)
As a direct consequence of the formula above, we have

**Corollary 4.** For each \( n \geq 3 \), \( G(S^2 \times S^n) \leq n^2 - 1 \).

**Remark 3.** If \( n = 3 \) the statement of Corollary 4 is actually an equality, as proved in [1, Corollary I].

Proposition 4 and Corollary 4, together with Remarks 2 and 3, suggest the following:

**Conjecture.** For each \( n \geq 3 \), \( G(S^2 \times S^n) = n^2 - 1 \).

**Proposition 5.** For each \( n \geq 1 \), \( \rho(G^{(3,n)}) = \frac{2}{3}n^3 + n^2 - \frac{2}{3}n \).

**Proof.**

\[
\begin{align*}
r_{3,n} &= r_{2,n} + r_{2,n-1} + r_{2,n-2} + \cdots + r_{2,2} + r_{3,1} \\
&= (n^2 + 2) + ((n - 1)^2 + 2) + \cdots + (4 + 2) + 5 \\
&= \sum_{i=1}^{n} i^2 + 2n + 2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{13}{6}n + 2 \\
s_{3,n} &= (n(n + 1) + 2) + ((n - 1)n + 2) + \cdots + (6 + 2) + 6 \\
&= \sum_{i=1}^{n} i(i + 1) + 2n + 2 \\
&= \frac{1}{3}n^3 + \frac{8}{3}n + 2 \\
u_{3,n} &= (n^2 - n + 4) + ((n - 1)^2 - (n - 1) + 4) + \cdots + (4 - 2 + 4) + 4 \\
&= \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i + 4n \\
&= \frac{1}{3}n^3 + \frac{11}{3}n
\end{align*}
\]

The result follows directly from Proposition 3.

Hence we have the following:

**Corollary 5.** For each \( n \geq 3 \), \( G(S^3 \times S^n) \leq \frac{2}{3}n^3 + n^2 - \frac{2}{3}n \).

**Remark 4.** Again by [1, Corollary 1], the statement of Corollary 5 is an equality for \( n = 2 \).
On the genus of $S^m \times S^n$

References


Dipartimento di Matematica Pura ed Applicata
Università di Modena e Reggio Emilia
Via Campi 213 B
I-41100 MODENA, Italy
E-mail: cristofori.paola@unimo.it